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SPACE-TIME SPECTRAL METHOD FOR AN OPTIMAL CONTROL PROBLEM GOVERNED BY A TWO-DIMENSIONAL PDE CONSTRAINT

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Abstract. In this paper, we solve a two-dimensional optimal control problem with a parabolic partial differential equation (PDE) constraint. First, the space-time spectral method is used to discretize time derivative and space derivative. Then the aforementioned problem is transformed into a solvable algebraic system. Since the spectral methods converge spectrally in both space and time, they have gained a significant attention in the last few decades. We prove that our method has exponential rates of convergence in both space and time.

1. Introduction

Many phenomena in nature and industry can be modeled by using partial differential equations. Among them, we can mention processes such as heat distribution, wave propagation, flow equation. The goal of the optimal control of PDEs is to move the system state to the desired state with the help of the control variable. This process can be done by minimizing an appropriate objective function, including the state variable and control variable. One of the main motivations for solving the optimal control problems is their applications. Readers can refer to Reference [8] for further understanding.

The optimal control problems of partial differential equations (PDEs) are of interest to researchers engaged in the field of PDEs and optimization. These problems have many applications in various industries and sciences, including fluid mechanics, materials engineering, etc. Given their importance, researchers are always trying to find effective solutions for them, some of which are mentioned in the following. A. Rezazadeh et al. proposed a solution for the optimal control problem governed by a parabolic PDE by using the space-time spectral collocation method (see [13]). The reduced basis method was employed by Rezazadeh et al. (see [14]) to solve fractional PDE constrained optimization. Artificial neural network was used in [4] for the optimal control of fractional parabolic. Darehmiraki et al. [5] combined interpolation methods and Barycentric polynomials to solve the optimal control of elliptic convection diffusion equation. Moving least square and radial basis function were employed for fractional distributed optimal control problems in [3]. Recently, Shojaeizadeh et al. investigated a solution for time-fractional convection-diffusion-reaction by compact integrated radial basis functions [12]. Also, they applied shifted Jacobi polynomials to solve the optimal control problem of fractal-fractional advection-diffusion-reaction [19].

This paper considers the following parabolic constrained optimization problem [9]:

$$\begin{aligned} \min \quad J(y,u) &= \frac{1}{2}\int_0^T\int_\Omega (y-y_d)^2 d\Omega dt + \frac{\gamma}{2}\int_0^T\int_\Omega u^2 d\Omega dt, \\ &\text{s.t} \\ -y_t + \Delta y &= u + f, \qquad \text{in } Q, \\ y &= 0, \qquad \text{on } \Sigma, \\ y(.,0) &= y_0, \qquad \text{in } \Omega. \end{aligned} \tag{1.1}$$

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$y \in H^1(Q)$ and $u \in L_2(Q)$ are called a state variable and a control variable, respectively, to be determined. $y_d \in L_2(Q)$ is the desired state, $J(y,u)$ is called a cost functional. The goal of the optimal control problems is to come as nearer as possible to the desired state with minimal control so that the constraints are true. $\gamma \geq 0$ is a given regularization parameter that is used to solve a well-defined problem. Due to the extra-ordinary ability of the Legender–Galerkin method in solving partial equation problems, we were motivated to solve the mentioned optimal control problem with the help of this method. Here, we first extract the necessary and sufficient conditions for the optimality of the problem and then use the Legender–Galerkin method to solve these conditions numerically. As far as we know, this is the first time this method has been used to solve the above problem in a two-dimensional case. The paper is organized as follows.

The optimal conditions of problem (1.1) are derived in Section 2. The space-time spectral collocation method is discussed in Sections 3 and 4. Description of the Legendre–Galerkin spectral method is given in Section 5. Multiple lemmas and theorems applied to obtain the error bound are in Section 6 and, lastly, several numerical cases are presented in Section 7 to show the efficiency of the proposed method.

2. The Optimal System

In this section, the optimal conditions for (1.1) are obtained. The Lagrangian functional L related to (1.1) is defined as follows:

$$\begin{aligned} L(y,u,p_\Omega,p_{\partial\Omega}) = \frac{1}{2}\int_0^T\int_\Omega (y-y_d)^2 d\Omega dt + \frac{\gamma}{2}\int_0^T\int_\Omega u^2 d\Omega dt \\ + \int_0^T\int_\Omega (-y_t + \Delta y - u - f)p_\Omega d\Omega dt + \int_0^T\int_{\partial\Omega} y p_{\partial\Omega} ds dt. \end{aligned}$$

Using Fréchet derivative and differentiating of L with respect to p_Ω, $p_{\partial\Omega}$, we have

$$\begin{cases} -y_t + \Delta y = u + f, & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(.,0) = y_0, & \text{in } \Omega. \end{cases} \tag{2.1}$$

The adjoint equations are gained by derivative of L with respect to y:

$$\begin{cases} p_t + \Delta p - y = -y_d, & \text{in } Q, \\ p = 0, & \text{on } \Sigma, \\ p(.,T) = 0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Differentiating L with respect to u, we get the gradient equation:

$$\gamma u + p = 0, \quad \text{in } \ Q. \tag{2.3}$$

These equations are discretized by the Legendre–Galerkin mathod and the spectral collocation method. For more details, see [1, 2, 6].

3. Chebyshev Spectral Collocation Method

Now, the Chebyshev–Gauss–Lobatto points in $\Lambda = [-1,1]$ are introduced as

$$\bar{x}_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0,1,\dots,N.$$

Differentiating and calculating the polynomial at the Chebyshev–Gauss–Lobatto points, we have

$$\bar{G}_N(\bar{x}) = \sum_{k=0}^{N} g_k \bar{L}_k(\bar{x}), \tag{3.1}$$

where $g_k = g(\bar{x}_k)$ and $\bar{L}_k$ are the Lagrange interpolation polynomials defined as follows:

$$\bar{L}_k(\bar{x}) = \prod_{j=0, j\neq k}^{N} \frac{\bar{x} - \bar{x}_j}{\bar{x}_k - \bar{x}_j},$$

assuring

$$\bar{L}_k(\bar{x}_j) = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Let $\bar{G} = [g(\bar{x}_0), \dots, g(\bar{x}_N)]$ and $\bar{G}^{(m)} = [g^{(m)}(\bar{x}_0), \dots, g^{(m)}(\bar{x}_N)]^T$. The m derivative of $\bar{G}$ at the points $\bar{x}_j$ is approximated by differentiating equation (3.1). Hence

$$\bar{G}^{(m)} = \sum_{k=0}^{N} g_k \bar{L}_k^{(m)}(\bar{x}), \qquad m \in \mathbb{N}. \tag{3.2}$$

