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POTENTIAL METHOD IN THE QUASI-STATIC PROBLEMS OF THE COUPLED LINEAR THEORY OF ELASTIC MATERIALS WITH DOUBLE POROSITY

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this paper, the coupled linear quasi-static theory of elasticity for materials with double porosity is considered in which the concepts of Darcy's law and volume fractions are proposed. The system of general governing equations is expressed in terms of the displacement vector field, the changes of the volume fractions of pores and fissures, and the changes of the fluid pressures in pores and fissures networks. By virtue of Green's identity, the uniqueness theorems of the basic internal and external boundary value problems (BVPs) are proved. The fundamental solution of the system of steady vibration equations in the theory under consideration is constructed. Then, the surface and volume potentials are constructed and their basic properties are given. Some useful singular integral operators are studied. Finally, on the basis of these results the existence theorems for classical solutions of the BVPs are proved by means of the potential method (boundary integral equation method) and the theory of singular integral equations.

1. Introduction

The theory of porous media is an important research area of continuum mechanics. The first quasi-static theory of poroelasticity based on Darcy's law was proposed by Biot [3] in which a coupling effect between fluid pressure and mechanical stress is shown. In this paper, the independent kinematic variables are the displacement vector and the pressure in a pore network.

In the last decades, Biot's classical theory of poroelasticity is developed by using several coupling processes and considerable progress has been made in the study of these coupled effects by many research groups. The basic results and historical information on the poroelasticity and thermoporoelasticity for single-porosity materials can be found in the books by Cheng [6], Coussy [8], Selvadurai and Suvorov [22], Wang [32] (see also references therein).

Moreover, the first quasi-static mathematical model of elastic solids with double porosity, as extensions of Biot's theory, was developed by Wilson and Aifantis [33]. More general models of the theories of elasticity and thermoelasticity for double porosity materials by using Darcy's law have been proposed by several investigators (see, Bai and Roegiers [1], Berryman and Wang [2], Gelet et al. [10], Khalili et al. [14], Khalili and Selvadurai [15], Masters et al. [18], Svanadze [25]). Then these models of double porosity materials are investigated extensively by various researchers. A comprehensive review of the basic results in the theories for double-porosity materials based on the concept of Darcy's law may be found in the books by Straughan [24] and Svanadze [26].

On the other hand, applying the concept of volume fraction, the theory of elasticity for materials with single-porosity is presented by Nunziato and Cowin [9, 21]. The theories of thermoelasticity for materials with single-and double-porosity structures as an extension of the Nunziato–Cowin theory, are developed by Ieşan [11] and Ieşan and Quintanilla [13], respectively. The governing equations of this theory involve the displacement vector field, the volume fraction fields associated with the pores and fissures and also the change of temperature. The important problems of the theories of elasticity and thermoelasticity for materials with a double-porosity structure are investigated by several authors

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and the basic results on this subject of research are given in the books by Ciarletta and Ieşan [7], Ieşan [12].

Meanwhile, as the physical properties of porous materials are coupled in nature, the investigation of these effects of coupling processes is important for modern theories of porous media. For instance, the coupled phenomena for this kind of materials usually play an important role in several applications of engineering, geological and biological porous materials. Extensive references on the coupled effects in porous media can be found in the books by Liu [17] and Stephanson et al. [23].

Recently, the linear models for elastic and thermoelastic single-porosity materials have been proposed by Svanadze [27, 28] in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is considered. The basic BVPs of steady vibrations in the quasi-static case of these models are studied by Bitsadze [4,5] and Mikelashvili [19,20]. The steady vibration problems of the viscoelastic single-porosity materials are considered by Svanadze [29].

More recently, in the papers [30] and [31], this coupled phenomenon is extended to the double-porosity elastic and viscoelastic materials, respectively, and the basic BVPs are investigated by using the potential method.

It is noteworthy that the potential method plays a pivotal role in the investigation of BVPs of mathematical physics and continuum mechanics. An extensive review of works, the historical and bibliographical materials on the potential method can be found in the books by Kupradze et al. [16] and Svanadze [26].

