Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 137–143

ON $|A|_k$ SUMMABILITY OF FACTORABLE FOURIER SERIES

MEHMET ALI SARIGÖL

Abstract. Some results on the absolute weighted summability of factored Fourier series have recently been proved by Bor [1]. In this paper, using an arbitrary triangle matrix instead of weighted mean matrix, we extend his results to the absolute matrix summability and give some its applications.

1. Introduction

Consider an infinite series Σa_v with the sequence of partial sums $s = (s_n)$ and let (p_n) be a sequence of positive numbers with $P_n = p_0+p_1+\cdots+p_n \to \infty$. The series Σa_v is absolutely weighted summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \left|T_n - T_{n-1}\right|^k < \infty,$$

where T is $(\overline{N}, p_n)$-weighted mean of the sequence s, i.e., $T_n = (1/P_n) \sum_{v=0}^{n} p_v s_v$. This definition was extended by the author to the matrix summability (see [15, 17, 18]) as: let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. A series is summable $|A|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k} \left|A_n(s) - A_{n-1}(s)\right|^k < \infty,$$

where $(A_n(s))$ is an A-transform sequence of sequence s, i.e.,

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n \geq 0.$$

Note that in a special case, where A is a weighted mean and a Cesàro matrix of order $\alpha > -1$, the method $|A|_k$ reduces to the methods $\left|\overline{N}, p_n\right|_k$ and $|C, \alpha, (\alpha - 1)(1 - 1/k)|_k$, $k \geq 1$, in Flett's notation [8], respectively, where $a_{nv} = p_v/P_n$ and $a_{nv} = E_{n-v}^{\alpha-1}/E_n^{\alpha}$, $0 \leq v \leq n$, and zero otherwise. Here, also

$$E_0^{\alpha} = 0, E_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \cong \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \quad n \geq 1.$$

By t_n we denote a Cesàro mean $(C, 1)$ of the sequence (na_n) and write $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ for any sequence $\lambda = (\lambda_n)$ and $n \geq 0$. Also, the sequence (λ_n) is said to be of bounded variation denoted by $(\lambda_n) \in BV$, if $(\Delta\lambda_n)$ is an absolutely convergent series.

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. The Fourier series of f is defined by

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \sum_{n=1}^{\infty} c_n(x),$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx,$$

2020 *Mathematics Subject Classification.* 26D15, 40D15, 42A24, 46A45.

Key words and phrases. Absolute matrix summability; Summability factors; Fourier series; Hölder and Minkowski Inequalities.

and we also write

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u) du, \quad \alpha > 0,$$

where

$$\phi(t) = \frac{1}{2} \left[f(x-t) + f(x+t) \right].$$

Fourier series and summability theory play important role in analysis and applied mathematics, especially in quantum mechanics and approximation theory. The summability factors of infinite series and Fourier series are one of their oldest research topics that has intensively been studied by now. For more information on the topic, the readers may refer to papers [1–4, 6, 7, 10–14, 16–22] et al.The following results have recently been proved by Bor [1].

Theorem 1.1. *Let (p_n) be a positive sequence with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$ and $X_n = \sum_{v=0}^{n} p_v/P_v$ for $n \geq 0$. Then the series $\Sigma \lambda_v a_v$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if for $\lambda_n \to 0$, $P_n = O(np_n)$, the conditions*

$$\sum_{v=1}^{\infty} v X_v \left| \Delta^2 \lambda_v \right| < \infty,$$

$$\sum_{v=1}^{n} \frac{p_v}{P_v} \frac{|t_v|^k}{X_v^{k-1}} = O(X_n) \quad \text{as} \quad n \to \infty$$

are satisfied.

Theorem 1.2. *The factored Fourier series $\Sigma a_n \lambda_n$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, if $\phi_1(t) \in BV$ and the conditions of Theorem 1.1 are satisfied.*

2. Main Results

In this paper, using an arbitrary triangle matrix instead of a weighted mean matrix, we extend Theorem 1.1 and Theorem 1.2 to the summability method $|A|_k$, $k \geq 1$, and also give some of its applications.