Equation (3.2) can be written as:

$$\bar{G}^{(m)} = \bar{D}^{(m)} \bar{G}, \qquad m \in \mathbb{N},$$

where

$$\bar{D}_{jk}^{(m)} = \bar{L}_k^{(m)}(\bar{x}_j) \qquad j, k = 0, 1, 2, \dots .$$

$\bar{D}^{(1)} = \bar{D} = (\bar{d}_{kj})$ is the first-order Chebyshev differentiation matrix [20] which

$$\bar{d}_{kj} = \begin{cases} \frac{2N^2+1}{6}, & k = j = 0, \\ -\frac{c_k}{2c_j} \frac{(-1)^{j+k}}{\sin((k+j)\frac{\pi}{2N}) \sin((k-j)\frac{\pi}{2N})}, & k \neq j, \\ -\frac{1}{2}\cos(\frac{k\pi}{N})(1 + \cot^2(\frac{k\pi}{N})), & k = j, k \neq 0, N, \\ -\frac{2N^2+1}{6}, & k = j = N, \end{cases}$$

where

$$c_k = \begin{cases} 2, & k = 0, N, \\ 1, & \text{o.w.} \end{cases}$$

Suppose that $x_j = a - \frac{b-a}{2}(\bar{x}_j + 1)$ are the Legendre–Gauss–Lobatto points in $[a, b]$ such that

$$\bar{x}_j = 1 - \frac{2}{b-a}(x_j - a),$$

$$G^{(m)} = D^{(m)} G, \qquad m \in \mathbb{N},$$

$$G = \Big[\bar{g}\Big(1 - \frac{2}{b-a}(x_0 - a)\Big), \dots, \bar{g}\Big(1 - \frac{2}{b-a}(x_N - a)\Big)\Big]^T,$$

and

$$G^{(m)} = \Big[\bar{G}^{(m)}\Big(1 - \frac{2}{b-a}(x_0 - a)\Big), \dots, \bar{G}^{(m)}\Big(1 - \frac{2}{b-a}(x_N - a)\Big)\Big]^T,$$

$$D_{jk}^{(m)} = \bar{L}_k^{(m)}\Big(1 - \frac{2}{b-a}(x_j - a)\Big), \qquad j, k = 0, 1, \dots, N,$$

$$D_{jk}^{(m)} = \Big(-\frac{2}{b-a}\Big)^m \bar{L}_k^{(m)}(\bar{x}_j), \qquad j, k = 0, 1, \dots N.$$

Thus $D^{(m)} = (-\frac{2}{b-a})^m \bar{D}^{(m)}$ is the Chebyshev differentiation matrix for the Chebyshev–Gauss–Lobatto points in $[a, b]$.

Definition 3.1 ([11]). Assume $C = (c_{ij})_{m\times n}$ and D are two arbitrary matrices. The matrix

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1n}D \\ c_{21}D & c_{22}D & \cdots & c_{2n}D \\ \vdots & \vdots \ddots & \vdots & \\ c_{m1}D & c_{m2}D & \cdots & c_{mn}D \end{bmatrix}$$

is called Kronecker product of C and D.

Definition 3.2 ([11]). Let $C = (c_{ij})_{m\times n}$ be a given matrix, then $\mathrm{vec}(C)$ is a column vector of size $m \times n$ and defined as:

$$\mathrm{vec}(C) = (c_{11}, c_{12}, \dots, c_{1n}, c_{21}, c_{22}, \dots, c_{m1}, \dots, c_{mn})^T.$$

4. Spectral Collocation Method for First-order Ordinary Differential Equation

In this section, the first-order initial value problem is solved with a spectral collocation method.

$$\begin{aligned} y' &= f(t, y(t)), \\ y(t_0) &= y_0, \quad t_0 \le t \le T. \end{aligned} \tag{4.1}$$

Suppose that N_t is a non-negetive integer, $z_j = \cos(\frac{j\pi}{N_t})$ are the Chebyshev–Gauss–Lobatto points in $[-1, 1]$ which $0 \le j \le N_t$. $t_j = t_0 - \frac{T-t_0}{2}(z_j + 1)$ are the Chebyshev–Gauss–Lobatto points in $[t_0, T]$.

We discretize equation (4.1) by the Chebyshev spectral collocation method that is described in Section 3. Therefore

$$\bar{A}\bar{Y} = F(Y), \tag{4.2}$$

where $\bar{A}$ is the first-order differentiation matrix in $[t_0, T]$. Thus

$$\begin{aligned} \bar{A} &= -\frac{2}{T - t_0}\bar{D}^{(1)}(2 : N_t + 1, 1 : N_t + 1), \\ \bar{Y} &= [y_0, Y]^T, \\ Y &= [y(t_1), \dots, y(t_N)]^T, \\ F(Y) &= [f(t_1, y(t_1)), \dots, f(t_{N_t}, y(t_{N_t}))]^T. \end{aligned}$$

Using $\bar{A} = [a_0, A]$, $\bar{Y} = [y_0, Y]^T$ and equation (4.2), we achieve

$$AY + a_0 y_0 = F(Y).$$

5. Legendre–Galerkin Spectral Method

Here, the two-dimensional parabolic equations (2.1)–(2.3) are discretized by using the Galerkin–Legendre spectral method in a space. First, these equations are transformed from $[c, d]$ into $[-1, 1]$ by the change of variable technique. Then, the Legendre–Galerkin method is applied.

By the following variable transformation [10], we have:

$$\begin{cases} x_1 = c - \frac{\bar{x}_1+1}{2}(d-c), \qquad x_2 = c - \frac{\bar{x}_2+1}{2}(d-c), \\ \bar{y}(\bar{x}_1, \bar{x}_2, t) = y\left(c - \frac{\bar{x}_1+1}{2}(d-c), c - \frac{\bar{x}_2+1}{2}(d-c), t\right), \\ \bar{f}(\bar{x}_1, \bar{x}_2, t) = f\left(c - \frac{\bar{x}_1+1}{2}(d-c), c - \frac{\bar{x}_2+1}{2}(d-c), t\right), \\ \bar{y}_d(\bar{x}_1, \bar{x}_2, t) = y_d\left(c - \frac{\bar{x}_1+1}{2}(d-c), c - \frac{\bar{x}_2+1}{2}(d-c), t\right), \\ \bar{u}(\bar{x}_1, \bar{x}_2, t) = u\left(c - \frac{\bar{x}_1+1}{2}(d-c), c - \frac{\bar{x}_2+1}{2}(d-c), t\right), \\ \bar{p}(\bar{x}_1, \bar{x}_2, t) = p\left(c - \frac{\bar{x}_1+1}{2}(d-c), c - \frac{\bar{x}_2+1}{2}(d-c), t\right). \end{cases}$$

