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CAPACITY INEQUALITIES AND LIPSCHITZ CONTINUITY OF MAPPINGS

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Abstract. In this paper, we consider homeomorphic mappings defined by p-capacity inequalities in domains of $\mathbb{R}^n$. In the case $p=n-1$, we prove the Lipschitz continuity of such mappings, i.e., the continuity that extends the result due to F. W. Gehring.

1. Introduction

This article is devoted to the study of mappings defined by capacity (moduli) inequalities, which have been actively studied in the recent years (see, for example, [1, 2, 4, 6, 14, 20] and [18]). In this article, we consider homeomorphic mappings $\varphi:\Omega\to\widetilde{\Omega}$, where $\Omega,\widetilde{\Omega}$ are the domains in $\mathbb{R}^n$, defined by the p-capacity inequalities

$$\operatorname{cap}_p(\varphi(F_0),\varphi(F_1);\widetilde{\Omega})\leqslant K_p^p\operatorname{cap}_p(F_0,F_1;\Omega),\quad 1<p<\infty. \tag{1.1}$$

In the case $p=n$, we have usual quasiconformal mappings [22] and in the case $p\neq n$, this class of mappings was introduced in [4]. In accordance with [28, 29], we define a homeomorphic mapping $\varphi:\Omega\to\widetilde{\Omega}$ as the mapping of bounded p-capacitory distortion if inequality (1.1) holds for any condenser $(F_0,F_1)\subset\Omega$.

The first topic of the article is devoted to the characterization of homeomorphic mappings defined by p-capacity inequalities (1.1) in terms of the inner p-dilatation. In the case of mappings with the conformal moduli inequalities of the Poletsky type, the estimates of the inner dilatation were obtained in [19]. Similar estimates of dilatation in the case of the p-modulus, $n-1<p\leqslant n$, were obtained in [5] and [7] for respectively homeomorphisms and mappings with a branching. In this article, we prove:

Let $\varphi:\Omega\to\widetilde{\Omega}$ be a homeomorphic mapping. Then φ is the mapping of bounded p-capacitory distortion, $p>n-1$, if and only if $\varphi\in W^1_{p',\mathrm{loc}}(\Omega)$, $p'=p/(p-n+1)$, has a finite distortion and

$$\operatorname*{ess\,sup}_{x\in\Omega}\left(\frac{|J(x,\varphi)|}{l(D\varphi(x))^p}\right)^{\frac{1}{p}}=K_p<\infty,\quad p>n-1.$$

The second topic of the article is devoted to the continuity of mappings in the sense of Lipschitz. In [4], the Lipschitz continuity of the mapping of bounded p-capacitory distortion is proved in the case $n-1<p<n$ and an example that in the case $1<p<n-1$ the Lipschitz continuity does not hold, is given. The results of such a type have been obtained for the mappings of finite distortion with some restrictions (see, e.g., [12, 14] and [17]). In the present article, using the methods of the composition operators theory [28, 29], we study analytical properties of these mappings and prove the Lipschitz continuity in the limit case $p=n-1$.

Let $\varphi:\Omega\to\widetilde{\Omega}$ be a homeomorphic mapping of bounded $(n-1)$-capacitory distortion. Then φ belongs to the Sobolev space $L^1_\infty(\Omega)$.

This result extends the result by F. W. Gehring [4] to the limit case $p=n-1$.

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2. Composition Operators and Capacity Inequalities

2.1. **Sobolev spaces and composition operators.** Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 2$, the Sobolev space $W_p^1(\Omega)$, $1 \leqslant p \leqslant \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W_p^1(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|.$$

The Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$ is defined as a space of functions $f \in W^1_p(U)$ for every open and bounded set $U \subset \Omega$ such that $\overline{U} \subset \Omega$.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leqslant p \leqslant \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

Recall that in Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, Sobolev spaces $W^1_p(\Omega)$ and $L^1_p(\Omega)$ coincide (see, for example, [15]).

In accordance with the non-linear capacity theory [16], we consider the elements of Sobolev spaces $W^{1,p}(\Omega)$ as classes of equivalence up to a set of p-capacity zero [15].

