Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 175–177

FULL EQUIPMENT OF THE MANIFOLD TANGENT FIBER SPACE $TT(T(Vn))$

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Abstract. We consider the manifold tangent space $T(T(Vn))$ of the tangent fiber space $T(Vn)$. Invariant I-forms of the space $T(T(Vn))$ are defined and their structural equations are obtained. Linear connections of the space $T(T(Vn))$ are considered when it is fully equipped.

Let us consider the manifold tangent fiber space $T(T(Vn))$ with local coordinates $(x^i, y^{\bar{i}}, y^i, z^{\bar{i}})$, $i, j, k = \overline{1,n}$, $\bar{i}, \bar{j}, \bar{k} = \overline{1,n}$, where $x^i, y^{\bar{i}}$ are the coordinates of the basis $T(Vn)$, and $y^i, z^{\bar{i}}$ are the coordinates of the fiber T_z, $z \in T(Vn)$ In other words, the vector fields $\mathfrak{X}$

$$\mathfrak{X} = y^i \frac{\partial}{\partial x^i} + z^{\bar{i}} \frac{\partial}{\partial y^{\bar{i}}}$$

generate the fiber space $T(T(Vn))$. It is obvious that the local coordinates $(x^i, y^{\bar{i}}, y^i, z^{\bar{i}})$ of a point of the fiber space $T(T(Vn))$ are transformed as follows:

$$\overline{x}^i = \overline{x}^i(x^k), \quad \overline{y}^i = x^i_k y^k, \quad \overline{y}^{\bar{i}} = x^{\bar{i}}_{\bar{k}} y^{\bar{k}}, \quad \overline{z}^{\bar{i}} = x^{\bar{i}}_{\bar{k}} z^{\bar{k}} + x^{\bar{i}}_{\bar{k}j} y^{\bar{k}} y^j.$$

On the space $T(T(Vn))$ one can define the following I-forms:

$$\theta^i = dy^i + \omega^i_k y^k, \quad \theta^{\bar{i}} = dy^{\bar{i}} + \omega^{\bar{i}}_{\bar{k}} y^{\bar{k}}, \quad \vartheta^{\bar{i}} = dz^{\bar{i}} + \omega^{\bar{i}}_{\bar{k}} z^{\bar{k}} + \omega^{\bar{i}}_{\bar{k}j} y^{\bar{k}} y^j.$$

Differentiating these equalities externally and using structural equations for the I-form ω^i_k, $\omega^{\bar{i}}_{\bar{k}}$, $\omega^{\bar{i}}_{\bar{k}j}$ [1] we obtain:

$$D\theta^i = \theta^k \wedge \omega^i_k + \omega^k \wedge \theta^i_k,$$
$$D\theta^{\bar{i}} = \theta^{\bar{k}} \wedge \omega^{\bar{i}}_{\bar{k}} + \omega^k \wedge \theta^{\bar{i}}_k,$$
$$D\vartheta^{\bar{i}} = \vartheta^{\bar{k}} \wedge \omega^{\bar{i}}_{\bar{k}} + \theta^{\bar{k}} \wedge \theta^{\bar{i}}_{\bar{k}} + \theta^k \wedge \theta^{\bar{i}}_k + \omega^k \wedge \vartheta^{\bar{i}}_k,$$

where

$$\theta^{\bar{i}}_k = \omega^{\bar{i}}_{\bar{j}k} y^{\bar{j}}, \quad \theta^i_k = \omega^i_{kj} y^j, \quad \theta^{\bar{i}}_{\bar{k}} = \omega^{\bar{i}}_{\bar{k}j} y^j, \quad \vartheta^{\bar{i}}_j = \omega^{\bar{i}}_{\bar{k}j} z^{\bar{k}} + \omega^{\bar{i}}_{\bar{k}ij} y^{\bar{k}} y^i.$$

From the transformation law of the local coordinates of a point of the tangent fiber space $T(T(Vn))$ it follows that

$$d\overline{x}^i = x^i_k dx^k,$$
$$d\overline{y}^i = x^i_{kj} y^k dx^j + x^i_k dy^k,$$
$$d\overline{y}^{\bar{i}} = x^{\bar{i}}_{\bar{k}j} y^{\bar{k}} dx^j + x^{\bar{i}}_{\bar{k}} dy^{\bar{k}},$$
$$d\overline{z}^{\bar{i}} = (x^{\bar{i}}_{\bar{k}j} z^{\bar{k}} + x^{\bar{i}}_{\bar{k}pj} y^{\bar{k}} y^p) dx^j + x^{\bar{i}}_{\bar{k}j} y^j dy^{\bar{k}} + x^{\bar{i}}_{\bar{k}j} y^{\bar{k}} dy^j + x^{\bar{i}}_{\bar{k}} dz^{\bar{k}}.$$

The quantities $\{dx^k, dy^{\bar{k}}, dy^k, dz^{\bar{k}}\}$ define the co-basis of the co-tangent space $\overset{*}{T}T(T(Vn))$.

