Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 97–110

TRIGONOMETRIC APPROXIMATION BY DEFERRED VORONOI–NÖRLUND AND BY DEFERRED RIESZ MEANS IN THE WEIGHTED SPACE L^p_w

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Abstract. The degree of approximation of functions belonging to the generalized Lipschitz classes, is obtained by deferred Voronoi–Nörlund and deferred Riesz transforms of partial sums of a trigonometric Fourier series in the weighted Lebesgue spaces. Some results as particular cases are derived.

1. Introduction

The degree of approximation of an integrable 2π-periodic function $f(x) \in \mathrm{Lip}(\alpha, p)$ by n-th partial sums

$$S_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

of its Fourier series (at the point x)

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

in L^p-norm ($p \in [1, \infty)$), has been studied by E. S. Quade (see [33]), who examined the range of values of α and p for which the degree of approximation is of order $\mathcal{O}(n^{-\alpha})$. His results can be considered as a very good starting point for continuation of the publication of many of their generalisations obtained recently by other researchers. For example, after Quade's results, some other results are presented by Sahney and Rao in [34], Khan in [22], and Mohapatra and Russell in [31]. Later on, more systematic results gives by Chandra in [4] by using generalized de la Vallé-Poussin means, Nörlund means [5], Riesz means [6], Borel means [7] and [11], Euler means [9], Nörlund and Riesz means [8], and once again by Nörlund and Riesz means in [10]. In the same spirit, Leindler [28] has weakened the monotonicity conditions in Chandra's results. More information on replacing the monotonicity conditions in Chandra's results can be found in the paper of Szal [36]. The interested reader can find other results obtained by Mital et al. [29, 30], Smita and Munjal [35], Khatri and Mishra [23], and Jena et al. [21].

While studying the approximation of functions $f(x) \in \mathrm{Lip}(\alpha, p)$ in the L^p-norm, other authors attempted (successfully) to obtain the counterparts of the above-mentioned results in different setting. For example, such a setting is the generalized Lebesgue space $L^{p(x)}$ (see [24]). It is a known fact that if $p(x) = p$ is a constant ($p \in [1, \infty)$), then the space $L^{p(x)}$ is isometrically isomorphic to the ordinary Lebesgue space L^p. To my best knowledge, it was Guven and Israfilov [18], who defined the Lipschitz class $\mathrm{Lip}(\alpha, p(x))$ and proved that Theorem 1 (cases (i) and (ii)) of [28] also holds true, when the $L^{p(x)}$-norm is used instead of the L^p-norm. Then some generalizations of it (also the case (v) of Theorem 1 of [28] is examined) with different conditions are given by the present author in [26]. Similar results on this topic can be found in [12, 20] and [16].

Now, for our intention, we need to recall the weighted Lebesgue space L^p_w. A measurable $2\pi-$periodic function $w : [0, 2\pi] \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has the Lebesgue

2020 *Mathematics Subject Classification.* 42A10, 41A25, 40Gxx.
Key words and phrases. Fourier series; Deferred Voronoi–Nörlund transformation; Degree of approximation; Modulus of continuity; Lipschitz class.

measure zero. The space $L^p_w = L^p_w[0,2\pi]$, where $p \in [1,\infty)$ and w is a weight function, contains all measurable $2\pi-$periodic functions f for which

$$\|f\|_{p,w} := \left(\int_0^{2\pi} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

Let $p \in (1,\infty)$. A weight function w belongs to the class $\mathcal{A}_p$ if

$$\sup_I \left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I [w(x)]^{-1/p-1}dx\right)^{p-1} < \infty,$$

where supremum is taken over all intervals I with length $|I| \le 2\pi$ (see [32]).

Assumming $p \in (1,\infty)$, $w \in \mathcal{A}_p$ and $f \in L^p_w$, the modulus of continuity of the function f is defined by

$$\Omega(f,\delta)_{p,w} = \sup_{|h|\le\delta} \|\triangle_h(f)\|_{p,w}, \quad \delta > 0,$$

where

$$\triangle_h(f;x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|dt.$$

The modulus of continuity $\Omega(f,\delta)_{p,w}$, defined by Ky [27], is a nondecreasing, nonnegative, continuous function such that

$$\lim_{\delta\to 0}\Omega(f,\delta)_{p,w} = 0, \quad \Omega(f_1+f_2,\delta)_{p,w} \le \Omega(f_1,\delta)_{p,w} + \Omega(f_2,\delta)_{p,w}.$$

Guven [15] defined the Lipschitz class $\mathrm{Lip}(\alpha,p,w)$, $0<\alpha\le 1$, by

$$\mathrm{Lip}(\alpha,p,w) = \left\{f \in L^p_w : \Omega(f,\delta)_{p,w} = \mathcal{O}(\delta^\alpha),\ \delta > 0\right\},$$

and gave the weighted versions of Chandra's results [10] and the respective Leindler's results [28], whenever $p \in (1,\infty)$.

Before we recall Guven's results, we need first some preliminaries. Whenever is necessary, we use the $n-$th partial sums of Fourier series of $f(x)$ at the point x, in the form

$$S_n(f;x) = \sum_{k=0}^{n} A_k(f;x),$$

where

$$A_0(f;x) := \frac{a_0}{2}, \quad A_k(f;x) := a_k\cos kx + b_k \sin kx, \ \ (k=1,2,\dots).$$

Let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We consider two transformations (the so-called Nörlund and Riesz transforms) of the sums $S_n(f;x)$ defined by

$$N_n(f;x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}S_m(f;x)$$

and

$$R_n(f;x) = \frac{1}{P_n}\sum_{m=0}^{n} p_m S_m(f;x),$$

where $P_n := \sum_{m=0}^{n} p_m$, $p_{-1} := P_{-1} := 0$.

The following results are already known.

