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ON NONMEASURABLE SUBGROUPS IN GENERALIZED MEASURABLE STRUCTURES

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Abstract. In this paper, we prove a result on nonmeasurable subgroups in commutative Polish groups with respect to more generalized structures than σ-finite measures.

1. Introduction

It can be shown by using Continuum hypothesis that there exists a countable family of subgroups of $\mathbb{R}$ such that no nonzero, σ-finite, diffused measure exists which includes all of these sets in its domain. Kharazishvili [5] extended these results to commutative groups. He proved the following two theorems:

Theorem K1. *Assume Continuum hypothesis. Let G be a commutative group having the cardinality of the continuum. Then there exists a countable family $(G_i)_{i<\omega}$ (ω is the first infinite cardinal) of subgroups of G such that for any non-zero, σ-finite, diffused measure μ on G, at least one of the groups, is nonmeasurable with respect to μ.*

Theorem K2. *Let G be any commutative group of cardinality ω_1 (the first uncountable cardinal). Then there exists a countable family $(G_i)_{i<\omega}$ (ω is the first infinite cardinal) of subgroups of G such that for every non-zero, σ-finite, diffused measure μ on G, at least one of the groups G_i is nonmeasurable with respect to μ.*

It may be noted that Theorem K2 does not require Continuum hypothesis.

In proving the above two theorems, Kharazishvili used two different type of matrices, namely, the Banach–Kuratowski matrix [6] and the admissible transfinite matrix [6] over any set of cardinality ω_1.

Definition 1.1. Given any set E with $\operatorname{card}(E) = \omega_1$, a double sequence $(E_{m,n})_{m<\omega,n<\omega}$ is called a Banach–Kuratowski matrix if:

(i) $E_{m,0} \subseteq E_{m,1} \subseteq \cdots \subseteq E_{m,n} \subseteq \cdots$ for every $m < \omega$.

(b) $E = \cup\{E_{m,n} : n < \omega\}$ and

(c) for any arbitrarily chosen function $f : \omega \to \omega$, $E_{0,f(0)} \cap E_{1,f(1)} \cap \cdots E_{m,f(m)} \cap \cdots$ is at most countable.

In the family Let $F = \omega^\omega$ of all functions from ω into ω, we define a preordering as follows: $f \preceq g$ iff there exists a natural number $n = n(f, g)$ such that $f(m) \leq g(m)$ for all $m \geq n$. Then under the assumption of continuum hypothesis, a subset $E = \{f_\xi : \xi < \omega_1\}$ can be defined which satisfies the following two conditions:

(a) The set E is cofinal in F, i.e., for any arbitrarily chosen member f of F, there exists $\xi < \omega_1$ such that $f \prec f_\xi$.

(b) For any two ξ and ρ related by the inequality $\xi < \rho < \omega_1$, the relation $f_\rho \preceq f_\xi$ does not hold.

Conditions (a) and (b) imply that $\operatorname{card}(E) = \omega_1$ and if we put $E_{m,n} = \{f_\xi : f_\xi(m) \leq n\}$, then the double sequence $(E_{m,n})_{m<\omega,n<\omega}$ forms a Banach–Kuratowski matrix over E.

It may be checked that there does not exist [6] any nonzero, σ-finite, diffused measure defined simultaneously for all the sets $E_{m,n}$.

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Definition 1.2. Let card(E)$=\omega_1$. A double family $(E_{m,\xi})_{m<\omega,\xi<\omega_1}$ of the subsets of E is called an admissible transfinite matrix for E if:

(i) For each $\xi<\omega_1$, the partial family $(E_{n,\xi})_{n<\omega}$ is increasing by inclusion and $\bigcup_{n<\omega}E_{n,\xi}=E$.

(ii) For any natural number n, there exists a natural number $m=m(n)$ such that for any set $\mathcal{D}\subseteq\omega_1$ having card$(\mathcal{D})=m$, card$(\cap\{E_{n,\xi}:\xi\in\mathcal{D}\})\leq\omega$.

For any set E with card$(E)=\omega_1$, there always exists an admissible transfinite matrix for E. This follows from the existence of Ulam's matrix [6] for E. However, in the above two Definitions, we identify every infinite cardinal with the initial ordinal representing the cardinal.