It is obvious that the space $\overset{*}{T}T(T(Vn))$ always has an invariant subspace spanned over the co-basis $\{dx^k\}$. Let us consider the case when $\overset{*}{T}T(T(Vn))$ is a fully equipped space. Then the matrix of

2020 *Mathematics Subject Classification.* 53B05.

Key words and phrases. Tangent fiber space; Manifold tangent fiber space; I-forms; Linear connections.

reference frame transformation has the form

$$\begin{Vmatrix} x^i_k & 0 & 0 & 0 \\ 0 & x^{\overline{i}}_{\overline{k}} & 0 & 0 \\ 0 & 0 & x^i_k & 0 \\ 0 & 0 & 0 & x^{\overline{i}}_{\overline{k}} \end{Vmatrix}.$$

The object of linear triangular co-connection always generates an object of linear triangular connection. In our case in which we deal with the full equipment of the tangent or co-tangent space, the linear connection does not generate the linear connection. However, between them there all the same exists a certain connection and that is why we separately consider the full equipment of the tangent and co-tangent spaces.

For the co-tangent space $\overset{*}{T}T(T(Vn))$ to be fully equipped, it is necessary that all of its subspaces be invariant.

The space $\overset{*}{T}T(T(Vn))$ always has an invariant subspace spanned over the co-basis $\{dx^k\}$, while other subspaces can be defined by means of the co-bases $Dy^{\overline{i}}$, Dy^i, $Dz^{\overline{i}}$:

$$Dy^{\overline{i}} = dy^{\overline{i}} + G^{\overline{i}}_j dx^j,$$
$$Dy^i = dy^i + \Gamma^i_j dx^j,$$
$$Dz^{\overline{i}} = dz^{\overline{i}} + L^{\overline{i}}_k Dy^k + G^{\overline{i}}_k dy^k + G^{\overline{i}}_{\overline{k}} dy^{\overline{k}}.$$

From the invariance condition it follows that by the change of coordinates, the quantities $G^{\overline{i}}_j$, Γ^i_j, $L^{\overline{i}}_k$, $G^{\overline{i}}_{\overline{k}}$ generate a differential-geometric object of co-connection in compliance with the following law of transformation of its components, respectively:

$$x^{\overline{i}}_{\overline{k}} G^{\overline{k}}_j = x^{\overline{k}}_j \overline{G}^{\overline{i}}_{\overline{k}} + x^{\overline{i}}_{\overline{k}j} y^{\overline{k}}, \tag{1}$$
$$x^p_i \Gamma^i_k = x^i_k \overline{\Gamma}^p_i + x^p_{ki} y^i, \tag{2}$$
$$x^{\overline{i}}_{\overline{j}} L^{\overline{j}}_k = x^j_k \overline{L}^{\overline{i}}_j + \overline{G}^{\overline{i}}_{\overline{j}} x^{\overline{j}}_{\overline{p}k} y^{\overline{p}} + \overline{G}^{\overline{i}}_p x^p_{jk} y^j + x^{\overline{i}}_{\overline{j}k} z^{\overline{j}} + x^{\overline{i}}_{\overline{j}pk} y^{\overline{j}} y^p, \tag{3}$$
$$x^{\overline{i}}_{\overline{j}} G^{\overline{j}}_{\overline{k}} = x^{\overline{j}}_{\overline{k}} \overline{G}^{\overline{i}}_{\overline{j}} + x^{\overline{i}}_{\overline{k}j} y^j. \tag{4}$$

The full equipment of the space $TT(T(Vn))$ can be defined by using the vectors F_i, $D_{\overline{i}}$, D_k:

$$F_i = \frac{\partial}{\partial y^i} - Q^{\overline{k}}_i \frac{\partial}{\partial z^{\overline{k}}},$$
$$D_{\overline{i}} = \frac{\partial}{\partial y^{\overline{i}}} - E^{\overline{k}}_{\overline{i}} \frac{\partial}{\partial z^{\overline{k}}},$$
$$D_k = \frac{\partial}{\partial x^k} - C^{\overline{i}}_k \frac{\partial}{\partial z^{\overline{i}}} - Q^{\overline{i}}_k \frac{\partial}{\partial y^{\overline{i}}} - E^i_k \frac{\partial}{\partial y^i}.$$

From the invariance condition it follows that the quantities $Q^{\overline{k}}_i$, $E^{\overline{k}}_{\overline{i}}$, $C^{\overline{i}}_k, E^i_k$ form an object of connection of the space $T(T(Vn))$. The coordinates are changed in compliance with the following law of transformation of their components:

$$x^{\overline{i}}_{\overline{k}} Q^{\overline{k}}_j = x^{\overline{k}}_j \overline{Q}^{\overline{i}}_{\overline{k}} + x^{\overline{i}}_{\overline{k}j} y^{\overline{k}}, \tag{5}$$
$$x^p_i E^i_k = x^i_k \overline{E}^p_i + x^p_{ki} y^i, \tag{6}$$
$$x^i_k C^{\overline{j}}_i = x^{\overline{j}}_{\overline{i}} \overline{C}^{\overline{i}}_k + \overline{Q}^{\overline{p}}_k x^{\overline{j}}_{\overline{p}i} y^i + \overline{E}^i_k x^{\overline{j}}_{\overline{p}i} y^{\overline{p}} - x^{\overline{j}}_{\overline{i}k} z^{\overline{i}} - x^{\overline{j}}_{\overline{i}pk} y^{\overline{i}} y^p, \tag{7}$$
$$x^{\overline{i}}_{\overline{j}} E^{\overline{j}}_{\overline{k}} = x^{\overline{j}}_{\overline{k}} \overline{E}^{\overline{i}}_{\overline{j}} + x^{\overline{i}}_{\overline{k}j} y^j. \tag{8}$$