Theorem 1.1 ([15]). *Let $1<p<\infty$, $w \in \mathcal{A}_p$, $0<\alpha\le 1$, and let $(p_n)_{n=0}^{\infty}$ be a monotonic sequence of positive real numbers such that*

$$(n+1)p_n = \mathcal{O}(P_n).$$

Then for every $f \in \mathrm{Lip}(\alpha,p,w)$, the estimate

$$\|f - N_n(f)\|_{p,w} = \mathcal{O}(n^{-\alpha}), \ \ n = 1,2,\dots$$

holds.

Theorem 1.2 ([15]). *Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \leq 1$, and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers satisfying the relation*

$$\sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| = \mathcal{O}\left(\frac{P_n}{n+1} \right).$$

Then for every $f \in \operatorname{Lip}(\alpha, p, w)$, the estimate

$$\|f - R_n(f)\|_{p,w} = \mathcal{O}(n^{-\alpha}), \quad n = 1, 2, \ldots$$

is satisfied.

The degree of approximation in the L_w^p-norm, not worse than the above degrees, is obtained for a class, more general, than the class $\operatorname{Lip}(\alpha, p, w)$ by the present author [25], by Guven [17] and by Jafarov by using some specific means [20]; some more general results are obtained very recently by Avsar and Yildirir [3].

In order to reveal our intention, we recall some notations and notions.

Let $a = (a_n)$ and $b = (b_n)$ be the sequences of non-negative integers with the conditions

$$a_n < b_n, \quad n = 1, 2, \ldots \tag{1.1}$$

and

$$\lim_{n \to \infty} b_n = +\infty. \tag{1.2}$$

The deferred Cesàro mean (see [1]) determined by a and b is defined as

$$D_n := D_a^b := \frac{S_{a_n+1} + S_{a_n+2} + \cdots + S_{b_n}}{b_n - a_n},$$

where (S_m) is a sequence of real or complex numbers.

Since each D_a^b with conditions (1.1) and (1.2) satisfies the Silverman–Toeplitz conditions, each D_a^b is regular. It should be noted here that D_a^b involves the means of deferred terms of (S_m), except the case, where $a_n = 0$ for all n. Moreover, D_{n-1}^n is the identity transformation and D_0^n is the well-known $(C, 1)$ transformation.

Very recently, the authors of [14] have introduced some new deferred means with conditions (1.1) and (1.2). Indeed, let (p_n) be a sequence of positive real numbers written as follows:

$$D_a^b N_n(f; x) = \frac{1}{P_0^{b_n - a_n - 1}} \sum_{m=a_n+1}^{b_n} p_{b_n - m} S_m(f; x)$$

and

$$D_a^b R_n(f; x) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m S_m(f; x),$$

where

$$P_0^{b_n - a_n - 1} := \sum_{m=0}^{b_n - a_n - 1} p_m \neq 0, \qquad P_{a_n+1}^{b_n} := \sum_{m=a_n+1}^{b_n} p_m \neq 0.$$

These two methods are called the deferred Woronoi–Nörlund means, $(D_a^b N, p)$, and the deferred Riesz means, $(D_a^b R, p)$, respectively. In the special case for $b_n = n$ and $a_n = 0$ for all $n \geq 0$, the methods $D_a^b N_n(f; x)$ and $D_a^b R_n(f; x)$ give us the classical well-known Woronoi–Nörlund and Riesz means, respectively. Moreover, for $p_m = 1$ for all $n \geq 0$, both of them lead to the deferred Cesàro means

$$D_a^b(f; x) = \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} S_m(f; x)$$

of $S_m(f; x)$.

Let us point out here that if $b_n = n$, $a_n = 0$, and $p_n = 1$ for all $n \geq 0$ for these two methods, then they coincide with the Cesàro method $(C, 1)$. In a particular case of this, when $a_n = 0$, (b_n) is a strictly increasing sequence of positive integers with $b_0 = 0$ and $p_n = 1$, then they give us the Cesàro submethod which is obtained by deleting a set of rows from the Cesàro matrix (for details, see in [2]).

The aim of this paper is to prove the versions of Theorems 1.1–1.2 by using the Woronoi–Nörlund means, $(D_a^b N, p)$, and the deferred Riesz means, $(D_a^b R, p)$, respectively. In our results, the degree of approximation of functions belonging to the generalized Lipschitz class $\mathrm{Lip}(\alpha, p, w)$ is of order $\mathcal{O}((b_n - a_n)^{-\alpha})$, which is, in general, not worse than $\mathcal{O}(n^{-\alpha})$. The importance of our results also comes from the paper of R. P. Agnew (see [1]) who verified that the deferred Cesàro transformation has some properties not possessed by the classical Cesàro transformation. To achieve this aim, we need first to recall some helpful statements given in the next section.

2. Auxiliary Lemmas

The following statements are needed for the proofs of our results.

Lemma 2.1 ([15])**.** *Let $1 < p < \infty$, $0 < \alpha \le 1$, and $w \in \mathcal{A}_p$. Then the estimate*

$$\|f - S_n(f)\|_{p,w} = \mathcal{O}\left(n^{-\alpha}\right), \quad n = 1, 2, \dots,$$

holds for every $f \in \mathrm{Lip}(\alpha, p, w)$.

Lemma 2.2 ([15])**.** *Let $1 < p < \infty$ and $w \in \mathcal{A}_p$. Then for $f \in \mathrm{Lip}(1, p, w)$ the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{p,w} = \mathcal{O}\left(n^{-1}\right), \quad (n = 1, 2, \dots),$$

holds.

Now, we need to recall two known classes of sequences.

