Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 155–159

ON THE WEIGHTED RELLICH–SOBOLEV AND HARDY–SOBOLEV INEQUALITIES IN VARIABLE EXPONENT LEBESGUE SPACES

DAVID EDMUNDS[1] AND ALEXANDER MESKHI[2,3]

Abstract. In this note, we present the weighted Rellich–Sobolev and Hardy–Sobolev inequalities in variable exponent Lebesgue spaces $L^{p(\cdot)}$ defined on homogeneous stratified groups $\mathbb{G}$. The results are new even for the Abelian (Euclidean) case and for the Heisenberg groups.

1. Introduction

Rellich's classical inequality states that if $u \in C_0^\infty\left(\mathbb{R}^d \backslash\{0\}\right)$ and $d \neq 2$, then

$$\frac{d^2(d-4)^2}{16} \int\limits_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^4} dx \leq \int\limits_{\mathbb{R}^d} |\Delta u(x)|^2 dx,$$

and the constant $d^2(d-4)^2/16$ is sharp. This inequality was announced by Rellich in 1954 (see [19]). When $d=2$, the inequality still holds but only for a restricted class of functions (see [1]).

For further progress regarding Rellich-type inequalities in the classical Lebesgue spaces we refer, e.g., to [1, 6, 9] and references cited therein.

We present Rellich–type inequalities in variable exponent Lebesgue spaces ($VELS$ briefly) $L^{p(\cdot)}$ in the higher dimensional case. We studied the problem in homogeneous groups $\mathbb{G}$, but the results are new even for the Abelian (Euclidean) case $\mathbb{G}=\left(\mathbb{R}^d,+\right)$ and for the Heisenberg groups $\mathbb{G}=\mathbb{H}^n$. Rellich inequalities in $L^{p(\cdot)}(I)$ spaces, where I is an interval, were studied in [10] (see also [11] for two–weighted Rellich type inequalities in these spaces). The results are obtained under the condition that the Hardy–Littlewood maximal operator is bounded in appropriate unweighed $VELS$, which, for example, is guaranteed if the variable exponent satisfies a log-Hölder continuity condition and a decay condition at infinity. We are also interested in Hardy-type estimates in the variable exponent setting. Similar results were derived in [13, 18] under different conditions on variable exponents.

A Lie group (on $\mathbb{R}^d$) $\mathbb{G}$ is said to be homogeneous if there is a dilation $D_\lambda(x)$ such that

$$D_\lambda(x):=\left(\lambda^{\nu_1} x_1, \ldots, \lambda^{\nu_d} x_d\right), \quad \nu_1, \ldots, \nu_d>0, \quad D_\lambda: \mathbb{R}^d \rightarrow \mathbb{R}^d,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda>0$. In the sequel, we use the notation λx for the dilation $D_\lambda(x)$. The number $Q:=\nu_1+\cdots+\nu_d$ is called the homogeneous dimension of $\mathbb{G}$. A homogeneous quasi–norm on $\mathbb{G}$ is a continuous non-negative function $r: \mathbb{G} \mapsto[0, \infty)$ such that

i) $r(x)=r\left(x^{-1}\right)$ for all $x \in \mathbb{G}$,
ii) $r(\lambda x)=\lambda r(x)$ for all $x \in \mathbb{G}$ and $\lambda>0$,
iii) $r(x)=0$ if and only $x=0$.

The quasi-ball centred at $x \in \mathbb{G}$ with radius $R>0$ is defined by

$$B(x, R):=\left\{y \in \mathbb{G}: r\left(x^{-1} y\right)<R\right\}.$$

A homogeneous group is necessarily nilpotent and the Haar measure on $\mathbb{G}$ coincides with the Lebesgue measure; we denote it by dx. If $|E|$ denotes the measure of a measurable set $E \subset \mathbb{G}$, then

$$\left|D_\lambda(E)\right|=\lambda^Q|E| \quad \text{and} \quad \int\limits_{\mathbb{G}} f(\lambda x) dx=\lambda^{-Q} \int\limits_{\mathbb{G}} f(x) dx.$$

2020 *Mathematics Subject Classification.* 22E30, 42B25, 43A80, 47B38.

Key words and phrases. Rellich inequality; Hardy inequality; Weighted Sobolev inequality; Variable exponent Lebesgue spaces; Homogeneous groups.

Hence, we have that the Haar measure of the quasi–ball has the following property: there is a constant $A \geq 1$ such that

$$A^{-1}R^Q \leq |B(x,R)| \leq AR^Q.$$

A homogeneous group with the quasi-norm $r(\cdot)$ and Haar measure dx is an example of a quasi-metric measure space with a doubling measure, which is also called a space of homogeneous type (SHT briefly).

Let us now recall the definition of a homogeneous stratified group (or homogeneous Carnot group). These form an important class of homogeneous groups. We refer, e.g., to [2, 12, 20].