In this paper, we obtain a modified generalization of the above two theorems in commutative Polish group (i.e., in commutative groups whose underlying topology is separable and completely metrizable). Our development follows a pattern that is similar to that followed by Kharazishvili. The difference is that in our case the nonmeasurable subgroups are the Bernstein subgroups. We also avoid the use of measures and instead replace them by a suitably designed measurable structure formed out of a σ-algebra which is admissible (in a sense defined in the following section) with respect to a small system of sets. It is worth mentioning here that the notion of a small system was originally introduced by Riécan and Neubrunn [10] and subsequently used by several authors to give abstract formulations of many well-known results of the classical measure and integration theory (see [3,4,9,11]).

2. Preliminaries and Results

In a topological space X, a set is called totally imperfect if it contains no nonempty perfect set. A Bernstein set [6] is that which together with its complement are both totally imperfect. It was first constructed by Bernstein in 1908. Bernstein sets are nonmeasurable with respect to the completion of every nonzero σ-finite diffused Borel measure and this phenomenon also characterises a Bernstein set. Since in every uncountable Polish space the family of all nonempty perfect sets has the cardinality of the continuum, Bernstein sets can be constructed in such spaces.

Let X be a nonempty set and $\mathcal{S}$ be a σ-algebra of subsets of X. $\mathcal{S}$ is called diffused if $\{x\}\in\mathcal{S}$ for every $x\in X$. By a small system on X (the Definition is a modified version of the one given in [10]) we mean

Definition 2.1. A sequence $\{\mathcal{N}_n\}_{n<\omega}$, where each $\mathcal{N}_n$ is a class of subsets of X satisfying the following properties:

(i) $\emptyset\in\mathcal{N}_n$.

(ii) $E\in\mathcal{N}_n$ and $F\subseteq E$ implies $F\in\mathcal{N}_n$. In other words, each $\mathcal{N}_n$ is a hereditary class.

(iii) $E\in\mathcal{N}_n$ and $F\in\bigcap_{n=1}^{\infty}\mathcal{N}_n$ implies $E\cup F\in\mathcal{N}_n$.

(iv) For any $m=1,2\ldots$, there exists a natural number $m'>m$ such that for any one-to-one correspondence $k\to n_k$ with $n_k>m'$, $\bigcup_{k=1}^{\infty}E_{n_k}\in\mathcal{N}_m$ whenever $E_{n_k}\in\mathcal{N}_{n_k}$.

(v) For any p,q, there exists $m>p,q$ such that $\mathcal{N}_m\subseteq\mathcal{N}_p,\mathcal{N}_q$ In other words, the system is directed.

We further define a small system $\{\mathcal{N}_n\}_{n<\omega}$ on X to be diffused if $\{x\}\in\mathcal{N}_n$ for all n and $x\in X$.

Definition 2.2. A σ-algebra $\mathcal{S}$ on X is called admissible with respect to a small system $\{\mathcal{N}_n\}_{n<\omega}$ if:

(i) $\mathcal{S}\setminus\mathcal{N}_n\neq\emptyset\neq\mathcal{S}\cap\mathcal{N}_n$.

(ii) Each $\mathcal{N}_n$ has an S-base which means that each $E\in\mathcal{N}_n$ is contained in some $F\in\mathcal{S}\cap\mathcal{N}_n$.

(iii) $\mathcal{S}\setminus\mathcal{N}_n$ satisfies the countable chain condition, which means that the cardinality of any arbitrary collection of mutually disjoint sets from $\mathcal{S}\setminus\mathcal{N}_n$ is atmost ω.

More general Definitions than the above are given in [1,2] and [12]. We set $\mathcal{N}_\infty=\bigcap_{n<\omega}\mathcal{N}_n$. By virtue of conditions (ii), (iv) of Definition 2.1, $\mathcal{N}_\infty$ is a σ-ideal. This σ-algebra $\mathcal{S}$ together with the σ-ideal $\mathcal{N}_\infty$ generate the σ-algebra $\widetilde{\mathcal{S}}$ whose members are of the form $(X\setminus Y)\cup Z$, where $X\in\mathcal{S}$ and $Y,Z\in\mathcal{N}_\infty$ and this gives rise to the measurable structure $(\widetilde{\mathcal{S}},\mathcal{N}_\infty)$. From admissibility of $\mathcal{S}$ with respect to $\{\mathcal{N}_n\}_{n<\omega}$ it follows that $(\widetilde{\mathcal{S}},\mathcal{N}_\infty)$ satisfies the countable chain condition.