Let $A = (a_{nv})$ be a normal matrix, we define the normal semi-matrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ by

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nv}, \quad \text{for } n, v \geq 0,$$

$$\widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v} \quad \text{and} \quad \widehat{a}_{00} = \overline{a}_{00} = a_{00}.$$

Then it may be noticed that $\widehat{A}$ and $\overline{A}$ are series-to-series and series-to-sequence transformations, respectively, and also,

$$A_n(s) = \sum_{v=0}^{n} \overline{a}_{nv} a_v \quad \text{and} \quad \widehat{A}_n(s) = \overline{A}_n(s) - \overline{A}_{n-1}(s), \quad n \geq 0.$$

So, we establish the following

Theorem 2.1. *Suppose that A is a positive normal matrix such that*

$$(v a_{vv})^{-1} = O(1) \quad \text{as} \quad n \to \infty, \tag{2.1}$$

$$a_{nv} \leq a_{n-1,v} \quad \text{for} \quad 0 \leq v \leq n-1, \tag{2.2}$$

$$\overline{a}_{n0} = 1 \quad \text{for} \quad n \geq 0, \tag{2.3}$$

$$\sum_{v=1}^{n-1} a_{vv} \widehat{a}_{n,v+1} = O(a_{nn}) \quad \text{as} \quad v \to \infty. \tag{2.4}$$

The series $\Sigma\lambda_v a_v$ is then summable $|A|_k$, $k \geq 1$, if the following conditions:

$$\lambda_n \to 0 \quad \text{as} \quad n \to \infty, \tag{2.5}$$

$$l_n = \sum_{v=1}^{n} a_{vv} \to \infty \quad \text{as} \quad n \to \infty,$$

$$\sum_{v=1}^{\infty} v l_v \left|\Delta^2 \lambda_v\right| < \infty, \tag{2.6}$$

$$\sum_{v=1}^{n} \frac{a_{vv} |t_v|^k}{l_v^{k-1}} = O(l_n) \quad \text{as} \quad n \to \infty$$

are satisfied. Note that condition (2.4) can be omitted for $k = 1$.

Also, if $\phi_1 \in BV$, then it is well known (see [6]) that $t_n = O(1)$, where t_n is the Cesàro mean of $(C;1)$ of the sequence (na_n). Hence, the following result is immediately obtained.

Theorem 2.2. *The factored Fourier series $\Sigma a_n \lambda_n$ is summable $|A|_k$, $k \geq 1$, if $\phi_1(t) \in BV$ and the conditions of Theorem* 1.2 *hold.*

It may be noticed that in the special case $A = \left(\overline{N}, p_n\right)$, Theorem 2.1 and Theorem 2.2 are reduced to Theorem 1.1 and Theorem 1.2, respectively.

We require the following lemma to prove our theorems.

Lemma 2.3. *Under the conditions of Theorem* 2.1*, we have*

$$\widehat{a}_{n,v} \geq 0 \quad \textit{for} \quad n, v \geq 0,$$

$$\overline{a}_{n,v} \leq 1 \quad \textit{for} \quad n, v \geq 0, \tag{2.7}$$

$$\sum_{v=1}^{n-1} |a_{nv} - a_{n-1,v}| = O(a_{nn}) \quad \textit{as} \quad n \to \infty, \tag{2.8}$$

$$|\lambda_n| \, |l_n| = O(1) \quad \textit{as} \quad n \to \infty, \tag{2.9}$$

$$\sum_{v=1}^{\infty} l_v |\Delta \lambda_v| < \infty, \tag{2.10}$$

$$n l_n |\Delta \lambda_n| = O(1) \quad \textit{as} \quad n \to \infty. \tag{2.11}$$