Therefore we obtain the following equations:

$$\begin{cases} -\bar{y}_t + (-\frac{2}{d-c})^2\Delta\bar{y} = -\frac{1}{\gamma}\bar{p} + \bar{f}, & \text{in } \bar{Q}, \\ \bar{y}(., t) = 0, & \text{on } \bar{\Sigma}, \\ \bar{y}(., 0) = \bar{y}_0, \end{cases} \tag{5.1}$$

$$\begin{cases} \bar{p}_t + (-\frac{2}{d-c})^2\Delta\bar{p} - \bar{y} = -\bar{y}_d, & \text{in } \bar{Q}, \\ \bar{p}(., t) = 0, & \text{on } \bar{\Sigma}, \\ \bar{p}(., T) = 0, \end{cases} \tag{5.2}$$

where

$$\begin{cases} \bar{\Omega} = [-1,1] \times [-1,1], & \bar{\Gamma} = \partial\bar{\Omega}, \\ \bar{Q} = \bar{\Omega} \times (t_0, T), & \bar{\Sigma} = \bar{\Gamma} \times (0,T), \end{cases}$$

$L_n(\bar{x})$ denotes the nth degree Legendre polynomial [16, 18] and

$$P_N = \operatorname{span}\{L_0(\bar{x}), \dots, L_N(\bar{x})\}, \qquad V_N = \{v \in P_N : v(\pm 1) = 0\}.$$

Our purpose is to find the Legendre–Galerkin approximation for systems (5.1) and (5.2) such that

$$\begin{cases} -(\bar{y}_{Nt}, v) - (\frac{2}{d-c})^2 (\nabla \bar{y}_N, \nabla v) + \frac{1}{\gamma}(\bar{p}_N, v) = (\bar{f}, v), & \text{in } \bar{Q}, \\ \bar{y}_N(.,t) = 0, & \text{on } \bar{\Sigma}, \\ (\bar{y}_N(\bar{x}, \bar{y}, 0) - \bar{y}_0, v) = 0, \quad \forall v \in V_N^2, \end{cases} \tag{5.3}$$

$$\begin{cases} (\bar{p}_{Nt}, v) - (\frac{2}{d-c})^2 (\nabla \bar{p}_N, \nabla v) - (\bar{y}_N, v) = -(\bar{y}_d, v), & \text{in } \bar{Q}, \\ \bar{p}_N(.,t) = 0, & \text{on } \bar{\Sigma}, \\ (\bar{p}_N(\bar{x}, \bar{y}, T), v) = 0, \quad \forall v \in V_N^2. \end{cases} \tag{5.4}$$

The following Lemma is useful to apply the Legendre–Galerkin spectral method.

Lemma 5.1 ([15])**.** *Let us denote*

$$c_k = \frac{1}{\sqrt{4k+6}}, \qquad \phi_k(\bar{x}) = c_k(L_k(\bar{x}) - L_{k+2}(\bar{x})),$$

$$\hat{a}_{jk} = \int_{-1}^{1} \phi_k'(\bar{x})\phi_j'(\bar{x}) d\bar{x}, \qquad \hat{b}_{jk} = \int_{-1}^{1} \phi_k(\bar{x})\phi_j(\bar{x}) d\bar{x}.$$

Then

$$\hat{a}_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases}, \qquad \hat{b}_{jk} = \hat{b}_{kj} = \begin{cases} c_k c_j (\frac{2}{2j+1} + \frac{2}{2j+5}), & k = j, \\ -c_k c_j \frac{2}{2k+1}, & k = j+2, \\ 0, & o.w. \end{cases}$$

and

$$V_N = \operatorname{span}\{\phi_0(\bar{x}), \dots, \phi_{N-2}(\bar{x})\}.$$

Set

$$\bar{y}_N(\bar{x}, \bar{y}, t) = \sum_{k,j=0}^{N-2} \hat{\alpha}_{k,j}(t) \phi_k(\bar{x}_1) \phi_j(\bar{x}_2),$$

$$\bar{p}_N(\bar{x}_1, \bar{x}_2, t) = \sum_{k,j=0}^{N-2} \hat{\beta}_{k,j}(t) \phi_k(\bar{x}_1) \phi_j(\bar{x}_2).$$

Taking

$$v = \phi_l(\bar{x}_1)\phi_m(\bar{x}_2), \qquad l, m = 0, 1, \dots, N-2,$$

we obtain

$$\begin{aligned} -(\bar{y}_{Nt}, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)) - \Big(\frac{2}{d-c}\Big)^2 (\nabla \bar{y}_N, \nabla(\phi_l(\bar{x}_1)\phi_m(\bar{x}_2))) \\ + \frac{1}{\gamma}(\bar{p}_N, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)) = (\bar{f}, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)), \\ (\bar{p}_{Nt}, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)) - \Big(\frac{2}{d-c}\Big)^2 (\nabla \bar{p}_N, \nabla(\phi_l(\bar{x}_1)\phi_m(\bar{x}_2))) \\ -(\bar{y}_N, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)) = -(\bar{y}_d, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)), \\ (\bar{y}_N(\bar{x}_1, \bar{x}_2, 0) - \bar{y}_0, \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)) = 0, \\ (\bar{p}_N(\bar{x}_1, \bar{x}_2, T), \phi_l(\bar{x}_1)\phi_m(\bar{x}_2)) = 0, \end{aligned}$$

where $l, m = 0, 1, \ldots, N-2$ and also,

$$\bar{y}(.,t) = 0, \quad \bar{p}(.,t) = 0, \quad \text{on } \bar{\Sigma}.$$

Suppose that

$$B = (\hat{b}_{k,j})_{k,j=0,1,\ldots,N-2}, \quad \alpha = (\hat{\alpha}_{k,j})_{k,j=0,1,N-2},$$
$$\beta = (\hat{\beta}_{k,j})_{k,j=0,1,\ldots,N-2}, \quad F = (\bar{f}(\bar{x}_1, \bar{x}_2, t), \phi_k(\bar{x}_1)\phi_j(\bar{x}_2))_{k,j=0,1,\ldots,N-2},$$
$$Y_d = (\bar{y_d}(\bar{x}_1, \bar{x}_2, t), \phi_k(\bar{x}_1)\phi_j(\bar{x}_2))_{k,j=0,1,\ldots,N-2}.$$