Definition 1.1. A Lie group $\mathbb{G} = \big(\mathbb{R}^d, \circ\big)$ is called a homogeneous stratified group if the following conditions hold:

(a) the decomposition $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_s}$ is valid for some natural numbers $d_1, \dots, d_s$ with $d_1 + \cdots + d_s = d$; the dilation $\delta_\lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ given by

$$\delta_\lambda(x) \equiv \delta_\lambda\big(x^{(1)}, \dots, x^{(s)}\big) := \big(\lambda x^{(1)}, \dots, \lambda^{(s)} x^{(r)}\big), \quad x^{(k)} \in \mathbb{R}^{d_k}, \quad k = 1, \dots, s,$$

is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$.

(b) If d_1 is as in (a) and $X_1, \dots, X_{d_1}$ are the left-invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}\Big|_0$ for $k = 1, \dots, d_1$, then

$$\operatorname{rank}\big(\operatorname{Lie}\{X_1, \dots, X_{d_1}\}\big) = d,$$

for every $x \in \mathbb{R}^d$. In other words, the iterated commutators of $X_1, \dots, X_{d_1}$ span the Lie algebra of $\mathbb{G}$.

In the sequel, by the symbol

$$\nabla_{\mathbb{G}} := (X_1, \dots, X_{d_1})$$

we denote the horizontal gradient on $\mathbb{G}$. Hence, the sub-Laplacian on (homogeneous) stratified groups is determined by the formula

$$\Delta_{\mathbb{G}} := \nabla_{\mathbb{G}} \cdot \nabla_{\mathbb{G}}.$$

We will assume that $\mathbb{G}$ is a stratified homogeneous group.

From the beginning of this century, the non-standard function spaces such as variable exponent function spaces, attracted a considerable interest of researchers. The main reason for that was to solve a number of contemporary problems arising naturally in non-linear theory of elasticity, fluid mechanics, image restoration, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations, etc. The $VELS$ (called also Nakano space) $L^{p(\cdot)}$ appeared first in a paper by Orlicz written in the 1930s. We refer to the monographs [3, 8, 15, 16] and the survey [14] for the recent results and progress regarding differential and integral operators in variable exponent function spaces.

For a positive continuous exponent function $p(\cdot)$ defined on a Haar-measurable set $E \subset \mathbb{G}$, we set

$$p_-(E) := \inf_E p(\cdot), \qquad p_+(E) := \sup_E p(\cdot).$$

We write p_- and p_+ for $p_-(E)$ and $p_+(E)$, respectively, if $E := \mathbb{G}$. Let $\mathcal{P}(E)$ be the class of continuous exponents $p(\cdot)$ defined on E, $E \subset \mathbb{G}$ such that

$$1 < p_-(E) \leq p_+(E) < \infty.$$

Let $p(\cdot) \in \mathcal{P}(E)$. A $VELS$, denoted by $L^{p(\cdot)}(E)$, is the linear space of all Haar-measurable functions f on E for which

$$S_{p(\cdot)}(E, f) := \int\limits_E |f(x)|^{p(x)} dx < \infty.$$

By the symbol $\int\limits_E g(x)dx$ we mean $\int\limits_{\mathbb{G}} g(x)\chi_E(x)dx$.

The norm in $L^{p(\cdot)}(E)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\big\{\lambda > 0 : \ S_{p(\cdot)}(E, f/\lambda) \leq 1\big\}.$$

If $p(\cdot) \equiv p_c \equiv \text{const}$, then $L^{p(\cdot)}(E)$ coincides with the classical Lebesgue space $L^{p_c}(E)$.

It is known (see, e.g., [17]) that $L^{p(\cdot)}(E)$ is a Banach space.

Definition 1.2. Let E be a measurable subset of $\mathbb{G}$. We denote by $\mathcal{P}_0^{\log}(E)$ the class of all positive continuous exponents $s(\cdot)$ on E satisfying the log–Hölder continuity condition on E, i.e., there is a positive constant L such that for all $x, y \in E$, $0 < r\left(xy^{-1}\right) \le \frac{1}{2}$,

$$|s(x) - s(y)| \le \frac{L}{\ln \frac{1}{r(xy^{-1})}}.$$

Definition 1.3. We say that an exponent function $s(\cdot) \in \mathcal{P}(\mathbb{G})$ satisfies the decay condition $\big(s(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{G})\big)$ if there exists the limit $s(\infty) = \lim\limits_{r(x)\to\infty} s(x)$ and a positive constant L_∞ such that

$$|s(x) - s(\infty)| \le \frac{L_\infty}{\ln(e + r(x))}, \quad x \in \mathbb{G}. \tag{1}$$

Let us call "decay constant" the best possible constant in (1).