Definition 2.3 ([3,10])**.** A small system $\{\mathcal{N}_n\}_{n<\omega}$ is said to be upper semicontinuous relative to a σ-algebra $\mathcal{S}$ if for every nested sequence $\{E_n\}_{n<\omega}$ of the sets from $\mathcal{S}$ satisfying $E_n\notin\mathcal{N}_m$ for some m and $n=1,2\ldots$, $\bigcap_{n<\omega}E_n\notin\mathcal{N}_\infty$.

More general Definition than the above is given in [3].

Below, we present some propositions which we will need for our purpose.

Proposition 2.4. *If $\{\mathcal{N}_n\}_{n<\omega}$ is upper semicontinuous relative to $\mathcal{S}$, then $\{\mathcal{N}_n\}_{n<\omega}$ is also upper semicontinuous relative to $\widetilde{\mathcal{S}}$.*

Proof. Let $\{E_n\}_{n<\omega}$ be a nested sequence of the sets from $\widetilde{\mathcal{S}}$ such that $E_n \notin \mathcal{N}_m$ for some m and $n = 1, 2 \ldots.$ We write $E_n = F_n \Delta P_n$, where $F_n \in \mathcal{S}$ and $P_n \in \mathcal{N}_\infty$. As $\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_n\}_{n<\omega}$, so, $P_n \subseteq Q_n \in \mathcal{S} \cap \mathcal{N}_\infty$. Hence $\{F_n \setminus \bigcup_{k=1}^n Q_k\}_{n<\omega}$ forms a nested sequence with $F_n \setminus \bigcup_{k=1}^n Q_k \notin \mathcal{N}_m$ for $n = 1, 2 \ldots$ by conditions (ii) and (iii) of Definition 2.1. But $F_n \setminus \bigcup_{i=1}^n Q_i \in \mathcal{S}$ and $\{\mathcal{N}_n\}_{n<\omega}$ is upper semicontinuous relative to $\mathcal{S}$, so, $\bigcap_{n<\omega}(F_n \setminus \bigcup_{i=1}^n Q_i) \notin \mathcal{N}_\infty$ which again implies that $\bigcap_{n<\omega} E_n \notin \mathcal{N}_\infty$. This proves the proposition. □

Proposition 2.5. *If $Y \notin \mathcal{N}_\infty$, then there exists a natural number m such that no subset M of Y may belong to $\mathcal{N}_m$ if its complement in Y, i.e., $Y \setminus M$ belong to $\mathcal{N}_m$.*

Proof. Choose a natural number r such that $Y \notin \mathcal{N}_r$. Since there exist p, q (by condition (iv) of Definition 2.1) such that $\mathcal{N}_p \cup \mathcal{N}_q \subseteq \mathcal{N}_r$ and also m (by condition (v) of Definition 2.1) such that $\mathcal{N}_m \subseteq \mathcal{N}_p$ and $\mathcal{N}_m \subseteq \mathcal{N}_q$, so, if M and $Y \setminus M$ both belong to $\mathcal{N}_m$, then their union belongs to $\mathcal{N}_p \cup \mathcal{N}_q$ and so to $\mathcal{N}_r$, which is a contradiction. □

Proposition 2.6. *If a σ-algebra $\mathcal{S}$ is admissible with respect to a small system $\{\mathcal{N}_n\}_{n<\omega}$, then $\widetilde{\mathcal{S}}$ is also admissible with respect to $\{\mathcal{N}_n\}_{n<\omega}$.*