Then we have

$$-[B \otimes B]\mathrm{vec}(\alpha)' - \left(\frac{2}{d-c}\right)^2 [B \otimes I_{N-1} + I_{N-1} \otimes B]\mathrm{vec}(\alpha) + \frac{1}{\gamma}[B \otimes B]\mathrm{vec}(\beta) = \mathrm{vec}(F),$$
$$[B \otimes B]\mathrm{vec}(\beta)' - \left(\frac{2}{d-c}\right)^2 [B \otimes I_{N-1} + I_{N-1} \otimes B]\mathrm{vec}(\beta) - [B \otimes B]\mathrm{vec}(\alpha) = -\mathrm{vec}(Y_d).$$

Through

$$(\bar{y}(\bar{x}, \bar{y}, 0) - \bar{y}_0, \phi_l(\bar{x})\phi_m(\bar{y})) = 0,$$

we achieve

$$[B \otimes B]\mathrm{vec}(\alpha(t_0) = \mathrm{vec}(Y_0),$$

where

$$Y_0 = (\bar{y}_0(\bar{x}_1, \bar{x}_2, t), \phi_k(\bar{x}_1)\phi_j(\bar{x}_2)), \quad (\beta(T))_{k,j} = 0, \ \ \forall k, j = 0, 1, \ldots, N-2.$$

Therefore we obtain the following ordinary differential equation system:

$$\begin{cases} -[B \otimes B]\mathrm{vec}(\alpha)' - \left(\frac{2}{d-c}\right)^2 [B \otimes I_{N-1} + I_{N-1} \otimes B]\mathrm{vec}(\alpha) + \frac{1}{\gamma}[B \otimes B]\mathrm{vec}(\beta) = \mathrm{vec}(F), \\ [B \otimes B]\mathrm{vec}(\beta)' - \left(\frac{2}{d-c}\right)^2 [B \otimes I_{N-1} + I_{N-1} \otimes B]\mathrm{vec}(\beta) - [B \otimes B]\mathrm{vec}(\alpha) = -\mathrm{vec}(Y_d), \\ [B \otimes B]\mathrm{vec}(\alpha(t_0)) = \mathrm{vec}(Y_0), \qquad \mathrm{vec}(\beta(T)) = 0 \quad t_0 \leq t \leq T. \end{cases}$$

Now, we can use the space-time spectral collocation method in Section 4.

6. A Priori Error Estimates

In this section, we obtain a priori error bound for problem (5.1), (5.2). Denote

$$L^2(\bar{\Omega}) = \{v | \|v\| < \infty\}, \qquad \|v\| = \|v\|_{L^2} = (v, v)^{\frac{1}{2}},$$
$$H^r(\bar{\Omega}) = \{v | \|v\|_r < \infty\}, \qquad \|v\|_r = \|v\|_{H^r} = \left(\sum_{|\alpha| \leq r} \|D^\alpha\|^2\right)^{\frac{1}{2}}.$$

Let $B = L^2(\bar{\Omega})$ or $B = H^r(\bar{\Omega})$. The Bochner space $L^p(J; B)$ is introduced as follows:

$$\|v\|_{L^p;B} = \begin{cases} \left(\int\limits_J \|v\|_B^p dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in J} \|v\|_B, & p = \infty. \end{cases}$$

Suppose that $H_0^1(\bar{\Omega}) = H^1(\bar{\Omega}) \bigcap \{v | v(\partial\bar{\Omega}) = 0\}$. The $L^2(\bar{\Omega})$ orthogonal projection $\mathbb{P}_N^0 : H_0^1(\bar{\Omega}) \to V_N^2$ is defined by

$$(\mathbb{P}_N^0 v - v, \phi) = 0, \quad \forall \phi \in V_N^2.$$

Lemma 6.1 ([2,7]). *For $s \geqslant 1$ and $v \in H^s(\bar{\Omega}) \bigcap H_0^1(\bar{\Omega})$, the following inequality*

$$|v - \mathbb{P}_N^0 v|_{H^1(\bar{\Omega})} + N\|v - \mathbb{P}_N^0 v\|_{L^2(\bar{\Omega})} \leqslant CN^{1-s}\|v\|_{H^s(\bar{\Omega})}$$

holds, where C is independent of N.

Suppose that $a(u,v) = (\nabla u, \nabla v)$ is a bilinear form and $\prod_N^0 : H_0^1(\bar{\Omega}) \to V_N^2$ is the elliptic projection such that

$$a\bigg(\prod_N^0 \bar{u} - \bar{u}, v\bigg) = 0, \quad \forall v \in V_N^2, \tag{6.1}$$

which is continious and coercive in $H_0^1(\bar{\Omega}) \times H_0^1(\bar{\Omega})$.

Lemma 6.2 ([10])**.** *The bilinear form* (6.1) *holds and we have the following inequality:*

$$a(u,v) \leqslant \|\nabla u\| \|\nabla v\|, \quad \forall u, v \in H_0^1(\bar{\Omega}),$$
$$a(u,u) = \|\nabla u\|^2, \quad \forall u \in H_0^1(\bar{\Omega}).$$

Lemma 6.3 ([10])**.** *Assume that* $\bar{u} \in H^s(\bar{\Omega} \bigcap H_0^1(\bar{\Omega}))$*. Then*

$$\bigg\|\nabla\bigg(\prod_N^0 \bar{u} - \bar{u}\bigg)\bigg\| \leq CN^{1-s}\|\bar{u}\|_s,$$
$$\bigg\|\prod_N^0 \bar{u} - \bar{u}\bigg\| \leq CN^{-s}\|\bar{u}\|_s.$$

Now, we introduce Gronwall's Lemma.