We denote $\mathcal{P}^{\log}(E) := \mathcal{P}_0^{\log}(E) \cap \mathcal{P}_\infty^{\log}(\mathbb{G})$.

It is known that the condition $p \in \mathcal{P}(E) \cap \mathcal{P}^{\log}(E)$ (resp., $p \in \mathcal{P}(E) \cap \mathcal{P}_0^{\log}(E)$) guarantees the boundedness of various important operators of Harmonic Analysis in $VELS$ defined on an unbounded set E (resp., bounded set E). The same is true for an SHT with a finite measure (see [3, 7, 8, 15, 16]).

Let $Mf(x)$ be the Hardy–Littlewood maximal function on $\mathbb{G}$ defined by the formula

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int\limits_B |f(y)| dy, \quad f \in L_{\mathrm{loc}}(\mathbb{G}),$$

where the supremum is taken over all balls B containing $x \in \mathbb{G}$.

It is known that the condition $p(\cdot) \in \mathcal{P}^{\mathrm{loc}}(\mathbb{G})$ guarantees the boundedness of the operator M in $L^{p(\cdot)}(\mathbb{G})$ (see [4, 7] for Euclidean spaces, and [5] for an quasi-metric measure space with doubling measure).

2. Main Results

Now, we formulate the main results of this paper. We will use the following notation:

$$\overline{q}(\cdot) := q(\cdot)/q_0, \quad q_0 < q_-, \quad r'(\cdot) = \frac{r(\cdot)}{r(\cdot) - 1}\,.$$

Theorem 2.1 (Weighted Rellich–Sobolev inequality)**.** *Let $\mathbb{G}$ be a stratified homogeneous group with homogeneous dimension $Q > 2$. Suppose that $1 < p_- \le p_+ < \frac{Q}{2}$ and $q(\cdot) = \frac{p(\cdot)Q}{Q - 2p(\cdot)}$. Let $-\frac{Q}{q_-} < \eta < \frac{Q}{(p_-)'}$. If $\overline{q}'(\cdot) \in \mathcal{B}(\mathbb{G})$ for some $q_0 < q_-$, then there is a positive constant C depending on $p(\cdot), \eta, A, Q, \|M\|_{L^{\overline{q}'(\cdot)}(\mathbb{G})}$ such that for all $u \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$,*

$$\Big\| r^\eta(\cdot) u \Big\|_{L^{q(\cdot)}(\mathbb{G})} \le C \Big\| r^\eta(\cdot) \Delta_{\mathbb{G}} u \Big\|_{L^{p(\cdot)}(G)}$$

holds.

Theorem 2.2 (Weighted Rellich Inequality)**.** *Let $\mathbb{G}$ be a stratified homogeneous group with homogeneous dimension $Q > 2$. Suppose that $1 < p_- \le p_+ < \frac{Q}{2}$. Let $\frac{2p_- - Q}{p_-} < \eta < \frac{Q}{(p_-)'}$. Let β be a constant such that $\beta < -2$. If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{G})$, then there is a positive constant C depending on $p(\cdot), \eta, A, Q, \|M\|_{L^{\overline{q}'(\cdot)}(\mathbb{G})}$ such that for all $u \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$,*

$$\Big\| \left(1 + r(\cdot)\right)^\beta r^\eta(\cdot) u(\cdot) \Big\|_{L^{p(\cdot)}(\mathbb{G})} \le C_{p(\cdot),\eta,A,Q}\left(\|M\|_{L^{\overline{q}}(\mathbb{G})}\right) \|r(\cdot)^\eta \Delta u(\cdot)\|_{L^{p(\cdot)}(\mathbb{G})}$$

holds.

Theorem 2.3 (Weighted Hardy–Sobolev (Weighted Sobolev–Stein embedding))**.** *Let $\mathbb{G}$ be a stratified homogeneous group with homogeneous dimension Q. Let $1 < p_- \le p_+ < Q$ and $q(\cdot) = \frac{p(\cdot)Q}{Q - p(\cdot)}$. Suppose* *that* $-\frac{Q}{q_-} < \eta < \frac{Q}{(p_-)'}$. *If* $\overline{q}'(\cdot) \in \mathcal{B}(\mathbb{G})$ *for some* $q_0 < q_-$, *then there is a positive constant* C *depending on* $p(\cdot), \eta, A, Q, \|M\|_{L^{\overline{q}'(\cdot)}(\mathbb{G})}$ *such that the inequality*

$$\Big\|u(\cdot)r^{\eta}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{G})} \le C\Big\|\nabla_{\mathbb{G}}u(\cdot)r^{\eta}(\cdot)\Big\|_{L^{p(\cdot)}(\mathbb{G})}$$

holds for all $u \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$.