Let Ω and $\widetilde{\Omega}$ be the domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphic mapping $\varphi : \Omega \to \widetilde{\Omega}$ belongs to the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$ ($L^1_p(\Omega)$) if its coordinate functions belong to $W^1_{p,\mathrm{loc}}(\Omega)$ ($L^1_p(\Omega)$). In this case, the formal Jacobi matrix $D\varphi(x)$ and its determinant (Jacobian) $J(x, \varphi)$ are well defined at almost all points $x \in \Omega$. We denote

$$|D\varphi(x)| := \max_{|v|=1} |D\varphi(x) \cdot v| \text{ and } l(D\varphi(x)) := \min_{|v|=1} |D\varphi(x) \cdot v|$$

the maximal dilatation of the linear operator $D\varphi(x)$ and the minimal dilatation of the linear operator $D\varphi(x)$, respectively.

Let Ω and $\widetilde{\Omega}$ be the domains in $\mathbb{R}^n$, $n \geqslant 2$. Then a homeomorphic mapping $\varphi : \Omega \to \widetilde{\Omega}$ induces a bounded composition operator [28, 29]

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leqslant q \leqslant p \leqslant \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L^1_p(\widetilde{\Omega})$, the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in Ω and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f) \mid L^1_q(\Omega)\| \leqslant K_{p,q}(\Omega) \|f \mid L^1_p(\widetilde{\Omega})\|.$$

The problem of the characterization of mappings that generate bounded composition operators on Sobolev spaces traces back to the Reshetnyak Problem (1968) and is closely connected with the quasiconformal mappings theory [24]. The solution of this problem is given by the following theorem [21] (see also [28, 29] and [10] for the case $p = \infty$).

Recall that a p-distortion of a mapping φ at a point $x \in \Omega$ is defined as

$$K_p(x) = \inf \left\{ k(x) : |D\varphi(x)| \leqslant k(x) |J(x, \varphi)|^{\frac{1}{p}}, \ x \in \Omega \right\}.$$

In the case $p = n$, we have the usual conformal dilatation and in the case $p \neq n$, the p-dilatation arises in [4] (see also [23]).

Theorem 2.1. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping between two domains Ω and $\widetilde{\Omega}$. Then φ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leqslant q \leqslant p \leqslant \infty,$$

if and only if φ is a Sobolev mapping of the class $W^1_{q,\mathrm{loc}}(\Omega; \widetilde{\Omega})$, has a finite distortion and

$$K_{p,q}(\varphi; \Omega) = \|K_p \mid L_\kappa(\Omega)\| < \infty,$$

where $1/q - 1/p = 1/\kappa$ ($\kappa = \infty$, if $p = q$).

The following theorem gives the properties of mappings, which are inverse to mappings generating bounded composition operators on Sobolev spaces.

Theorem 2.2. *Let a homeomorphic mapping $\varphi : \Omega \to \widetilde{\Omega}$ between two domains Ω and $\widetilde{\Omega}$ generate a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_q(\Omega), \quad n-1 < q \leqslant p < \infty.$$

Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{q'}(\Omega) \to L^1_{p'}(\widetilde{\Omega}),$$

where $p' = p/(p-n+1)$, $q' = q/(q-n+1)$.

2.2. **Capacity inequalities.** Recall the notion of the variational p-capacity [9]. The condenser in the domain $\Omega \subset \mathbb{R}^n$ is the pair (F_0, F_1) of connected closed relatively to Ω sets $F_0, F_1 \subset \Omega$. A continuous function $u \in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0, F_1) if the set $F_i \cap \Omega$ is contained in some connected component of the set $\operatorname{Int}\{x | u(x) = i\}$, $i = 0, 1$. We call p-capacity of the condenser (F_0, F_1) relatively to domain Ω the value

$$\operatorname{cap}_p(F_0, F_1; \Omega) = \inf \|u | L^1_p(\Omega)\|^p,$$

where the greatest lower bond is taken over all admissible for the condenser $(F_0, F_1) \subset \Omega$ functions. If the condenser has no admissible functions, we put the capacity is equal to infinity.

Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping between two domains Ω and $\widetilde{\Omega}$. Then φ is called the mapping of bounded p-capacitory distortion if the inequality

$$\operatorname{cap}_p(\varphi(F_0), \varphi(F_1); \widetilde{\Omega}) \leqslant K_p^p \operatorname{cap}_p(F_0, F_1; \Omega), \quad 1 < p < \infty \tag{2.1}$$

holds for any condenser $(F_0, F_1) \subset \Omega$.

Theorem 2.3. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping. Then φ is the mapping of bounded p-capacitory distortion, $p > n-1$, if and only if $\varphi \in W^1_{p',\mathrm{loc}}(\Omega)$, $p' = p/(p-n+1)$, has finite distortion and*

$$\operatorname*{ess\,sup}_{x \in \Omega} \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))^p} \right)^{\frac{1}{p}} = K_p < \infty, \quad p > n-1.$$

Proof. Consider the inverse mapping $\psi := \varphi^{-1} : \widetilde{\Omega} \to \Omega$. Inequality (2.1) is equivalent to the inequality

$$\operatorname{cap}_p(\psi^{-1}(F_0), \psi^{-1}(F_1); \widetilde{\Omega}) \leqslant K_p^p \operatorname{cap}_p(F_0, F_1; \Omega).$$

So, by [21, 26], the inverse mapping φ^{-1} generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_p(\Omega) \to L^1_p(\widetilde{\Omega})$$

and is a p-quasiconformal mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ [8, 21]. Hence the mapping φ^{-1} has the following properties [21, 27]:

1. The mapping $\varphi^{-1} \in W^1_{p,\mathrm{loc}}(\widetilde{\Omega})$, has finite distortion and

$$\left(\frac{|D\varphi^{-1}(y)|^p}{|J(y,\varphi^{-1})|} \right)^{\frac{1}{p}} \leqslant K_p \ \text{ for almost all } \ y \in \widetilde{\Omega}.$$

2. The mapping φ^{-1} is differentiable a.e. in $\widetilde{\Omega}$.

3. The mapping φ^{-1} possesses the Luzin N^{-1}-property if $n-1 < p < n$ (φ possesses the Luzin N-property).

4. The mapping φ^{-1} possesses the Luzin N-property if $n < p < \infty$ (φ possesses the Luzin N^{-1}-property).

5. The mapping φ^{-1} possesses the Luzin N-property and the Luzin N^{-1}-property if $p = n$ [22] (φ possesses the Luzin N^{-1}-property).

Now, by Theorem 2.2, the mapping φ generates a bounded composition operator

$$\varphi^* : L^1_{p'}(\widetilde{\Omega}) \to L^1_{p'}(\Omega), \qquad p' = p/(p-n+1).$$

Hence the mapping $\varphi \in W^1_{p',\mathrm{loc}}(\Omega)$, has finite distortion and is differentiable a.e. in Ω [21, 26].

Denote $\widetilde{Z} = \{y \in \widetilde{\Omega} : J(y, \varphi^{-1}) = 0\}$. The set $\widetilde{S} \subset \widetilde{\Omega}$, $|\widetilde{S}| = 0$, is the set such that on set $\widetilde{\Omega} \setminus \widetilde{S}$, the mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ has the Luzin N-property [13].

Then by the change of variables formula [3, 13], $|\varphi^{-1}(\widetilde{Z} \setminus \widetilde{S})| = 0$ and on the set $\varphi^{-1}(\widetilde{S} \setminus \widetilde{Z})$, we have $J(x, \varphi) = 0$ for almost all $x \in \varphi^{-1}(\widetilde{S} \setminus \widetilde{Z})$. Hence, for almost all $x \in \Omega \setminus \varphi^{-1}(\widetilde{Z} \cup \widetilde{S})$, we have