Lemma 6.4 ([18])**.** *Suppose that* $f(t)$ *is a non-negetive function on the interval* $(t_0, T]$ *which is integrable, and* $g(t)$ *and* $F(t)$ *are the continuous functions on* $(t_0, T]$*. If* $F(t)$ *satisfies the inequality*

$$F(t) \leqslant g(t) + \int_{t_0}^{t} f(\tau)F(\tau)d\tau, \quad \forall t \in [t_0, T],$$

then we have

$$F(t) \leqslant g(t) + \int_{t_0}^{t} f(s)g(s)\exp\bigg(\int_{s}^{t} f(\tau)d\tau\bigg)ds, \quad \forall t \in [t_0, T].$$

Also, if g is non-decreasing, then we have

$$F(t) \leqslant g(t)\exp\bigg(\int_{t_0}^{t} f(\tau)d\tau\bigg), \quad \forall t \in [t_0, T].$$

Now, the error of the state and the adjoint variables can be written as:

$$\bar{y}_N - \bar{y} = \bigg(\bar{y}_N - \prod_N^0 \bar{y}\bigg) + \bigg(\prod_N^0 \bar{y} - \bar{y}\bigg) = \theta + \rho,$$
$$\bar{p}_N - \bar{p} = \bigg(\bar{p}_N - \prod_N^0 \bar{p}\bigg) + \bigg(\prod_N^0 \bar{p} - \bar{p}\bigg) = \theta' + \rho',$$

where

$$\theta = \bar{y}_N - \prod_N^0 \bar{y}, \qquad \rho = \prod_N^0 \bar{y} - \bar{y},$$
$$\theta' = \bar{p}_N - \prod_N^0 \bar{p}, \qquad \rho' = \prod_N^0 \bar{p} - \bar{p}.$$

Lemma 6.5 ([10])**.** *For* $\rho = \prod_N^0 u - u$ *and* $u, u_t \in L^\infty(J, H^m(\bar{\Omega}) \bigcap H_0^1(\bar{\Omega}))$*, we have*

$$a(\rho, v) = 0, \quad \forall v \in V_N^2,$$
$$N\|\rho(t)\| + |\rho(t)|_1 \leq CN^{1-m}\|u(t)\|_{L^\infty(J;H^m(\bar{\Omega}))}, \quad t \in \bar{J},$$
$$N\|\rho_t(t)\| + |\rho_t(t)|_1 \leq CN^{1-m}\|u_t(t)\|_{L^\infty(J;H^m(\bar{\Omega}))}, \quad t \in \bar{J},$$

where C is independent of N.

Let the interpolation operator $\bar{I}_N$ from $H^1(\Omega)$ onto P_N, we assure that

$$\bar{I}_N f(\bar{x}) = \sum_{j=0}^{N} f(\bar{x}_j)\bar{L}_j(\bar{x}).$$

Then we have the following

Lemma 6.6 ([2, 7, 17, 18])**.** *For any $f \in H^s(\Omega), s > \frac{1}{2}$, we have*

$$\|f - I_N f\| \leqslant CN^{-s}\|f\|_s.$$

Theorem 6.7. *Let $\bar{y}$, $\bar{p}$ be the solutions of equations* (5.1)*,* (5.2)*, $\bar{y}_N$ and $\bar{p}_N$ be the solutions of equations*(5.3)*,* (5.4)*, respectively. Assume that $\bar{y}, \bar{y}_t, \bar{p}, \bar{p}_t \in L^\infty(J, H^s(\bar{\Omega}) \bigcap H_0^1(\bar{\Omega})$. Then*

$$\|\bar{y}_N(t) - \bar{y}(t)\| \le C_1 N^{-s}(\|\bar{y}\|_{L^\infty(J,H^s(\bar{\Omega}))} + \|\bar{y}_t\|_{L^\infty(J,H^s(\bar{\Omega}))}), \quad \textit{for } t \in \bar{J},$$
$$\|\bar{p}_N(t) - \bar{p}(t)\| \le C_2 N^{-s}(\|\bar{p}\|_{L^\infty(J,H^s(\bar{\Omega}))} + \|\bar{p}_t\|_{L^\infty(J,H^s(\bar{\Omega}))}), \quad \textit{for } t \in \bar{J}.$$

Proof. Since $\bar{y}$, $\bar{y}_N$, $\bar{p}$, $\bar{p}_N$ satisfy the following equations:

$$\begin{cases} -\left(\bar{y}_t, v\right) - \left(\frac{2}{d-c}\right)^2 \left(\nabla \bar{y}, \nabla v\right) + \frac{1}{\gamma}\left(\bar{p}, v\right) = \left(\bar{f}, v\right), & \forall v \in V_N^2, \\ \left(\bar{p}_t, v\right) - \left(\frac{2}{d-c}\right)^2 \left(\nabla \bar{p}, \nabla v\right) - \left(\bar{y}, v\right) = -\left(\bar{y}_d, v\right), & \forall v \in V_N^2, \end{cases}$$

$$\begin{cases} -\left(\bar{y}_{Nt}, v\right) - \left(\frac{2}{d-c}\right)^2 \left(\nabla \bar{y}_N, \nabla v\right) + \frac{1}{\gamma}\left(\bar{p}, v\right) = \left(\bar{f}, v\right), & \forall v \in V_N^2, \\ \left(\bar{p}_{Nt}, v\right) - \left(\frac{2}{d-c}\right)^2 \left(\nabla \bar{p}_N, \nabla v\right) - \left(\bar{y}, v\right) = -\left(\bar{y}_d, v\right), & \forall v \in V_N^2, \end{cases}$$

we have

$$\begin{cases} \left(\theta_t + \rho_t, v\right) + \left(\frac{2}{d-c}\right)^2 \left(\nabla\theta + \nabla\rho, \nabla v\right) = 0, \\ -\left(\theta_t' + \rho_t', v\right) + \left(\frac{2}{d-c}\right)^2 \left(\nabla\theta' + \nabla\rho', \nabla v\right) = 0. \end{cases}$$

Using Lemma 6.5, we get

$$\left(\theta_t, v\right) + \left(\rho_t, v\right) + \left(\frac{2}{d-c}\right)^2 a\left(\theta, v\right) = 0, \tag{6.2}$$

$$-\left(\theta_t', v\right) - \left(\rho_t', v\right) + \left(\frac{2}{d-c}\right)^2 a\left(\theta', v\right) = 0. \tag{6.3}$$

Set $v = \theta$ in equation (6.2) and $v = \theta'$ in equation (6.3)

$$\left(\theta_t, \theta\right) + \left(\rho_t, \theta\right) + \left(\frac{2}{d-c}\right)^2 a\left(\theta, \theta\right) = 0,$$
$$-\left(\theta_t', \theta'\right) - \left(\rho_t', \theta'\right) + \left(\frac{2}{d-c}\right)^2 a\left(\theta', \theta'\right) = 0.$$

Since $\left(\frac{2}{d-c}\right)^2 a\left(\theta, \theta\right) \geqslant 0$, we have

$$\left(\theta_t, \theta\right) + \left(\rho_t, \theta\right) \le 0,$$
$$\left(\theta_t', \theta'\right) + \left(\rho_t', \theta'\right) \ge 0.$$

It is obvious that

$-\|x\|\|y\| \le (x, y) \leqslant \|x\|\|y\|, \;\; \|x\|\|y\| \leqslant \dfrac{1}{2}\left(\|x\|^2 + \|y\|^2\right)$ **are the Cauchy–Schwarz inequalities**.