A positive sequence $\mathbf{c} := (c_n)$ is called almost monotone decreasing (increasing) if there exists a constant $K := K(\mathbf{c})$, depending only on the sequence $\mathbf{c}$ such that for all $n \ge m$

$$c_n \le K c_m \quad (c_n \ge K c_m).$$

To symbolize these classes, we denote them by $\mathbf{c} \in AMDS$ and $\mathbf{c} \in AMIS$, respectively. The following lemma has a key role in the proof of our main results.

Lemma 2.3. *Let $(p_n) \in AMDS$ or $(p_n) \in AMIS$ and*

$$b_n p_{b_n - a_n - 1} = \mathcal{O}\big(P_0^{b_n - a_n - 1}\big). \tag{2.1}$$

Then

$$\sum_{m=0}^{b_n - a_n - 1} p_{b_n - a_n - 1 - m}(a_n + 1 + m)^{-\alpha} = \mathcal{O}\big((b_n - a_n)^{-\alpha} P_0^{b_n - a_n - 1}\big)$$

for $0 < \alpha < 1$.

Proof. Let r denote the integral part of $\frac{b_n - a_n - 1}{2}$ and $(p_n) \in AMDS$. Then we have

$$\begin{aligned}
&\sum_{m=0}^{b_n - a_n - 1} p_{b_n - a_n - 1 - m}(a_n + 1 + m)^{-\alpha} \\
&\quad = \sum_{m=0}^{r} p_{b_n - a_n - 1 - m}(a_n + 1 + m)^{-\alpha} + \sum_{m=r+1}^{b_n - a_n - 1} p_{b_n - a_n - 1 - m}(a_n + 1 + m)^{-\alpha} \\
&\quad = \mathcal{O}\left(p_{b_n - a_n - 1 - r}\right) \sum_{m=0}^{r} (a_n + 1 + m)^{-\alpha} + (a_n + r + 2)^{-\alpha} \sum_{m=r+1}^{b_n - a_n - 1} p_{b_n - a_n - 1 - m} \\
&\quad = \mathcal{O}\left(p_{b_n - a_n - 1 - r}\right) \sum_{m=1}^{b_n - a_n} (a_n + m)^{-\alpha} + \mathcal{O}\left(b_n^{-\alpha}\right) \sum_{m=0}^{b_n - a_n - 1} p_{b_n - a_n - 1 - m} \\
&\quad = \mathcal{O}\left((b_n)^{1-\alpha}\right) p_{b_n - a_n - 1 - r} + \mathcal{O}\left((b_n)^{-\alpha}\right) P_0^{b_n - a_n - 1} \\
&\quad = \mathcal{O}\left((b_n - a_n)^{-\alpha}\right) P_0^{b_n - a_n - 1}.
\end{aligned}$$

Now, let $(p_n) \in AMIS$ and (2.1) be satisfied. Then we have

$$\begin{aligned}
&\sum_{m=0}^{b_n-a_n-1} p_{b_n-a_n-1-m}(a_n+1+m)^{-\alpha}\\
&\quad= \sum_{m=0}^{r} p_{b_n-a_n-1-m}(a_n+1+m)^{-\alpha} + \sum_{m=r+1}^{b_n-a_n-1} p_{b_n-a_n-1-m}(a_n+1+m)^{-\alpha}\\
&\quad= \mathcal{O}\left(p_{b_n-a_n-1}\right)\sum_{m=1}^{b_n-a_n}(a_n+m)^{-\alpha} + (a_n+r+2)^{-\alpha}\sum_{m=r+1}^{b_n-a_n-1} p_{b_n-a_n-1-m}\\
&\quad= \mathcal{O}\Big(\frac{P_0^{b_n-a_n-1}}{b_n}\Big)\mathcal{O}\left((b_n)^{1-\alpha}\right) + \mathcal{O}\Big((b_n+a_n)^{-\alpha}P_0^{b_n-a_n-1}\Big)\\
&\quad= \mathcal{O}\Big((b_n-a_n)^{-\alpha}P_0^{b_n-a_n-1}\Big). \qquad \square
\end{aligned}$$

Remark 2.1. If we put $b_n = n+1$ and $a_n = 0$ for all $n = 1, 2, \ldots$ in Lemma 2.3, we obtain Lemma 4 from [28].

3. Main Results

Our first main result is the following

Theorem 3.1. *Let $p \in (1,\infty)$, $w \in \mathcal{A}_p$ and $f \in \mathrm{Lip}(\alpha, p, w)$. If one of the conditions*

(i) $0 < \alpha < 1$, *and* $(p_n) \in AMDS$
(ii) $0 < \alpha < 1$, $(p_n) \in AMIS$, *and* (2.1) *holds*
(iii) $\alpha = 1$, $\sum_{j=1}^{b_n-a_n-1} |\Delta(p_j)| = \mathcal{O}\Big(\frac{P_0^{b_n-a_n-1}}{b_n-a_n}\Big)$, *and* (2.1) *holds*
(iv) $\alpha = 1$, $\sum_{j=1}^{b_n-a_n-1} j\,|\Delta(p_j)| = \mathcal{O}\big(P_0^{b_n-a_n-1}\big)$

is true, where $\Delta(p_j) = p_j - p_{j+1}$, then

$$\|f - D_a^b N_n(f)\|_{p,w} = \mathcal{O}((b_n-a_n)^{-\alpha}), \quad (n = 1, 2, \ldots).$$