Theorem 2.4 (Weighted Hardy Inequality)**.** *Let* $\mathbb{G}$ *be a stratified homogeneous group with homogeneous dimension* Q. *Let* $1 < p_- \le p_+ < Q$. *Suppose that* $\beta < -1$ *and* $\frac{p_- - Q}{p_-} < \eta < \frac{Q}{(p_-)'}$. *If* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{G})$, *then there is a positive constant* C *depending on* $p(\cdot), \eta, A, Q, \|M\|_{L^{\overline{q}'(\cdot)}(\mathbb{G})}$ *such that*

$$\Big\|u(\cdot)(1+r(\cdot))^{\beta}r^{\eta}(\cdot)\Big\|_{L^{p(\cdot)}(\mathbb{G})} \le C\Big\|\nabla_{\mathbb{G}}u(\cdot)r^{\eta}(\cdot)\Big\|_{L^{p(\cdot)}(\mathbb{G})}$$

holds for all $u \in C_0^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$.

Remark 2.5. Theorems 2.1 and 2.3 remain valid under the condition $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{G})$, since it implies the condition.

Acknowledgement

The second author was supported by Batumi Shota Rustaveli State University Grant (Resolution No. 06-01/04, 31.01.2023).

References

1. A. A. Balinsky, W. D. Evans, R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality.* Universitext. Springer, Cham, 2015.
2. A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
3. D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces.* Foundations and harmonic analysis. Springer Science & Business Media, 2013.
4. D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer, The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), no. 1, 223–238.
5. D. Cruz-Uribe, P. Shukla, The boundedness of fractional maximal operators on variable Lebesgue spaces over spaces of homogeneous type. *Studia Math.* **242** (2018), no. 2, 109–139.
6. E. B. Davies, A. M. Hinz, Explicit constants for Rellich inequalities in $L_p(\Omega)$. *Math. Z.* **227** (1998), no. 3, 511–523.
7. L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.* **7** (2004), no. 2, 245–253.
8. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
9. D. E. Edmunds, W. D. Evans, The Rellich inequality. *Rev. Mat. Complut.* **29** (2016), no. 3, 511–530.
10. D. E. Edmunds, A. Meskhi, On the Rellich inequality in $L^{p(\cdot)}(a,b)$. *Georgian Math. J.* **25** (2018), no. 2, 207–216. DOI:https://doi.org/10.1515/gmj-2018-0024.
11. D. E. Edmunds, A. Meskhi, Two-weighted Hardy operators in $L^{p(\cdot)}$ spaces and applications. *Studia Math.* **249** (2019), no. 2, 143–162. DOI:10.4064/sm180204-20-8.
12. V. Fischer, M. Ruzhansky, *Quantization on Nilpotent Lie Groups.* Progress in Mathematics, 314. Birkhäuser/Springer, [Cham], 2016.
13. P. Harjulehto, P. Hästö, M. Koskenoja, Hardy's inequality in a variable exponent Sobolev space. *Georgian Math. J.* **12** (2005), no. 3, 431–442.
14. M. Izuki, E. Nakai, Y. Sawano, Function spaces with variable exponents an introduction. *Sci. Math. Jpn.* **77** (2014), no. 2, 187–315.
15. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral operators in nonstandard function spaces: Variable exponent Lebesgue and amalgam spaces*, Volume 1, Birkäuser/Springer, Heidelberg, 2016.
16. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-standard Function Spaces.* vol. 2. Variable exponent Hölder, Morrey-Campanato and grand spaces. Operator Theory: Advances and Applications, 249. Birkhäuser/Springer, [Cham], 2016.
17. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.* **41(116)** (1991), no. 4, 592–618.
18. H. Rafeiro, S. Samko, Hardy type inequality in variable Lebesgue spaces. *Ann. Acad. Sci. Fenn. Math.* **34** (2009), no. 1, 279–289.
19. F. Rellich, Halbbeschränkte Differentialoperatoren höherer Ordnung. (German) In: *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam*, vol. III, 243–250, Erven P. Noordhoff N. V., Groningen, 1956.
20. M. Ruzhansky, D. Suragan, Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups. *Adv. Math.* **317** (2017), 799–822.

21. M. Ruzhansky, D. Suragan, *Hardy Inequalities on Homogeneous Groups*. 100 years of Hardy inequalities. Progress in Mathematics, 327. Birkhäuser/Springer, Cham, 2019.
22. J.-O. Strömberg, A. Torchinsky, *Weighted Hardy Spaces*. Lecture Notes in Mathematics, 1381. Springer-Verlag, Berlin, 1989.

(Received 12.06.2023)

[1]Department of Mathematics, University of Sussex, Falmer, Brighton BN1 9QH, U.K.

[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

[3]Kutaisi International University, Youth Avenue, Turn 5/7, 4600 Kutaisi, Georgia

Email address: email:davideedmunds@aol.com

Email address: alexander.meskhi@kiu.edu.ge; alexander.meskhi@tsu.ge