By the triangle inequality and the Cauchy–Schwarz inequality, we obtain

$$\frac{d}{dt}\|\theta\|^2 \leqslant 2\|\rho_t\|\|\|\theta\| \leqslant \|\rho_t\| + \|\theta\|^2,$$
$$\frac{d}{dt}\|\theta'\|^2 \geqslant -2\|\rho_t'\|\|\|\theta'\| \geqslant -\|\rho_t'\|^2 - \|\theta'\|^2.$$

Integrating over $[t_0, t]$ for the first equation and $[T, t]$ for the second equation, we have

$$\int_{t_0}^{t} \frac{d}{dt}\|\theta\|^2 \leq \int_{t_0}^{t} \|\rho_\tau\|^2 d\tau + \int_{t_0}^{t} \|\theta\|^2 d\tau,$$

$$\int_{T}^{t} \frac{d}{dt}\|\theta'\|^2 \leqslant -\int_{T}^{t} \|\rho'_\tau\|^2 d\tau - \int_{T}^{t} \|\theta'\|^2 d\tau.$$

So,

$$\|\theta(t)\|^2 \leq \|\theta(t_0)\|^2 + \int_{t_0}^{t} \left(\|\rho_\tau\|^2 + \|\theta\|^2\right) d\tau,$$

$$\|\theta'(t)\|^2 \leq \|\theta'(T)\|^2 + \int_{t}^{T} \left(\|\rho'_\tau\|^2 + \|\theta'\|^2\right) d\tau.$$

Using the Gronwall inequality, we obtain

$$\|\theta(t)\|^2 \leq \left(\|\theta(t_0)\|^2 + \int_{t_0}^{t} \|\rho_\tau\|^2 d\tau\right) \exp(t - t_0),$$

$$\|\theta'(t)\|^2 \leq \left(\|\theta'(T)\|^2 + \int_{t}^{T} \|\rho'_\tau\|^2 d\tau\right) \exp(T - t).$$

Thus

$$\|\theta(t)\|^2 \leq C_1\left(\|\theta(t_0)\|^2 + \int_{t_0}^{t} \|\rho_\tau\|^2 d\tau\right),$$

$$\|\theta'(t)\|^2 \leq C_2\left(\|\theta'(T)\|^2 + \int_{T}^{t} \|\rho'_\tau\|^2 d\tau\right).$$

Using Lemma 6.1 and Lemma 6.5, we achieve

$$\begin{aligned}\|\theta(t_0)\| = \left\|\mathbb{P}_N^0 \bar{y}(t_0) - \prod_N^0 \bar{y}(t_0)\right\| &\leq \|\mathbb{P}_N^0 \bar{y}(t_0) - \bar{y}(t_0)\| + \left\|\prod_N^0 \bar{y}(t_0) - \bar{y}(t_0)\right\| \\ &\leq CN^{-s}\|\bar{y}\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)},\end{aligned}$$

$$\|\theta'(T)\| = \left\|\mathbb{P}_N^0 \bar{p}(T) - \prod_N^0 \bar{p}(T)\right\| = 0.$$

Therefore

$$\|\theta(t)\| \leq C_1 N^{-s}\Big(\|\bar{y}\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)} + \|\bar{y}_t\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)}\Big),$$

$$\|\theta'(t)\| \leq C_2 N^{-s}\Big(\|\bar{p}\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)} + \|\bar{p}_t\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)}\Big).$$

Owing to Lemma 6.6,

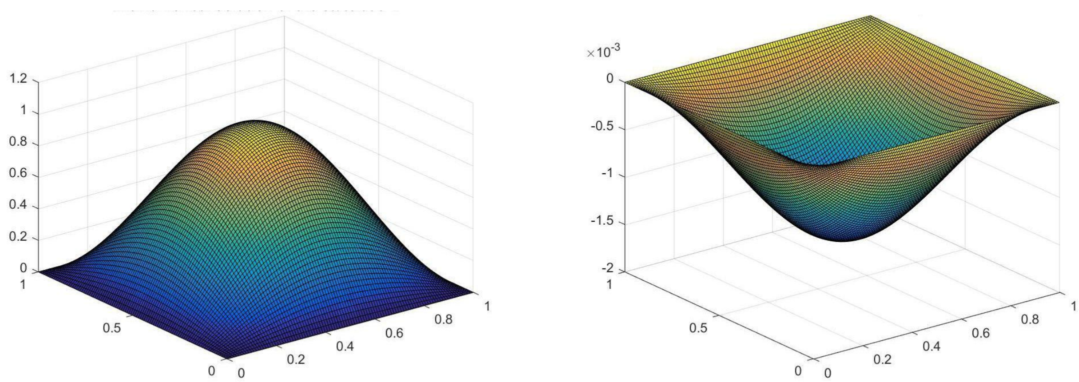
$$\|\bar{y}_N(t) - \bar{y}(t)\| \leq C_1 N^{-s}\Big(\|\bar{y}\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)} + \|\bar{y}_t\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)}\Big),$$

$$\|\bar{p}_N(t) - \bar{p}(t)\| \leq C_2 N^{-s}\Big(\|\bar{p}\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)} + \|\bar{p}_t\|_{L^\infty\left(J, H^s(\bar{\Omega})\right)}\Big).$$

□

7. Numerical Simulations

In this section, numerical samples are presented to investigate the efficiency of the proposed method. Two distributed optimal control problems are solved.



(a) (b)

Figure 1. Plots of the approximated solutions of $y(x,t)$ (a) and $p(x,t)$ (b) in $t = 1$s with $\gamma = 10^{-2}$ in Example 1.

Example 1. In problem (1.1), we take $T = 2$, $x = (x_1, x_2) \in [0,1]^2$ and

$$f = (\pi \sin(\pi t) - 2\pi^2 \cos(\pi t) - \sin(\pi t)) \sin(\pi x_1) \sin(\pi x_2),$$
$$y_d = (\gamma\pi \cos(\pi t) - 2\gamma\pi^2 \sin(\pi t) + \cos(\pi t)) \sin(\pi x_1) \sin(\pi x_2).$$

The initial condition is $y_0(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$. The exact solutions of the state variable and adjoint variable are

$$y(x,t) = \cos(\pi t) \sin(\pi x_1) \sin(\pi x_2),$$
$$p(x,t) = \gamma \sin(\pi t) \sin(\pi x_1) \sin(\pi x_2).$$

This example is similar to the test case of [9].