Proof. It can be easily obtained by (2.2) and (2.3), and for $0 \leq v \leq n-1$,

$$\widehat{a}_{nv} = \overline{a}_{n0} - \overline{a}_{n-1,0} + \sum_{r=0}^{v-1} (a_{n-1,r} - a_{nr}) = \sum_{r=0}^{v-1} (a_{n-1,r} - a_{nr}) \geq 0,$$

$$\overline{a}_{nv} = 1 - \sum_{r=0}^{v-1} a_{nr} \leq 1,$$

$$\sum_{v=1}^{n-1} |a_{nv} - a_{n-1,v}| = (1 - 1 + a_{n,0} - a_{n-1,0} + a_{nn}) \leq a_{nn}.$$

Also, conditions (2.11), (2.10) and (2.9) are deduced by (2.6) as follow:

$$n l_n |\Delta \lambda_n| \leq \sum_{v=n}^{\infty} v l_v \left|\Delta^2 \lambda_v\right| \leq \sum_{v=1}^{\infty} v l_v \left|\Delta^2 \lambda_v\right| < \infty,$$

$$\sum_{v=1}^{n} l_v |\Delta \lambda_v| \leq \sum_{v=1}^{n-1} \left|\Delta^2 \lambda_v\right| \sum_{r=1}^{v} l_r + |\Delta \lambda_n| \sum_{r=1}^{n} l_r$$

$$\leq \sum_{v=1}^{n-1} v l_v \left|\Delta^2 \lambda_v\right| + n l_n |\Delta \lambda_n| = O(1) \quad \text{as} \quad n \to \infty,$$

$$l_n\,|\lambda_n| \le \sum_{v=n}^{\infty} l_v\,|\Delta\lambda_v| \le \sum_{v=1}^{\infty} l_v\,|\Delta\lambda_v| < \infty.$$

Proof of Theorem 2.1. By $A_n(s)$, we denote an A-transform of the series $\Sigma\lambda_v a_v$. We have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} \sum_{r=0}^{v} \lambda_r a_r = \sum_{v=0}^{n} \overline{a}_{nv}\lambda_v a_v$$

which implies

$$\widehat{A}_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=1}^{n} \widehat{a}_{nv}\lambda_v a_v.$$

Applying Abel's summation to this sum, we arrive at

$$\begin{aligned}\sum_{v=1}^{n} \frac{\widehat{a}_{nv}}{v} v\lambda_v a_v &= \sum_{v=1}^{n-1} \Delta\Big(\frac{\widehat{a}_{nv}\lambda_v}{v}\Big)\sum_{r=1}^{v} r a_r + \frac{\widehat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n} r a_r \\ &= \sum_{v=1}^{n-1} (v+1)\,t_v \Delta\Big(\frac{\widehat{a}_{nv}\lambda_v}{v}\Big) + \frac{\widehat{a}_{nn}\lambda_n\,(n+1)\,t_n}{n}.\end{aligned}$$

By the formula for the difference of the products of sequences (see [9]), we obtain

$$\begin{aligned}\Delta\Big(\frac{\widehat{a}_{nv}\lambda_v}{v}\Big) &= \frac{\lambda_v}{v}\Delta\left(\widehat{a}_{nv}\right) + \widehat{a}_{n,v+1}\Delta\Big(\frac{\lambda_v}{v}\Big) \\ &= \left(a_{nv} - a_{n-1,v}\right)\frac{\lambda_v}{v} + \widehat{a}_{n,v+1}\frac{\Delta\lambda_v}{v} + \frac{\widehat{a}_{n,v+1}\lambda_{v+1}}{v(v+1)}\end{aligned}$$

and hence

$$\begin{aligned}A_n(s) - A_{n-1}(s) &= \frac{a_{nn}\lambda_n(n+1)t_n}{n} + \sum_{v=1}^{n-1}\left(a_{nv} - a_{n-1,v}\right)t_v\lambda_v\frac{v+1}{v} \\ &\quad + \sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\Delta\lambda_v t_v\frac{v+1}{v} + \sum_{v=1}^{n-1}\frac{\widehat{a}_{n,v+1}}{v}\lambda_{v+1}t_v \\ &= L_n^{(1)} + L_n^{(2)} + L_n^{(3)} + L_n^{(4)}, \ \text{ say.}\end{aligned}$$