$$J(x, \varphi)| = |J(y, \varphi^{-1})|^{-1}, \quad y = \varphi(x),$$

and

$$l(D\varphi(x)) = |D\varphi^{-1}(y)|^{-1}, \quad y = \varphi(x).$$

Hence, by setting

$$\left(\frac{|J(x, \varphi)|}{l(D\varphi(x))^p}\right)^{\frac{1}{p}} = 0$$

on the set $Z = \{x \in \Omega : J(x, \varphi) = 0\}$, we obtain

$$\operatorname*{ess\,sup}_{x\in\Omega} \left(\frac{|J(x, \varphi)|}{l(D\varphi(x))^p}\right)^{\frac{1}{p}} = \operatorname*{ess\,sup}_{y\in\widetilde{\Omega}} \left(\frac{|D\varphi^{-1}(\varphi^{-1}(y))|^p}{|J((\varphi^{-1}(y)), \varphi^{-1})|}\right)^{\frac{1}{p}} \leqslant K_p < \infty. \qquad \square$$

Remark 2.4. The assertion of Theorem 2.3 is correct in the case $1 \leqslant p \leqslant n-1$ with additional assumptions that $\varphi \in W^1_{1,\mathrm{loc}}(\Omega)$ and φ is differentiable a.e. in Ω.

3. On the Lipschitz Continuity of Mapping of Bounded p-capacitory Distortion

Now, we consider the Lipschitz continuity of homeomorphic mappings of bounded p-capacitory distortion in the case $p = n-1$.

Let (F_0, F_1) be a condenser in the domain $\Omega \subset \mathbb{R}^n$ such that $\mathrm{cap}_p(F_0, F_1; \Omega) < \infty$. Suppose that a function v belonging to $L^1_p(\Omega)$ is admissible for the condenser (F_0, F_1). Then v is called *an extremal function for the condenser* (F_0, F_1) [25] if

$$\int\limits_{\Omega\setminus(F_0\cup F_1)} |\nabla v|^p \, dx = \mathrm{cap}_p(F_0, F_1; \Omega).$$

Note that for any $1 < p < \infty$ and any condenser (F_0, F_1) with the $\mathrm{cap}_p(F_0, F_1; \Omega) < \infty$, the extremal function exists and is unique.

The set of extremal functions for p-capacity of every possible pairs of n-dimensional connected compacts $F_0,\ F_1 \subset \Omega$, having smooth boundaries, we denote by the symbol $E_p(\Omega)$. Then the following approximation holds.

Theorem 3.1 ([25]). *Let $1 < p < \infty$. Then there exists a countable collection of functions $v_i \in E_p(\Omega)$, $i \in \mathbb{N}$, such that for every function $u \in L^1_p(\Omega)$ and for any $\varepsilon > 0$, there exists a presentation of u in the form $u = c_0 + \sum\limits_{i=1}^{\infty} c_i v_i$, for which the inequalities*

$$\|u \mid L^1_p(\Omega)\|^p \leqslant \sum_{i=1}^{\infty} \|c_i v_i \mid L^1_p(\Omega)\|^p \leqslant \|u \mid L^1_p(\Omega)\|^p + \varepsilon$$

hold.

The following theorem was not formulated, but proved in [21] by using the approximation by extremal functions (see, also, [26]).

Theorem 3.2. *Let $1 < p < \infty$. A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator*

$$\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_p(\Omega)$$

if and only if for every condenser $(F_0, F_1) \subset \widetilde{\Omega}$, the inequality

$$\mathrm{cap}_p^{\frac{1}{p}}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); \Omega) \leqslant K_{p,p}(\varphi; \Omega)\, \mathrm{cap}_p^{\frac{1}{p}}(F_0, F_1; \widetilde{\Omega})$$

holds.

Now, using the capacitory characterization of composition operators on Sobolev spaces, we prove the Lipschitz continuity of homeomorphic mappings of bounded $(n-1)$-capacitory distortion that, extends the result by F. W. Gehring [4].