The graphs of the estimated solutions of $y(x,t)$ and $p(x,t)$ for $t = 1$ with $N = 10$ and $\gamma = 10^{-2}$ are plotted in Figure 1. Figure 2 illustrates the absolute error functions of the state and adjoint functions. Tables 1 and 2 illustrate the absolute errors of the state function and the adjoint function for some values of N and N_t. The reported results illustrate that one can obtain an excellent solution by increasing the number of the Legendre basis.

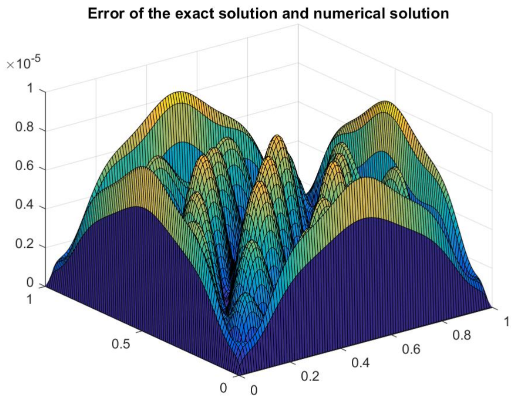
Example 2. In problem (1.1), we set $T = 1$, $x \in [0,1]^2$ and

$$f = 0,$$
$$y_d = ((t-1)^2 t^2 + 2\gamma(2\pi^4 t^4 - 4\pi^4 t^3 + 2(\pi^4 - 3)t^2 + 6t - 1)) \sin(\pi x_1) \sin(\pi x_2).$$

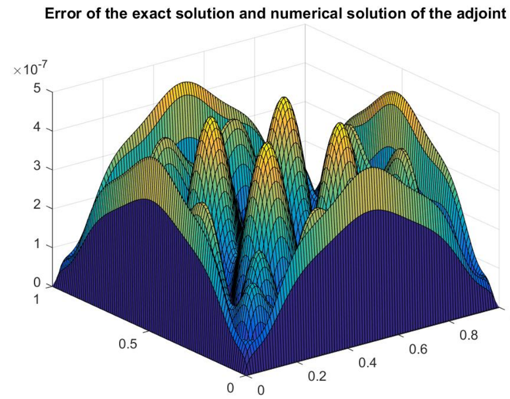
The initial condition is $y_0(x_1, x_2) = y(x_1, x_2, 0)$, the exact solutions of the state variable and the adjoint variable are

$$y(x,t) = t^2(1-t)^2 \sin(\pi x_1) \sin(\pi x_2),$$
$$p(x,t) = 2\gamma(1-t)t(\pi^2 t^2 - (\pi^2 - 2)t - 1) \sin(\pi x_1) \sin(\pi x_2).$$

This example is similar to the test case of [9]. The graphs of the estimated solutions of $y(x,t)$ and $p(x,t)$ for $t = 0.5$ with $N = 10$ and $\gamma = 10^{-4}$ are plotted in Figure 3. Figure 4 shows the error functions of the state and adjoint function with $\gamma = 10^{-4}$. Tables 3 and 4 display the absolute errors of the state function and the adjoint function for some values of N and N_t. According to the obtained results, by increasing the number of Legendre basis, the numerical solutions tend to the exact ones.



(a)



(b)

FIGURE 2. Plots of $y - y_N$ (a) and $p - p_N$ (b) with $\gamma = 10^{-2}$ in Example 1.

TABLE 1. Absolute error of the state function and the adjoint function at $N_t = 10$ for Example 1.

N	$\|y - y_N\|_{L^2}$	$\|p - p_N\|_{L^2}$
2	0.8979	0.0080
4	0.0183	1.5042×10^{-4}
6	1.7639×10^{-4}	2.9803×10^{-6}
8	9.5746×10^{-5}	4.0753×10^{-6}
10	1.3216×10^{-4}	5.7634×10^{-6}

TABLE 2. Absolute error of the state function and the adjoint function at $N = 10$ for Example 1.

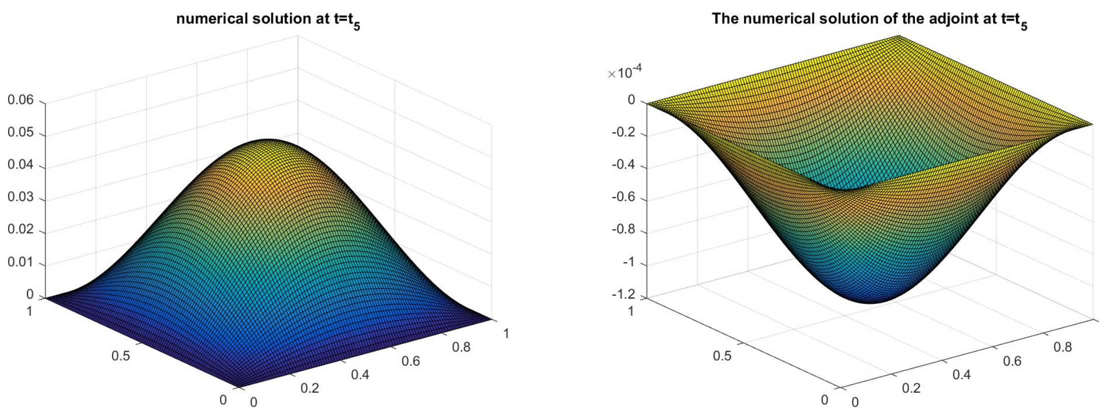
N_t	$\|y - y_N\|_{L^2}$	$\|p - p_N\|_{L^2}$
2	1.2573	0.0133
4	0.2406	0.0090
6	0.0160	9.3661×10^{-4}
8	6.1613×10^{-4}	4.5300×10^{-5}
10	1.3216×10^{-4}	5.7634×10^{-6}

TABLE 3. Absolute error of the state function and the adjoint function at $N_t = 10$ for Example 2.