Proof. Let $0 < \alpha < 1$. Using Lemma 2.1 and Lemma 2.3, the cases (i) and (ii) can be proved simultaneously. Namely, since

$$f(x) = \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-1} p_{b_n-a_n-1-m} f(x),$$

we can write

$$f(x) - D_a^b N_n(f;x) = \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-1} p_{b_n-a_n-1-m}[f(x) - S_{a_n+1+m}(f;x)],.$$

whence, using Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned}
\|f - D_a^b N_n(f)\|_{p,w} &\le \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-1} p_{b_n-a_n-1-m}\|f - S_{a_n+1+m}(f)\|_{p,w}\\
&= \frac{1}{P_0^{b_n-a_n-1}}\mathcal{O}\Big(\sum_{m=0}^{b_n-a_n-1} p_{b_n-a_n-1-m}(a_n+1+m)^{-\alpha}\Big)\\
&= \frac{1}{P_0^{b_n-a_n-1}}\mathcal{O}\Big((b_n-a_n)^{-\alpha}P_0^{b_n-a_n-1}\Big) = \mathcal{O}\Big((b_n-a_n)^{-\alpha}\Big).
\end{aligned}$$

(iii) Obviously, here we have to consider the case $\alpha = 1$. Namely, the use of Abel's transformation gives

$$D_a^b N_n(f;x) = \sum_{m=0}^{b_n-a_n-1} p_{b_n-a_n-1-m} S_{a_n+1+m}(f;x)$$

$$= \frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{m=0}^{b_n-a_n-2}\left[S_{a_n+m+1}(f;x)-S_{a_n+m+2}(f;x)\right]\sum_{j=0}^{m}p_{b_n-a_n-1-j}\right.$$
$$\left. + S_{b_n}(f;x)\sum_{j=0}^{b_n-a_n-1}p_{b_n-a_n-1-j}\right\}$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{m=0}^{b_n-a_n-2}A_{a_n+m+2}(f;x)\left(P_0^{b_n-a_n-1-(m+1)}-P_0^{b_n-a_n-1}\right)\right.$$
$$\left. + S_{b_n}(f;x)P_0^{b_n-a_n-1}\right\}$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-2}P_0^{b_n-a_n-1-(m+1)}A_{a_n+m+2}(f;x)$$
$$- \sum_{m=0}^{b_n-a_n-2}A_{a_n+m+2}(f;x)+\sum_{m=0}^{b_n}A_m(f;x)$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}P_0^{b_n-a_n-1-m}A_{a_n+m+1}(f;x)$$
$$+ \sum_{m=0}^{b_n}A_m(f;x)-\sum_{m=a_n+2}^{b_n}A_m(f;x)$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}P_0^{b_n-a_n-1-m}A_{a_n+m+1}(f;x)+\sum_{m=0}^{a_n+1}A_m(f;x)$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-1}P_0^{b_n-a_n-1-m}A_{a_n+1+m}(f;x)+S_{a_n}(f;x).$$

We use the obvious equality

$$S_{b_n}(f;x)=\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=0}^{b_n-a_n-1}P_0^{b_n-a_n-1}A_{a_n+1+m}(f;x)+\sum_{m=0}^{a_n}A_m(f;x)$$

and Abel's transformation to obtain

$$S_{b_n}(f;x)-D_a^bN_n(f;x)$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}mA_{a_n+1+m}(f;x)$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{m=1}^{b_n-a_n-2}\Delta\left(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}\right)\sum_{j=1}^{m}jA_{a_n+1+j}(f;x)\right.$$
$$\left. + \frac{P_0^{b_n-a_n-1}-P_0^{0}}{b_n-a_n-1}\sum_{j=1}^{b_n-a_n-1}jA_{a_n+1+j}(f;x)\right\}$$
$$= \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}\Delta\left(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}\right)\sum_{j=1}^{m}jA_{a_n+1+j}(f;x)$$
$$+ \frac{1}{b_n-a_n}\sum_{j=1}^{b_n-a_n-1}jA_{a_n+1+j}(f;x),$$

where $\Delta(c_{n,m}) := c_{n,m}-c_{n,m+1}$, and we agree with $P_0^{-1} := 0$.

Hence, we have

$$\begin{aligned}&\|S_{b_n}(f)-D_a^bN_n(f)\|_{p,w}\\ &\leq \frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}\left|\Delta\left(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}\right)\right|\left\|\sum_{j=1}^{m} jA_{a_n+1+j}(f)\right\|_{p,w}\\ &+\frac{1}{b_n-a_n}\left\|\sum_{j=1}^{b_n-a_n-1} jA_{a_n+1+j}(f)\right\|_{p,w}.\end{aligned}$$

Elementary calculations imply

$$\begin{aligned}S_{b_n-a_n-1}(f;x)-D_a^b(f;x)&=S_{b_n-a_n-1}(f;x)-\frac{1}{b_n-a_n}\sum_{j=0}^{b_n-a_n-1}S_{a_n+1+j}(f;x)\\ &=\frac{1}{b_n-a_n}\sum_{j=0}^{b_n-a_n-1}(S_{b_n-a_n-1}(f;x)-S_{a_n+1+j}(f;x))\\ &=\frac{1}{b_n-a_n}\sum_{j=0}^{b_n-a_n-1}\left(\sum_{m=a_n+2+j}^{b_n-a_n-1}A_m(f;x)\right)\\ &=\frac{1}{b_n-a_n}\sum_{j=1}^{b_n-a_n-1}jA_{a_n+1+j}(f;x),\end{aligned}$$

or

$$\sum_{j=1}^{b_n-a_n-1}jA_{a_n+1+j}(f;x)=(b_n-a_n)[S_{b_n-a_n-1}(f;x)-D_a^b(f;x)].$$

Thus, Lemma 2.3 gives

$$\left\|\sum_{j=1}^{b_n-a_n-1}jA_{a_n+1+j}(f;x)\right\|_{p,w}=(b_n-a_n)\|S_{b_n-a_n-1}(f;x)-D_a^b(f;x)\|_{p,w}=\mathcal{O}(1),$$

and therefore

$$\begin{aligned}&\|S_{b_n}(f)-D_a^bN_n(f)\|_{p,w}\\ &=\mathcal{O}\left(\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}\left|\Delta\left(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}\right)\right|+\frac{1}{b_n-a_n}\right).\end{aligned}$$

Moreover, the equality

$$\Delta\left(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}\right)=\frac{\sum\limits_{j=b_n-a_n-m}^{b_n-a_n-1}p_j-mp_{b_n-a_n-1-m}}{m(m+1)}$$

holds true.