Proof. The first two conditions of Definition 2.2 are obvious. Let $E = F \Delta P \in \widetilde{\mathcal{S}} \setminus \mathcal{N}_n$ where $F \in \mathcal{S}$ and $P \in \mathcal{N}_\infty$. By condition (ii) of Definition 2.2, there exists $Q \in \mathcal{N}_\infty \cap \mathcal{S}$ such that $P \subseteq Q$. Clearly, $F \setminus Q \in \mathcal{S} \setminus \mathcal{N}_n$, otherwise by condition (ii) and (iii) of Definition 2.1, $E \subseteq (F \setminus Q) \cup Q \in \mathcal{N}_n$. Thus every set in $\widetilde{\mathcal{S}} \setminus \mathcal{N}_n$ contains a set in $\mathcal{S} \setminus \mathcal{N}_n$. But $\mathcal{S}$, being admissible with respect to $\{\mathcal{N}_n\}_{n<\omega}$, $\mathcal{S} \setminus \mathcal{N}_n$, satisfies the countable chain condition. Hence $\widetilde{\mathcal{S}} \setminus \mathcal{N}_\infty$ also satisfies the countable chain condition which proves condition (iii) of Definition 2.2. □

Proposition 2.7. *Let $\{Z_\alpha : \alpha < \omega_1\}$ be an uncountable collection of sets from $\mathcal{S}$ and m be a positive integer such that $\bigcap_{\alpha \in \mathcal{D}} Z_\alpha \in \mathcal{N}_\infty$ for every m-element subset $\mathcal{D}$, where $\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_n\}_{n<\omega}$. Then there exists an uncountable subset A of ω_1 such that $Z_\alpha \in \mathcal{N}_\infty$ for every $\alpha \in A$.*

A proof of the above proposition follows from the notion of admissibility and the inductive argument used in proving Lemma 4 [6, Chapter 13].

Proposition 2.8. *Assuming Continuum hypothesis, any uncountable Polish group can be expressed as the direct sum of two Bernstein subgroups; i.e., if $(G, +)$ is an uncountable Polish group, then $G = H + B$, where $H \cap B = \{0\}$ and H, B are the Bernstein subgroups of G.*

Proof. Since G is an uncountable Polish group, card(G)=c (the cardinality of the continuum), which is again the cardinality of the class of all nonempty perfect subsets of G. By virtue of Continuum hypothesis, we can now consider an enumeration of this class $\{P_\xi : \xi < \omega_1\}$ and, using a transfinite recursion, construct two ω_1-sequences $\{x_\xi : \xi < \omega_1\}$ and $\{y_\xi : \xi < \omega_1\}$ such that (i) $x_\xi, y_\xi \in P_\xi$ for $\xi < \omega_1$; (ii) the family $\{x_\xi : \xi < \omega_1\} \cup \{y_\xi : \xi < \omega_1\}$ is independent in the sense that if $m_1 z_{i_1} + m_2 z_{i_2} + \cdots m_k z_{i_k} = 0$, where $\{i_1, i_2 \ldots i_k\} \subseteq \omega$; $m_1, m_2 \ldots m_k$ are all integers and $\{z_{i_1}, z_{i_2} \ldots z_{i_k}\} \subseteq \{x_\xi : \xi < \omega_1\} \cup \{y_\xi : \xi < \omega_1\}$, then $m_1 = m_2 = \cdots = m_k = 0$.

Let H be the subgroup generated by $\{x_\xi : \xi < \omega_1\}$ and B be the maximal subgroup containing $\{y_\xi : \xi < \omega_1\}$ such that $H \cap B = \{0\}$. Then H and B are the required Bernstein subgroups of G.

The above proof is similar to that of Lemma 1 [7, Chapter 18].

Consider the ideal $\mathcal{I}$ generated by the family $\left\{ \bigcup_{g \in \Sigma} g + B : \Sigma \text{ is a countable subset of } H \right\}$. It is a σ-ideal and also proper because for each member Z of this family, there exists an uncountable family $\{g_\alpha : \alpha < \omega_1\}$ of elements of H such that the sets of the family $\{g_\alpha + Z : \alpha < \omega_1\}$ are pairwise disjoint. The following theorem generalizes Theorem K1 and Theorem K2 in which we assume Continuum hypothesis. □

Theorem 2.9. *In every uncountable commutative Polish group G, there exist a proper σ-ideal $\mathcal{I}$ and a countable family $(B_i)_{i<\omega}$ of Bernstein subgroups such that if $\mathcal{S}$ is a σ-algebra and $\{\mathcal{N}_n\}_{n<\omega}$ is a small system on G satisfying the conditions:*