N	$\|y - y_N\|$	$\|p - p_N\|$
2	0.0369	7.6006×10^{-5}
4	$7.2094\ \times10^{-4}$	1.8641×10^{-6}
6	7.2338×10^{-6}	1.2197×10^{-7}
8	3.1607×10^{-6}	1.8567×10^{-7}
10	2.6105×10^{-6}	2.6399×10^{-7}

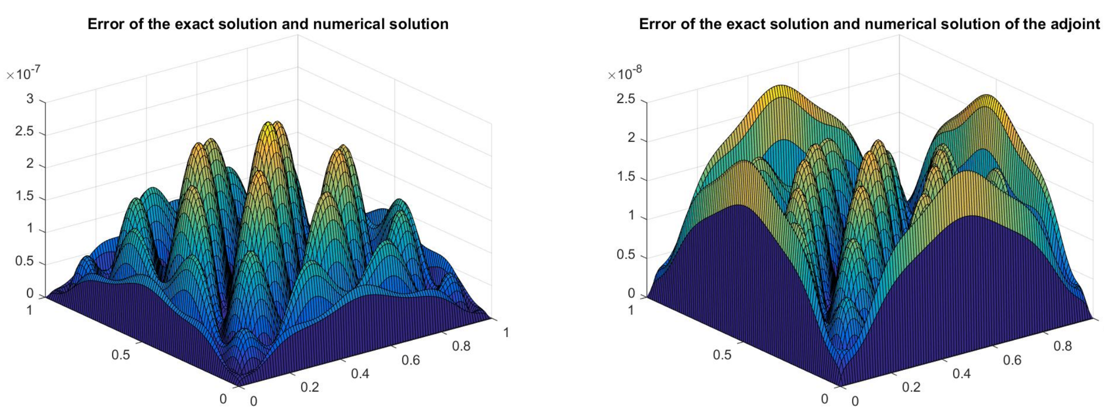
TABLE 4. Absolute error of the state function and the adjoint function at $N = 10$ for Example 2.

N_t	$\|y - y_N\|$	$\|p - p_N\|$
2	0.0734	8.7849×10^{-5}
4	1.6744×10^{-6}	1.6933×10^{-7}
6	2.022×10^{-6}	2.0449×10^{-7}
8	2.3349×10^{-6}	2.3612×10^{-7}
10	2.6105×10^{-6}	2.6399×10^{-7}



(a) (b)

FIGURE 3. Plots of the approximated solutions of $y(x,t)$ (a) and $p(x,t)$ (b) in $t = 0.5$s with $\gamma = 10^{-4}$ in Example 2.



(a) (b)

FIGURE 4. Plots of $y - y_N$ (a) and $p - p_N$ (b) with $\gamma = 10^{-4}$ in Example 2.

8. CONCLUSION

In this paper, we proposed a high-order space-time spectral method to solve a two-dimensional parabolic optimal control problem by combining the spectral collocation method for time derivative and the Legendre–Galerkin method for the space derivative. We have obtained a priori error bound in the L^2-norm for the semidiscrete formulation. Numerical examples are presented to show that the convergence rate of our method is of exponential order in both space and time. In our future work, we intend to apply our technique for three-dimensional cases with even non-classic boundary conditions.

References

1. H. Antil, E. Otárola, A. J. Salgado, A space-time fractional optimal control problem: analysis and discretization. *SIAM J. Control Optim.* **54** (2016), no. 3, 1295–1328.
2. C. Caruto, M. Y. Hussaini, A. M. Quarteroni, T. A. Zang, *Spectral Method in Fluid Mechanics*. Springer-Verlag, Berlin, 1987.
3. M. Darehmiraki, M. H. Farahi, S. Effati, A novel method to solve a class of distributed optimal control problems using Bezier curves. *Journal of Computational and Nonlinear Dynamics*, **11** (2016), no. 6, 13pp.
4. M. Darehmiraki, A. Rezazadeh, A. Ahmadian, An artificial neural network-based method for the optimal control problem governed by the fractional parabolic equation. *Numer. Methods Partial Differential Equations* **37** (2021), no. 3, 2296–2316.
5. M. Darehmiraki, A. Rezazadeh, A. Ahmadian, S. Salahshour, An interpolation method for the optimal control problem governed by the elliptic convection–diffusion equation. *Numerical Methods for Partial Differential Equations* **38** (2022), no. 2, 137–159.
6. M. J. Gander, S. Vandewalle, Analysis of the parareal time-parallel time-integration method. *SIAM J. Sci. Comput.* **29** (2007), no. 2, 556–578.
7. B. Y. Guo, *Spectral Methods and their Applications*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.
8. V. Karl, *Augmented Lagrangian Methods for State Constrained Optimal Control Problems*. PhD diss., Universität Würzburg, 2020.
9. B. Li, J. Liu, M. Xiao, A new multigrid method for unconstrained parabolic optimal control problems. *J. Comput. Appl. Math.* **326** (2017), 358–373.
10. W. Liu, J. Sun, B. Wu, Space-time spectral method for two-dimensional semilinear parabolic equations. *Math. Methods Appl. Sci.* **39** (2016), no. 7, 1646–1661.
11. WJ. Liu, B. Wu, J. Sun, Some numerical algorithms for solving the highly oscillatory second-order initial value problems. *J. Comput. Phys.* **276** (2014), 235–251.
12. M. Mahmoudi, T. Shojaeizadeh, M. Darehmiraki, Optimal control of time-fractional convection-diffusion-reaction problem employing compact integrated RBF method. *Math. Sci. (Springer)* **17** (2023), no. 1, 1–14.
13. A. Rezazadeh, M. Mahmoudi, M. Darehmiraki, Space-time spectral collocation method for one-dimensional PDE constrained optimisation. *Internat. J. Control* **93** (2020), no. 5, 1231–1241.
14. A. Rezazadeh, M. Mahmoudi, M. Darehmiraki, A solution for fractional PDE constrained optimization problems using reduced basis method. *Comput. Appl. Math.* **39** (2020), no. 2, paper no. 82, 17 pp.
15. J. Shen, Efficient spectral-Galerkin method. I. Direct solvers of second- and fourth-order equations using Legendre polynomials. *SIAM J. Sci. Comput.* **15** (1994), no. 6, 1489–1505.
16. J. Shen, T. Tang, *Spectral and High-order Methods with Applications*. Mathematics Monograph Series, 3. Science Press Beijing, Beijing, 2006.
17. J. Shen, T. Tang, *Spectral and High-order Methods with Applications*. Mathematics Monograph Series, 3. Science Press Beijing, Beijing, 2006.
18. J. Shen, T. Tang, L. L. Wang, *Spectral methods: Algorithms, Analysis and Applications*. vol. 41. Springer Science & Business Media, 2011.
19. T. Shojaeizadeh, M. Mahmoudi, M. Darehmiraki, Optimal control problem of advection-diffusion-reaction equation of kind fractal-fractional applying shifted Jacobi polynomials. *Chaos Solitons Fractals* **143** (2021), paper no. 110568, 14 pp.
20. L. N. Trefethen, *Spectral Methods in MATLAB*. Software, Environments, and Tools, 10. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

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