Now, we prove by induction with respect to m that

$$\left|\sum_{j=b_n-a_n-m}^{b_n-a_n-1}p_j-mp_{b_n-a_n-1-m}\right|\leq\sum_{j=1}^{m}j\left|p_{b_n-a_n-j-1}-p_{b_n-a_n-j}\right|.$$

Indeed, for $m=1$, the equality

$$|p_{b_n-a_n-1}-p_{b_n-a_n-2}|=|p_{b_n-a_n-2}-p_{b_n-a_n-1}|$$

is true.

Assume that the above inequality holds true for m and we prove it for $m+1$:

$$\begin{aligned}
&\Big|\sum_{j=b_n-a_n-(m+1)}^{b_n-a_n-1} p_j-(m+1)p_{b_n-a_n-1-(m+1)}\Big| = \Big|\sum_{j=b_n-a_n-m}^{b_n-a_n-1} p_j-mp_{b_n-a_n-1-(m+1)}\Big| \\
&\leq \Big|\sum_{j=b_n-a_n-m}^{b_n-a_n-1} p_j-mp_{b_n-a_n-1-m}\Big| + m\left|p_{b_n-a_n-1-m}-p_{b_n-a_n-1-(m+1)}\right| \\
&\leq \sum_{j=1}^{m} j\left|p_{b_n-a_n-j-1}-p_{b_n-a_n-j}\right| + (m+1)\left|p_{b_n-a_n-1-m}-p_{b_n-a_n-1-(m+1)}\right| \\
&= \sum_{j=1}^{m+1} j\left|\Delta(p_{b_n-a_n-j})\right|.
\end{aligned}$$

Using this inequality and assumptions of our theorem, we obtain

$$\begin{aligned}
&\|S_{b_n}(f)-D_a^bN_n(f)\|_{p,w} \\
&= \mathcal{O}\bigg(\frac{1}{P_0^{b_n-a_n-1}}\sum_{m=1}^{b_n-a_n-1}\frac{1}{m(m+1)}\sum_{j=1}^{m} j\left|\Delta(p_{b_n-a_n-j})\right| + \frac{1}{b_n-a_n}\bigg) \\
&= \mathcal{O}\bigg(\frac{1}{P_0^{b_n-a_n-1}}\sum_{j=1}^{b_n-a_n-1} j\left|\Delta(p_{b_n-a_n-j})\right|\sum_{m=j}^{\infty}\frac{1}{m(m+1)} + \frac{1}{b_n-a_n}\bigg) \\
&= \mathcal{O}\bigg(\frac{1}{P_0^{b_n-a_n-1}}\sum_{j=1}^{b_n-a_n-1}\left|\Delta(p_{b_n-a_n-j})\right| + \frac{1}{b_n-a_n}\bigg) = \mathcal{O}\bigg(\frac{1}{b_n-a_n}\bigg).
\end{aligned}$$

To complete the proof of our theorem, we have to consider the remaining case (iv). First, we show that under the assumption

$$\sum_{j=1}^{b_n-a_n-1} j\left|\Delta(p_j)\right| = \mathcal{O}\big(P_0^{b_n-a_n-1}\big),$$

the inequality

$$K_{a_n,b_n} := \sum_{m=1}^{b_n-a_n-1}\left|\Delta\bigg(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-a_n-1-m}}{m}\bigg)\right| = \mathcal{O}\bigg(\frac{P_0^{b_n-a_n-1}}{b_n-a_n}\bigg) \tag{3.1}$$

holds true.

We already have showen, during the proof of the case (iii), that

$$\begin{aligned}
K_{a,b} &\leq \sum_{m=1}^{b_n-a_n-1}\frac{1}{m(m+1)}\sum_{j=1}^{m} j\left|\Delta(p_{b_n-a_n-j})\right| \\
&= \sum_{m=1}^{r}(\cdot) + \sum_{m=r+1}^{b_n-a_n-1}(\cdot) =: K_{a,b}^1 + K_{a,b}^2.
\end{aligned}$$

As during this paper, we take r to be the integral part of $\frac{b_n-a_n-1}{2}$. Then, by Abel's transformation and assumption on (iv), we obtain

$$\begin{aligned}
K_{a,b}^1 &= \sum_{m=1}^{r}\frac{1}{m(m+1)}\sum_{j=1}^{m} j\left|\Delta(p_{b_n-a_n-j})\right| \\
&\leq \sum_{j=1}^{r} j\left|\Delta(p_{b_n-a_n-j})\right|\sum_{r=j}^{\infty}\frac{1}{r(r+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r} |\Delta(p_{b_n-a_n-j})| \\
&= \sum_{j=b_n-a_n-r}^{p_{b_n-a_n-1}} |\Delta(p_j)| \\
&\le \frac{2}{b_n-a_n+1} \sum_{j=1}^{p_{b_n-a_n-1}} j\,|\Delta(p_j)| = \mathcal{O}\Big(\frac{P_0^{b_n-a_n-1}}{b_n-a_n}\Big).
\end{aligned}
$$

Now, we can write

$$
K_{a,b}^2 \le \sum_{m=r}^{b_n-a_n-1} \frac{1}{m(m+1)}\Big[\sum_{j=1}^{r} j\,|\Delta(p_{b_n-a_n-j})| + \sum_{j=r}^{m} j\,|\Delta(p_{b_n-a_n-j})|\Big] := K_{a,b}^{21} + K_{a,b}^{22}.
$$