By Minkowski's inequality, it suffices to prove the theorem

$$\sum_{n=1}^{\infty} a_{nn}^{1-k}\big|L_n^{(r)}\big|^k < \infty, \quad r = 1,2,3,4.$$

Now, by (2.9) and (2.10), we get

$$\begin{aligned}\sum_{n=1}^{m} a_{nn}^{1-k}\big|L_n^{(1)}\big|^k &= \sum_{n=1}^{m} a_{nn}^{1-k}\left|\frac{a_{nn}\lambda_n(n+1)t_n}{n}\right|^k \\ &= O(1)\sum_{n=1}^{m} a_{nn}\,|\lambda_n t_n|^k = O(1)\sum_{n=1}^{m} a_{nn}\,|\lambda_n|\,\frac{|t_n|^k}{l_n^{k-1}} \\ &= O(1)\Big\{\sum_{n=1}^{m-1}\Delta\,|\lambda_n|\sum_{v=1}^{n}\frac{a_{vv}\,|t_v|^k}{l_v^{k-1}} + |\lambda_m|\sum_{v=1}^{m}\frac{a_{vv}\,|t_v|^k}{l_v^{k-1}}\Big\} \\ &= O(1)\Big\{\sum_{n=1}^{m-1} l_n\,|\Delta\lambda_n| + |\lambda_m|\,l_m\Big\} = O(1) \ \text{ as } \ n\to\infty.\end{aligned}$$

Applying Hölder's inequality for $k > 1$ (clearly, $k = 1$), it follows from (2.8), as in $L_n^{(1)}$, that

$$\sum_{n=2}^{m+1} a_{nn}^{1-k}\big|L_n^{(2)}\big|^k = O(1)\sum_{n=2}^{m+1} a_{nn}^{1-k}\Big\{\sum_{v=1}^{n-1}\left|a_{nv} - a_{n-1,v}\right|\left|\lambda_v t_v\right|\Big\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}|a_{nv}-a_{n-1,v}|\,|\lambda_v t_v|^k\left\{\frac{1}{a_{nn}}\sum_{v=1}^{n-1}|a_{nv}-a_{n-1,v}|\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}|a_{nv}-a_{n-1,v}|\,|\lambda_v t_v|^k$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v t_v|^k\sum_{n=v+1}^{m}(a_{n-1,v}-a_{nv})$$
$$= O(1)\sum_{v=1}^{\infty}a_{vv}\,|\lambda_v t_v|^k<\infty.$$

Also, using (2.4) and (2.7), we have

$$\sum_{n=2}^{m+1}a_{nn}^{1-k}\left|L_n^{(3)}\right|^k = O(1)\sum_{n=2}^{m+1}a_{nn}^{1-k}\left\{\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\frac{|\Delta\lambda_v t_v|}{a_{vv}}\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\left(\frac{|\Delta\lambda_v t_v|}{a_{vv}}\right)^k\left\{\frac{1}{a_{nn}}\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}a_{vv}\left(\frac{|\Delta\lambda_v t_v|}{a_{vv}}\right)^k\sum_{n=v+1}^{m}\widehat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}a_{vv}\left(\frac{|\Delta\lambda_v t_v|}{a_{vv}}\right)^k\overline{a}_{m,v+1} = O(1)\sum_{v=1}^{m}v\,|\Delta\lambda_v|\,\frac{a_{vv}\,|t_v|^k}{X_v^{k-1}}$$
$$= O(1)\left\{\sum_{v=1}^{m-1}\Delta(v\,|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{a_{rr}\,|t_r|^k}{X_r^{k-1}}+|m\Delta\lambda_m|\sum_{r=1}^{m}\frac{a_{rr}\,|t_r|^k}{rX_r^{k-1}}\right\}$$
$$= O(1)\left\{\sum_{v=1}^{m-1}vM_v\left|\Delta^2\lambda_v\right|+\sum_{v=1}^{m-1}M_v\,|\Delta\lambda_v|+mM_m\,|\Delta\lambda_m|\right\}$$
$$= O(1)\ \text{ as }\ n\to\infty$$

by virtue of (2.6), (2.10) and (2.11).