Formulas (1)–(4), and (5)–(8) show that the quantities $G^{\overline{k}}_j$ and $Q^{\overline{k}}_j$, $G^{\overline{j}}_{\overline{k}}$, and $E^{\overline{j}}_{\overline{k}}$, Γ^i_k and E^i_k generate one and the same object of connection. From the transformation laws (3) and (7) we see that the linear connection $L_k^{\bar{j}}$ and the linear connection $C_k^{\bar{j}}$ do not generate one and the same object of connection and note that between them there exists a certain connection which will be defined below.

Here there arises the same question as in the case of the partial equipment of the space $TT(T(Vn))$: whether from the triplet connection and its differential continuation it is possible to construct an object connection of the space $TT(T(Vn))$? It turns out that like in the case of partial equipment the answer is also positive. Let us introduce the following notations:

$$\begin{gathered} \check{G}_i^{\bar{k}} \equiv \Gamma_i^k, \quad \check{G}_{\bar{i}}^{\bar{k}} \equiv \Gamma_{ip}^k y^p, \quad \check{\Gamma}_{\bar{j}}^i \equiv \Gamma_{jk}^i y^k, \\ \check{L}_i^{\bar{k}} \equiv \Gamma_{i\bar{p}}^{\bar{k}} z^{\bar{p}} + \overset{\Gamma}{\partial}_i \, \Gamma_p^{\bar{k}} \, y^p + \Gamma_{i\bar{q}}^{\bar{k}} \Gamma_p^{\bar{q}} \, y^p + \Gamma_i^{\bar{q}} \Gamma_{j\bar{q}}^{\bar{k}} y^j, \\ \check{C}_i^{\bar{k}} \equiv \Gamma_{i\bar{p}}^{\bar{k}} z^{\bar{p}} + \overset{\Gamma}{\partial}_i \, \Gamma_p^{\bar{k}} \, y^p + \Gamma_{i\bar{q}}^{\bar{k}} \Gamma_p^{\bar{q}} \, y^p + \Gamma_j^{\bar{k}} \Gamma_{pi}^j y^p, \end{gathered} \tag{9}$$

where the quantities $\Gamma_j^{\bar{k}}$ are functions only of x^i and $y^{\bar{i}}$, i.e. $\Gamma_j^{\bar{k}} \equiv \Gamma_j^{\bar{k}}(x^i, y^{\bar{i}})$, while for other quantities, it is a assumed that

$$\begin{gathered} \check{G}_{\bar{i}}^{\bar{k}} \equiv \check{G}_{\bar{i}}^{\bar{k}}(x^i, y^{\bar{i}}, y^i), \quad \check{\Gamma}_{\bar{j}}^i \equiv \check{\Gamma}_{\bar{j}}^i(x^i, y^{\bar{i}}, y^i), \\ \check{L}_i^{\bar{k}} \equiv \check{L}_i^{\bar{k}}(x^i, y^{\bar{i}}, y^i, z^{\bar{i}}), \quad \check{C}_i^{\bar{k}} \equiv \check{C}_i^{\bar{k}}(x^i, y^{\bar{i}}, y^i, z^{\bar{i}}). \end{gathered}$$

By the change of coordinates, the quantities $\check{G}_{\bar{i}}^{\bar{k}}$, $\check{G}_i^{\bar{k}}$, $\check{\Gamma}_{\bar{j}}^i$, $\check{L}_i^{\bar{k}}$, $\check{C}_i^{\bar{k}}$ according to the same law, are transformed to the quantities $G_{\bar{k}}^{\bar{j}}$, $G_k^{\bar{j}}$, Γ_j^i, $L_i^{\bar{k}}$, $C_i^{\bar{k}}$, respectively.

Hence the following theorem follows.

Theorem. *The object of triplet connection and the differential continuation of the object* $\Gamma_p^{\bar{k}} : \big(\overset{\Gamma}{\partial}_i \, \Gamma_p^{\bar{k}}\big)$ *always generate the full equipment of the manifold tangent fiber space* $TT(T(Vn))$.

From formulas (9) we see that the objects of connections $L_k^{\bar{j}}$ and $C_k^{\bar{j}}$ are related by the equality

$$L_k^{\bar{j}} = C_k^{\bar{j}} + \Gamma_k^{\bar{i}} G_{\bar{i}}^{\bar{j}} + \Gamma_i^{\bar{j}} \Gamma_k^i,$$

which was our aim to establish (see also [1–7] for related topics).

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(Received 29.05.2023)

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