Moreover, based on our assumption, we get

$$
\begin{aligned}
K_{a,b}^{21} &\le \sum_{m=r}^{b_n-a_n-1} \frac{1}{m(m+1)} \sum_{j=b_n-a_n-r}^{m} j\,|\Delta(p_j)| \\
&\le \sum_{m=r}^{b_n-a_n-1} \frac{1}{m+1} \sum_{j=b_n-a_n-1}^{m} |\Delta(p_j)| \\
&\le \frac{b_n-a_n-1-(r-1)}{r+1} \sum_{j=b_n-a_n-r}^{b_n-a_n-1} \frac{j}{j}\,|\Delta(p_j)| \\
&\le \frac{1}{b_n-a_n} \sum_{j=1}^{b_n-a_n-1} j\,|\Delta(p_j)| = \mathcal{O}\Big(\frac{P_0^{b_n-a_n-1}}{b_n-a_n}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
K_{a,b}^{22} &\le \sum_{m=r}^{b_n-a_n-1} \frac{1}{m+1} \sum_{j=r}^{m} |\Delta(p_{b_n-a_n-j})| \\
&\le \frac{1}{r+1} \sum_{m=r}^{b_n-a_n-1} \sum_{j=r}^{m} |\Delta(p_{b_n-a_n-j})| \\
&= \mathcal{O}\Big(\frac{1}{b_n-a_n}\Big)\big[\,|\Delta(p_1)| + 2|\Delta(p_2)| + \cdots + (b_n-a_n-r)|\Delta(p_{b_n-a_n-r})|\,\big] \\
&= \mathcal{O}\Big(\frac{1}{b_n-a_n} \sum_{j=1}^{b_n-a_n-1} j\,|\Delta(p_j)|\Big) = \mathcal{O}\Big(\frac{P_0^{b_n-a_n-1}}{b_n-a_n}\Big)
\end{aligned}
$$

which show that (3.1) holds true.

So, we have proved that

$$
\|S_{b_n}(f) - D_a^b N_n(f)\|_{p,w} = \mathcal{O}((b_n-a_n)^{-\alpha})
$$

and using Lemma 2.1, we obtain

$$
\|f - D_a^b N_n(f)\|_{p,w} \le \|f - S_{b_n}(f)\|_{p,w} + \|S_{b_n}(f) - D_a^b N_n(f)\|_{p,w} = \mathcal{O}((b_n-a_n)^{-\alpha}).
$$

The proof is completed. □

Theorem 3.2. *Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \le 1$, and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers satisfying the relation*

$$
\sum_{m=0}^{b_n-a_n-3} \Big|\frac{P_{a_n+1}^{m+a_n+1}}{a_n+2+m} - \frac{P_{a_n+1}^{m+a_n+2}}{a_n+3+m}\Big| = \mathcal{O}\Big(\frac{P_{a_n+1}^{b_n-1}}{b_n}\Big). \tag{3.2}
$$

Then for every $f\in \mathrm{Lip}(\alpha,p,w)$, the estimate

$$\|f-D_a^bR_n(f)\|_{p,w}=\mathcal{O}\left((b_n-a_n)^{-\alpha}\right),\quad n=1,2,\dots$$

is satisfied.

Proof. Let $0<\alpha<1$. First of all, we can write

$$f(x)-D_a^bR_n(f;x)=\frac{1}{P_{a_n+1}^{b_n}}\sum_{m=0}^{b_n-a_n-1}p_{a_n+1+m}[f(x)-S_{a_n+1+m}(f;x)].$$

Using Lemma 2.1,

$$\begin{aligned}\|f-D_a^bR_n(f)\|_{p,w}&\le \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=0}^{b_n-a_n-1}p_{a_n+1+m}\|f-S_{a_n+1+m}(f)\|_{p,w}\\&\le \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=0}^{b_n-a_n-1}p_{a_n+1+m}\mathcal{O}((a_n+1+m)^{-\alpha}).\end{aligned}\tag{3.3}$$

We use the summation by parts to obtain ($P_{a_n+1}^0:=0$):

$$\begin{aligned}&\sum_{m=0}^{b_n-a_n-1}p_{a_n+1+m}(a_n+1+m)^{-\alpha}\\&=\sum_{m=0}^{b_n-a_n-2}\left[(a_n+1+m)^{-\alpha}-(a_n+2+m)^{-\alpha}\right]P_{a_n+1}^{m+a_n+1}+b_n^{-\alpha}P_{a_n+1}^{b_n}\\&=\mathcal{O}(1)\sum_{m=0}^{b_n-a_n-2}(a_n+2+m)^{-1-\alpha}P_{a_n+1}^{m+a_n+1}+\frac{P_{a_n+1}^{b_n}}{(b_n-a_n)^{\alpha}}.\end{aligned}\tag{3.4}$$

Using the summation by parts again and condition (3.2), we get

$$\begin{aligned}&\sum_{m=0}^{b_n-a_n-2}(a_n+2+m)^{-\alpha}\frac{P_{a_n+1}^{m+a_n+1}}{a_n+2+m}\\&\quad\le\sum_{m=0}^{b_n-a_n-3}\left|\frac{P_{a_n+1}^{m+a_n+1}}{a_n+2+m}-\frac{P_{a_n+1}^{m+a_n+2}}{a_n+3+m}\right|\sum_{j=0}^{m}(a_n+2+j)^{-\alpha}\\&\qquad+\frac{P_{a_n+1}^{b_n-1}}{b_n}\sum_{j=0}^{b_n-a_n-2}(a_n+2+j)^{-\alpha}\\&\quad=\sum_{m=0}^{b_n-a_n-3}\left|\frac{P_{a_n+1}^{m+a_n+1}}{a_n+2+m}-\frac{P_{a_n+1}^{m+a_n+2}}{a_n+3+m}\right|\mathcal{O}\left((a_n+2+m)^{1-\alpha}\right)\\&\qquad+\frac{P_{a_n+1}^{b_n-1}}{b_n}\mathcal{O}\left(b_n^{1-\alpha}\right)\\&\quad=\mathcal{O}\left(\frac{P_{a_n+1}^{b_n-1}}{b_n}(b_n-1)^{1-\alpha}\right)+\mathcal{O}\left(\frac{P_{a_n+1}^{b_n-1}}{b_n^{\alpha}}\right)=\mathcal{O}\left(\frac{P_{a_n+1}^{b_n}}{(b_n-a_n)^{\alpha}}\right),\end{aligned}\tag{3.5}$$