(i) *$\mathcal{I} \subseteq \mathcal{N}_n$ for all n;*

(ii) *$\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_n\}_{n<\omega}$, then at least one member of the above family of subgroups do not belong to the σ-algebra $\widetilde{\mathcal{S}}$ generated by $\mathcal{S}$ and $\mathcal{N}_\infty$.*

Proof. We first assume that $\{\mathcal{N}_n\}_{n<\omega}$ is upper semicontinuous ralative to $\mathcal{S}$. We set $\mathcal{T} = \{T \subseteq H : T + B \in \widetilde{\mathcal{S}}\}$ and $\mathcal{M}_n = \{M \subseteq H : M + B \in \mathcal{N}_n\}$ $(n < \omega)$. Then it can be easily checked that $\mathcal{T}$ is a σ-algebra and $\{\mathcal{M}_n\}_{n<\omega}$ is a small system of sets in H. Again, as $x + B \subseteq \mathcal{I} \subseteq \mathcal{N}_n \subseteq \widetilde{\mathcal{S}}$ for all n and $x \in H$, so, $\mathcal{T}$ and $\{\mathcal{M}_n\}_{n<\omega}$ are diffused.

By the admissibility of $\mathcal{S}$ with respect to $\{\mathcal{N}_n\}_{n<\omega}$, $\mathcal{S} \setminus \mathcal{N}_n \neq \emptyset$. Therefore $G \notin \mathcal{N}_\infty$ and, consequently, $H \notin \mathcal{M}_\infty$. Now, H being commutative, by Kulikov's theorem (see [7, 8]), we can write $H = \bigcup_{i<\omega} \Gamma_i$, where $\{\Gamma_i\}_{i<\omega}$ is an increasing sequence of commutative subgroups of H and each Γ_i is again a direct sum of cyclic groups. Thus for every $i < \omega$, $\Gamma_i = \sum_{j<\omega_1} [e_{ij}]$, where $[e_{i,j}]$ is the cyclic group generated by the element $e_{i,j}$ in H. Let $E_i = \{e_{i,j} : j < \omega_1\}$ and $(E_{i,k,n})_{k<\omega_1, n<\omega}$ be the Banach Kuratowski matrix over E_i. We put $\Gamma_{i,k,n} = [E_{i,k,n}]$, where for any set E, $[E]$ represents the group generated by E. Then $(\Gamma_{i,k,n})_{k<\omega, n<\omega}$ is a Banach– Kuratowski matrix over Γ_i. We assert that the family $\{\Gamma_i : i < \omega\} \cup \{\Gamma_{i,k,n} : i < \omega, k < \omega, n < \omega\}$ is not contained in $\mathcal{T}$.

Suppose if possible, let all the members of the above family belong to $\mathcal{T}$. Since $\mathcal{M}_\infty$ is a σ-ideal, then there exists i^* such that $\Gamma_{i^*} \notin \mathcal{M}_\infty$. By Proposition 2.5, let m be the natural number such that no subset M of Γ_{i^*} belongs to $\mathcal{M}_m$ if its complement in Γ_{i^*} is in $\mathcal{M}_m$.

We set $G_{k,n} = \Gamma_{i^*} \setminus \Gamma_{i^*,k,n}$; $k, n < \omega$. Then $\bigcap_{n<\omega} G_{k,n} = \emptyset$ and therefore $\bigcap_{n<\omega} (G_{k,n} + B) = B \in \mathcal{N}_\infty$, where $\{G_{k,n} + B\}_{n<\omega}$ is a nested sequence of the sets from $\widetilde{\mathcal{S}}$. Since $\{\mathcal{N}_n\}_{n<\omega}$ is upper semicontinuous relative to $\widetilde{\mathcal{S}}$ (Proposition 2.4), there exists n_k such that $G_{k,n_k} + B \in \mathcal{N}_{n_k}$. Hence $G_{n_k} \in \mathcal{M}_{n_k}$. By condition (iv) of Definition 2.1, we may choose $n_k > m$ such that $\bigcup_{k<\omega} G_{n_k} \in \mathcal{M}_m$. Hence $\bigcap_{k<\omega} \Gamma_{i^*,k,n_k} \notin \mathcal{M}_m$. But $(\Gamma_{i^*,k,n})_{k<\omega, n<\omega}$, being a Banach–Kuratowski matrix, card $(\bigcap_{k<\omega} \Gamma_{i^*,k,n_k}) \leq \omega$. So, $\bigcap_{k<\omega} \Gamma_{i^*,k,n_k} \in \mathcal{M}_\infty$ because $\{\mathcal{M}_n\}_{n<\omega}$ is diffused. This is a contradiction. We denote the family $\{\Gamma_i : i < \omega\} \cup \{\Gamma_{i,k,n} : i < \omega, k < \omega, n < \omega\}$ by $\{G_i : i < \omega\}$. Consequently, not all members of the family $\{G_i + B : i < \omega\}$ which consists of Bernstein groups (a fact which can be easily checked) belong to $\widetilde{\mathcal{S}}$. This proves the theorem.