Theorem 3.3. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping of bounded $(n-1)$-capacitory distortion. Then φ belongs to the Sobolev space $L^1_\infty(\Omega)$.*

Proof. Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping of bounded $(n-1)$-capacitory distortion. Consider the inverse mapping $\psi := \varphi^{-1} : \widetilde{\Omega} \to \Omega$. Then inequality (2.1) is equivalent to the inequality

$$\operatorname{cap}_{n-1}(\psi^{-1}(F_0), \psi^{-1}(F_1); \widetilde{\Omega}) \leqslant K_{n-1}^{n-1} \operatorname{cap}_{n-1}(F_0, F_1; \Omega).$$

So, by Theorem 3.2, the inverse mapping φ^{-1} generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_{n-1}(\Omega) \to L^1_{n-1}(\widetilde{\Omega})$$

and is a $(n-1)$-quasiconformal mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ [8, 21]:

$$\operatorname*{ess\,sup}_{y \in \widetilde{\Omega}} \left(\frac{|D\varphi^{-1}(y)|^{n-1}}{|J(y, \varphi^{-1})|} \right)^{\frac{1}{n-1}} = K_{n-1}(\varphi^{-1}; \widetilde{\Omega}) < \infty.$$

Hence, by [11], the mapping $\varphi : \Omega \to \widetilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L^1_\infty(\widetilde{\Omega}) \to L^1_\infty(\Omega),$$

and the inequality

$$\|\varphi^*(f) \mid L^1_\infty(\Omega)\| \leqslant K_{n-1}^{n-1} \|f \mid L^1_\infty(\widetilde{\Omega})\| \tag{3.1}$$

holds for any function $f \mid L^1_\infty(\widetilde{\Omega})$.

Now, substituting in inequality (3.1) the test functions $f = y_i$, $y = 1, \dots, n$, where y_i is the i-coordinate of $y \in \widetilde{\Omega}$, we have $\varphi_i \in L^1_\infty(\Omega)$, $i = 1, \dots, n$, and

$$\|\varphi_i \mid L^1_\infty(\Omega)\| \leqslant K_{n-1}^{n-1}, \quad i = 1, \dots, n.$$

Hence the mapping $\varphi : \Omega \to \widetilde{\Omega}$ belongs to the Sobolev space $L^1_\infty(\Omega)$. □

4. On the Differentiability of Mapping of Bounded p-capacitory Distortion

Using Theorem 3.3 and [4, Theorem 3], we have the following assertion.

Theorem 4.1. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping of bounded p-capacitory distortion, $n-1 \leqslant p < n$. Then φ is a locally Lipschitz mapping, differentiable a.e. in Ω.*

In [4], the following estimate of Jacobians of a mapping of bounded p-capacitory distortion has been proved.

Lemma 4.2. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping of bounded p-capacitory distortion, $n-1 \leqslant p < n$. Then $|J(x, \varphi)| \leqslant (K_p^p)^{n/(n-p)}$ a.e. in Ω.*

Using Lemma 4.2, we obtain estimates of the Lipschitz constants of homeomorphic mapping of bounded p-capacitory distortion.

Corollary 4.3. *Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphic mapping of bounded p-capacitory distortion, $n-1 \leqslant p < n$. Then $|D\varphi(x)| \leqslant (K_p^p)^{1/(n-p)}$ a.e. in Ω.*

Proof. Indeed, it is easy to check that

$$|D\varphi(x)|^p \leqslant |J(x, \varphi)|^{p-n+1} \left(\frac{|J(x, \varphi)|}{l(D\varphi(x))^p} \right)^{n-1} \text{ a.e. in } \Omega.$$

By Theorem 2.3, Lemma 4.2 and Remark 2.4, we have

$$|D\varphi(x)|^p \leqslant |J(x, \varphi)|^{p-n+1} \left(\frac{|J(x, \varphi)|}{l(D\varphi(x))^p} \right)^{n-1} \leqslant \left(K_p^{\frac{p}{n-p}} \right)^p \text{ a.e. in } \Omega.$$ □

Now, using Theorem 4.1, we have

Corollary 4.4. *Let* $\varphi : \Omega \to \widetilde{\Omega}$ *be a homeomorphic mapping of bounded* p*-capacitory distortion,* $n-1 \leqslant p < n$. *Then*

$$\limsup_{x\to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \leqslant (K_p^p)^{1/(n-p)}$$

for almost all $x_0 \in \Omega$.

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