whence (3.3), (3.4) and (3.5) imply

$$\|f-D_a^bR_n(f)\|_{p,w}=\mathcal{O}\left(\frac{1}{(b_n-a_n)^{\alpha}}\right).$$

Now, we consider the case $\alpha=1$. Indeed, the use of the summation by parts gives

$$D_a^bR_n(f;x)=\frac{1}{P_{a_n+1}^{b_n}}\sum_{m=0}^{b_n-a_n-1}p_{a_n+1+m}S_{a_n+1+m}(f;x)$$

$$= \frac{1}{P_{a_n+1}^{b_n}} \Bigg[\sum_{m=0}^{b_n-a_n-2} (S_{a_n+1+m}(f;x) - S_{a_n+2+m}(f;x)) \sum_{j=0}^{m} p_{a_n+1+j} + S_{b_n}(f;x) \sum_{j=0}^{b_n-a_n-1} p_{a_n+1+j} \Bigg]$$

$$= -\frac{1}{P_{a_n+1}^{b_n}} \Bigg[\sum_{m=0}^{b_n-a_n-2} A_{a_n+2+m}(f;x) P_{a_n+1}^{a_n+1+m} - S_{b_n}(f;x) P_{a_n+1}^{b_n} \Bigg]$$

or

$$D_a^b R_n(f;x) - S_{b_n}(f;x) = -\frac{1}{P_{a_n+1}^{b_n}} \sum_{m=0}^{b_n-a_n-2} P_{a_n+1}^{a_n+1+m} A_{a_n+2+m}(f;x).$$

We use the summation by parts again to obtain

$$\begin{aligned}
&\sum_{m=0}^{b_n-a_n-2} P_{a_n+1}^{a_n+1+m} A_{a_n+2+m}(f;x) \\
&\quad= \sum_{m=0}^{b_n-a_n-2} \frac{P_{a_n+1}^{a_n+1+m}}{a_n+2+m} (a_n+2+m) A_{a_n+2+m}(f;x) \\
&\quad= \sum_{m=0}^{b_n-a_n-3} \left(\frac{P_{a_n+1}^{a_n+1+m}}{a_n+2+m} - \frac{P_{a_n+1}^{a_n+2+m}}{a_n+3+m} \right) \sum_{j=0}^{m} (a_n+2+j) A_{a_n+2+j}(f;x) \\
&\qquad + \frac{P_{a_n+1}^{b_n-1}}{b_n} \sum_{j=0}^{b_n-a_n-2} (a_n+2+j) A_{a_n+2+j}(f;x),
\end{aligned}$$

whence, using the equality

$$\begin{aligned}
\sum_{j=0}^{m} (a_n+2+j) A_{a_n+2+j}(f;x) &= \sum_{j=a_n+2}^{m+a_n+2} j A_j(f;x) \\
&= \sum_{j=0}^{m+a_n+2} j A_j(f;x) - \sum_{j=0}^{a_n+1} j A_j(f;x) \\
&= (m+a_n+3)(S_{m+a_n+2}(f;x) - \sigma_{m+a_n+2}(f;x)) \\
&\quad - (a_n+2)(S_{a_n+1}(f;x) - \sigma_{a_n+1}(f;x)),
\end{aligned}$$

we get

$$\begin{aligned}
&\left\| \sum_{m=0}^{b_n-a_n-2} P_{a_n+1}^{a_n+1+m} A_{a_n+2+m}(f) \right\|_{p,\omega} \\
&\le \sum_{m=0}^{b_n-a_n-3} \left| \frac{P_{a_n+1}^{a_n+1+m}}{a_n+2+m} - \frac{P_{a_n+1}^{a_n+2+m}}{a_n+3+m} \right| \left\| \sum_{j=0}^{m} (a_n+2+j) A_{a_n+2+j}(f) \right\|_{p,\omega} \\
&\quad + \frac{P_{a_n+1}^{b_n-1}}{b_n} \left\| \sum_{j=0}^{b_n-a_n-2} (a_n+2+j) A_{a_n+2+j}(f) \right\|_{p,\omega} \\
&\le \sum_{m=0}^{b_n-a_n-3} (m+a_n+3) \left| \frac{P_{a_n+1}^{a_n+1+m}}{a_n+2+m} - \frac{P_{a_n+1}^{a_n+2+m}}{a_n+3+m} \right| \, \| S_{m+a_n+2}(f) - \sigma_{m+a_n+2}(f) \|_{p,\omega} \\
&+ \sum_{m=0}^{b_n-a_n-3} (a_n+2) \left| \frac{P_{a_n+1}^{a_n+1+m}}{a_n+2+m} - \frac{P_{a_n+1}^{a_n+2+m}}{a_n+3+m} \right| \, \| S_{a_n+1}(f) - \sigma_{a_n+1}(f) \|_{p,\omega}
\end{aligned}$$

$$+\frac{P_{a_n+1}^{b_n-1}}{b_n}\left[(b_n+1)\left\|S_{b_n}(f)-\sigma_{b_n}(f)\right\|_{p,\omega}+(a_n+2)\left\|S_{a_n+1}(f)-\sigma_{a_n+1}(f)\right\|_{p,\omega}\right].$$