The above proof rests heavily on the notion of an upper semicontinuity of $\{\mathcal{N}_n\}_{n<\omega}$ relative to $\mathcal{S}$. If we remove this condition, then we can no longer use Banach–Kuratowski matrix. But still then we are able to prove the theorem by suitably constructing an admissible matrix on H. We will show this below.

As before, we use Kulikov's theorem and write $H = \bigcup_{i<\omega} \Gamma_i$, where $\Gamma_i = \sum_{j<\omega_1} [e_{i,j}]$. Let $A_i = \{e_{i,j} : j < \omega_1\}$ and $(A_{i,n,\xi})_{n<\omega_1, \xi<\omega_1}$ be an admissible matrix over A_i. By Lemma 6 [6, Chapter 13], for each i, there exists a countable family $\mathcal{S}_i$ of the sets from A_i such that $\{A_{i,n,\xi} : n < \omega_1, \xi < \omega_1\} \subseteq \sigma(\mathcal{S}_i)$, where $\sigma(\mathcal{S}_i)$ is the σ-algebra generated by $\mathcal{S}_i$. We may assume that $\mathcal{S}_i$ is an algebra, so, $\sigma(\mathcal{S}_i)$ is the monotone class generated by $\mathcal{S}_i$. Then all the subgroups $[A_{i,n,\xi}]$ $(n < \omega, \xi < \omega_1)$ belong to the monotone class generated by the family $\{[Z] : Z \in \mathcal{S}_i\}$. Since $\mathcal{M}_\infty$ is a σ-ideal, there exists i^* such that $\Gamma_{i^*} \notin \mathcal{M}_\infty$. Moreover, $\{[A_{i^*,n,\xi}] : n < \omega_1, \xi < \omega_1\}$ is an admissible matrix over Γ_{i^*}. Suppose all the members of the family $\{\Gamma_i : i < \omega\} \cup \{[Z] : Z \in \bigcup_{i<\omega} \mathcal{S}_i\}$ belong to $\mathcal{T}$. Then so are all the members of the family $\{[Z] : Z \in \mathcal{S}_{i^*}\}$ and, consequently, all the members $\{[A_{i^*,n,\xi}] : n < \omega, \xi < \omega_1\}$ belong to $\mathcal{T}$. Hence there exists a set $\Xi \subseteq \omega_1$ with card(Ξ)=ω_1 and n_0 such that $[A_{i^*,n_0,\xi}] \notin \mathcal{M}_\infty$ for all $\xi \in \Xi$ and $\bigcap\{[A_{i^*,n_0,\xi}] : \xi \in \mathcal{D}\} = \{0\}$ for any set $\mathcal{D} \subseteq \omega_1$ having card$(\mathcal{D})$= $n_0 + 2$. Consequently, $[A_{i^*,n_0,\xi}] + B \notin \mathcal{N}_\infty$ for all $\xi \in \Xi$ and $\bigcap\{[A_{i^*,n_0,\xi}] + B : \xi \in \mathcal{D}\} = \{0\} + B \in \mathcal{N}_\infty$. But this contradicts Proposition 2.7 because $\widetilde{\mathcal{S}}$ is admissible with respect to $\{\mathcal{N}_n\}_{n<\omega}$. Thus all the members of the above family cannot belong to $\mathcal{T}$. We denote this family by $\{G_i : i < \omega\}$. Consequently, not all members of the family $\{G_i + B : i < \omega\}$ which consists of Bernstein groups (a fact which can be easily checked) belong to $\widetilde{\mathcal{S}}$. $\square$

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