Finally, as in $L_n^{(3)}$, it follows from (2.1) that

$$\sum_{n=2}^{m+1}a_{nn}^{1-k}\left|L_n^{(4)}\right|^k = O(1)\sum_{n=2}^{m+1}a_{nn}^{1-k}\left\{\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\frac{|\lambda_{v+1}t_v|}{va_{vv}}\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\left(\frac{|\lambda_{v+1}t_v|}{va_{vv}}\right)^k\left\{\frac{1}{a_{nn}}\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}a_{vv}\left(\frac{|\lambda_{v+1}t_v|}{va_{vv}}\right)^k\sum_{n=v+1}^{m}\widehat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{1}{va_{vv}}\right)^k a_{vv}\left(|\lambda_{v+1}t_v|\right)^k\overline{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\,\frac{a_{vv}\,|t_v|^k}{l_v^{k-1}}$$
$$= O(1)\left\{\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\sum_{r=1}^{v}\frac{a_{rr}\,|t_r|^k}{l_r^{k-1}}+|\lambda_{m+1}|\sum_{r=1}^{m}\frac{a_{rr}\,|t_r|^k}{l_r^{k-1}}\right\}$$

$$= O(1)\left\{ \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}|\, l_{v+1} + l_{m+1}\, |\lambda_{m+1}| \right\} = O(1) \text{ as } n \to \infty.$$

which completes the proof. □

It should be remarked that to Theorem 2.1 and to Theorem 2.2 can be applied various matrices other than weighted mean matrices. In fact, we choose the matrix A as the matrix of Cesàro mean of order $0 < \alpha < 1$. Then, as is well known (see [5]),

$$\overline{a}_{nv} = E_{n-v}^{\alpha}/A_n^{\alpha} \quad \text{and} \quad \widehat{a}_{nv} = vE_{n-v}^{\alpha-1}/nE_n^{\alpha}.$$

Also, by considering $E_n^{\alpha} \sim n^{\alpha}/\Gamma(\alpha+1)$ (see [8]), conditions (2.1), (2.2), (2.3) and (2.4) are easily verified. Hence the following results are immediately obtained.

Corollary 2.4. i) *Let $0 < \alpha < 1$. Then the series $\Sigma\lambda_v a_v$ is summable $|C, \alpha, (\alpha-1)(1-1/k)|_k$, $k \geq 1$, if condition* (2.5) *and the following conditions:*

$$\sum_{v=1}^{\infty} v^{2-\alpha} \left|\Delta^2\lambda_v\right| < \infty,$$

$$\sum_{v=1}^{n} v^{(\alpha-1)k-2\alpha+1-k} \left|t_v\right|^k = O(n^{1-\alpha}) \text{ as } n \to \infty$$

hold.

ii) *Let $\alpha = 1$. Then the series $\Sigma\lambda_v a_v$ is summable $|C, 1|_k$, $k \geq 1$, if condition* (2.5) *and the following conditions:*

$$\sum_{v=2}^{\infty} v \log v \left|\Delta^2\lambda_v\right| < \infty,$$

$$\sum_{v=2}^{n} v^{-1} (\log v)^{1-k} \left|t_v\right|^k = O(\log n) \text{ as } n \to \infty$$

hold.