Consequently, using Lemma 2.2 and condition (3.2), we have

$$\begin{aligned}\left\|D_a^bR_n(f)-S_{b_n}(f)\right\|_{p,\omega}&=\frac{1}{P_{a_n+1}^{b_n}}\mathcal{O}\left(\sum_{m=0}^{b_n-a_n-3}\left|\frac{P_{a_n+1}^{a_n+1+m}}{a_n+2+m}-\frac{P_{a_n+1}^{a_n+2+m}}{a_n+3+m}\right|+\frac{P_{a_n+1}^{b_n-1}}{b_n}\right)\\&=\mathcal{O}\Big(\frac{1}{b_n}\Big)=\mathcal{O}\Big(\frac{1}{b_n-a_n}\Big).\end{aligned}$$

Finally, the latest estimate and Lemma 2.1 imply

$$\|f-D_a^bR_n(f)\|_{p,w}\leq\|f-S_{b_n}(f)\|_{p,w}+\|S_{b_n}(f)-D_a^bR_n(f)\|_{p,w}=\mathcal{O}\Big(\frac{1}{b_n-a_n}\Big).$$

The proof is completed. □

4. Few Remarks and Corollaries

As we have seen, for $p_m=1$, for all $n\geq 0$, the Voronoi–Nörlund and the deferred Riesz means reduce to the deferred Cesàro means

$$D_a^b(f;x)=\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}S_m(f;x)$$

of $S_m(f;x)$. So, both Theorems 3.1–3.2 imply the deviation

$$\|f-D_a^b(f)\|_{p,w}=\mathcal{O}\left((b_n-a_n)^{-\alpha}\right)\quad(n=1,2,\dots).$$

Moreover, if $b_n=n$, $a_n=0$, and $p_n=1$, for all $n\geq 1$, then $D_a^b(f;x)$ means coincide with the Cesàro means $\sigma_n(f;x)=\frac{1}{n}\sum_{j=1}^{n}S_m(f;x)$. Even in this case, we clearly have

$$\|f-\sigma_n(f)\|_{p,w}=\mathcal{O}\left(n^{-\alpha}\right)\quad(n=1,2,\dots).$$

Remark 4.1. If we take $a_n=0$ and $b_n=n$ $(n=1,2,\dots)$ in our theorems, then we obtain the results proved in [15].

Let us suppose that $\mathbb{F}$ is a subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of a strictly increasing sequence of positive integers, say $\mathbb{F}=(\lambda(n))_1^\infty$.

The polynomials

$$N_n^\lambda(f;x)=\frac{1}{P_{\lambda(n)}}\sum_{k=0}^{\lambda(n)}p_{\lambda(n)-k}s_k(f;x),$$

and

$$R_n^\lambda(f;x)=\frac{1}{Q_{\lambda(n)}}\sum_{k=0}^{\lambda(n)}q_ks_k(f;x),$$

introduced in [13], are the particular case (for $a_n=0$ and $b_n=\lambda(n)$) of the means $D_a^bN_n(f;x)$ and $D_a^bR_n(f;x)$, respectively. Therefore Theorem 3.1 implies:

Corollary 4.1 ([20])**.** *Let $p\in(1,\infty)$, $w\in\mathcal{A}_p$ and $f\in\operatorname{Lip}(\alpha,p,w)$. If one of the conditions*

(i) $0<\alpha<1$, *and* $(p_n)\in AMDS$

(ii) $0<\alpha<1$, $(p_n)\in AMIS$, *and* $(\lambda(n)+1)p_{\lambda(n)}=P_{\lambda(n)}$ *holds*

(iii) $\alpha=1$, $\sum_{j=1}^{\lambda(n)-1}|\Delta(p_j)|=\mathcal{O}\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right)$, *and* (2.1) *holds*

(iv) $\alpha=1$, $\sum_{j=1}^{\lambda(n)-1}j\,|\Delta(p_j)|=\mathcal{O}\left(P_{\lambda(n)}\right)$

is true, where $\Delta(p_j)=p_j-p_{j+1}$, then

$$\|f-N_n^\lambda(f)\|_{p,w}=\mathcal{O}((\lambda(n))^{-\alpha}),\quad(n=1,2,\dots).$$

Moreover, for $p_k = q_k = 1$, $k = 0, 1, \dots, \lambda(n)$, we obtain the polynomials (see [2, p. 195])

$$C_n^\lambda(f;x) = \frac{1}{\lambda(n)+1}\sum_{k=0}^{\lambda(n)} s_k(f;x),$$

which for $\lambda(n) = n$, as a particular case, they reduce to the ordinary Cesàro mean.

So, under the appropriate conditions, the deviation

$$\|f - C_n^\lambda(f)\|_{p,w} = \mathcal{O}\left((\lambda(n))^{-\alpha}\right) \quad (n = 1, 2, \dots)$$

is also implied from our results.

Finally, Theorem 3.2 implies the following

Corollary 4.2. *Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha \le 1$, and let $(p_n)_{n=0}^\infty$ be a sequence of positive real numbers satisfying the relation*

$$\sum_{m=0}^{\lambda(n)-3}\left|\frac{P_1^{m+1}}{m+2} - \frac{P_1^{m+2}}{m+3}\right| = \mathcal{O}\left(\frac{P_1^{\lambda(n)-1}}{\lambda(n)}\right).$$

Then for every $f \in \mathrm{Lip}(\alpha, p, w)$, the estimate

$$\|f - R_n^\lambda(f)\|_{p,w} = \mathcal{O}\left((\lambda(n))^{-\alpha}\right), \quad n = 1, 2, \dots$$

is satisfied.

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(Received 02.04.2022)

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