Corollary 2.5. *Let $0 < \alpha \leq 1$. Then the factored Fourier series $\Sigma a_v\lambda_v$ is summable $|C, \alpha, (\alpha-1)(1-1/k)|_k$, $k \geq 1$, if $\phi_1(t) \in BV$ and the conditions of Corollary* 2.4 *hold.*

Corollary 2.4 (ii) *was also given by Mazhar* [13].

References

1. H. Bor, Some new results on absolute Riesz summability of infinite series and Fourier series. *Positivity* **20** (2016), no. 3, 599–605.
2. H. Bor, Certain new factor theorems for infinite series and trigonometric Fourier series. *Quaest. Math.* **43** (2020), no. 4, 441–448.
3. H. Bor, R. P. Agarwal, A new application of almost increasing sequences to factored infinite series. *Anal. Math. Phys.* **10** (2020), no. 3, Paper No. 26, 7 pp.
4. L. S. Bosanquet, Note on the Absolute Summability (C) of a Fourier Series. *J. London Math. Soc.* **11** (1936), no. 1, 11–15.
5. L. S. Bosanquet, G. Das, Absolute summability factors for Nörlund means. *Proc. London Math. Soc. (3)* **38** (1979), no. 1, 1–52.
6. K. K. Chen, Functions of bounded variation and the Cesaro means of a Fourier series. *Acad. Sinica Science Record* **1** (1945), 283–289.
7. H. P. Dikshit, A. Kumar, Absolute summability of Fourier series with factors. *Pacific J. Math.* **61** (1975), no. 1, 59–69.
8. T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. London Math. Soc. (3)* **7** (1957), 113–141.
9. G. H. Hardy, *Divergent Series.* Oxford, at the Clarendon Press, 1949.
10. M. Izumi, S. Izumi, Absolute Nörlund summability factor of Fourier series. *Proc. Japan Acad.* **46** (1970), 642–646.
11. B. Jena, Vandana, S. K. Paikray, U. K. Misra, On Generalized Local Property of $|A;\delta|_k$-Summability of Factored Fourier Series. *Inter. J. Anal. Appl.* **16** (2018), no. 2 , 209–221.
12. K. Kanno, On the absolute Nörlund summability of the factored Fourier series. *Tohoku Math. J. (2)* **21** (1969), 434–447.

13. S. M. Mazhar, On C, 1_k summability factors of infinite series. *Indian J. Math.* **14** (1972), 45–48.
14. R. Mohanty, The absolute Cesàro summability of some series associated with a Fourier series and its allied series. *J. London Math. Soc.* **25** (1950), 63–67.
15. M. A. Sarıgöl, On absolute summability factors. *Comment. Math. Prace Mat.* **31** (1991), 157–163.
16. M. A. Sarıgöl, On absolute weighted mean summability methods. *Proc. Amer. Math. Soc.* **115** (1992), no. 1, 157–160.
17. M. A. Sarıgöl, On local property of $|A|_k$ summability of factored Fourier series. *J. Math. Anal. Appl.* **188** (1994), no. 1, 118–127.
18. M. A. Sarıgöl, On the local properties of factored Fourier series. *Appl. Math. Comput.* **216** (2010), no. 11, 3386–3390.
19. N. Singh, On the absolute Cesàro summability factors of Fourier series. *Ann. Inst. Fourier (Grenoble)* **18** (1968), fasc. 2, 17–30, v (1969).
20. Ş. Yıldız, On application of matrix summability to Fourier series. *Math. Methods Appl. Sci.* **41** (2018), no. 2, 664–670.
21. Ş. Yıldız, On the absolute matrix summability factors of Fourier series. *Math. Notes* **103** (2018), no. 1-2, 297–303.
22. Ş. Yıldız, On the generalizations of some factors theorems for infinite series and Fourier series. *Filomat* **33** (2019), no. 14, 4343–4351.

(Received 14.11.2021)

University of Pamukkale, Department of Mathematics, Denizli, Turkey
Email address: msarigol@pau.edu.tr