The goal of this work is to prove the uniqueness and existence theorems for classical solutions of the basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of double-porosity materials. This paper is articulated as follows. In Section 2, the governing equations of motion and steady vibrations of the considered theory are presented. In Section 3, the basic BVPs are formulated. In Section 4, on the basis of Green's identity, the uniqueness theorems are proved. Afterwards, in Section 5, the fundamental solution of the system of steady vibrations is constructed and its basic properties are established. In Section 6, the surface (single-layer and double-layer) and volume potentials are defined and their properties are established. Some useful singular integral operators are studied. Finally, in Section 7, the existence theorems for classical solutions of the BVPs of steady vibrations are proved.

2. Governing Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$ and let t denote the time variable, $t \geq 0$. We assume that an isotropic and homogeneous elastic solid body with double porosity structure occupies a region of $\mathbb{R}^3$. This structure of materials means that the skeleton of solid consists of pores on the macro-scale and pores on a much smaller micro-scale (called also fissures). Afterwards, in this section, the functions and vectors that depend on the space variable $\mathbf{x}$ and the time t will be denoted with a hat.

Let $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ be the displacement vector in a solid body, and let $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are the changes of the volume fractions of pores and fissures, respectively; $\hat{p}_1$ and $\hat{p}_2$ are the changes of the fluid pressures in pores and fissures networks, respectively. Moreover, throughout this paper, we employ the usual summation and differentiation conventions: (i) repeated Latin and Greek indices are summed over the ranges $(1, 2, 3)$ and $(1, 2)$, respectively; (ii) the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; (iii) a superposed dot denotes differentiation with respect to t.

Following [30], the governing system of field equations in the coupled linear quasi-static theory of elasticity for materials with double porosity consists of the following four sets of equations:

• *Constitutive equations*

$$\begin{gathered}\hat{t}_{lj} = 2\mu\, \hat{e}_{lj} + \lambda\, \hat{e}_{rr}\delta_{lj} + (b_\alpha\, \hat{\varphi}_\alpha - \beta_\alpha \hat{p}_\alpha)\delta_{lj},\\ \hat{\sigma}_l^{(1)} = a_1\hat{\varphi}_{1,l} + a_3\hat{\varphi}_{2,l}, \qquad \hat{\sigma}_l^{(2)} = a_3\hat{\varphi}_{1,l} + a_2\hat{\varphi}_{2,l},\\ l, j = 1, 2, 3.\end{gathered} \tag{2.1}$$

- *Equilibrium equations*

$$\hat{t}_{lj,j} = -\rho \hat{F}'_l, \qquad \hat{\sigma}^{(1)}_{j,j} + \hat{\xi}^{(1)} = -\rho \hat{s}_1,$$
$$\hat{\sigma}^{(2)}_{j,j} + \hat{\xi}^{(2)} = -\rho \hat{s}_2, \qquad l = 1,2,3. \tag{2.2}$$

- *Darcy's law for double-porosity materials*

$$\hat{\mathbf{v}}^{(1)} = -\frac{k'_1}{\mu'}\nabla \hat{p}_1 - \frac{k'_3}{\mu'}\nabla \hat{p}_2 - \rho_1 \hat{\mathbf{s}}_3,$$
$$\hat{\mathbf{v}}^{(2)} = -\frac{k'_3}{\mu'}\nabla \hat{p}_1 - \frac{k'_2}{\mu'}\nabla \hat{p}_2 - \rho_2 \hat{\mathbf{s}}_4. \tag{2.3}$$

- *Equations of fluid mass conservation*

$$\hat{v}^{(1)}_{j,j} + \dot{\hat{\zeta}}_1 + \beta_1 \dot{\hat{e}}_{rr} + \gamma_0(\hat{p}_1 - \hat{p}_2) = 0,$$
$$\hat{v}^{(2)}_{j,j} + \dot{\hat{\zeta}}_2 + \beta_2 \dot{\hat{e}}_{rr} - \gamma_0(\hat{p}_1 - \hat{p}_2) = 0. \tag{2.4}$$

In these equations, $\hat{t}_{lj}$ is the component of total stress tensor, ρ is the reference mass density, $\rho > 0$, $\hat{\mathbf{F}}' = (\hat{F}'_1, \hat{F}'_2, \hat{F}'_3)$ is the body force per unit mass, $\hat{\sigma}^{(1)}_j, \hat{\xi}^{(1)}, \hat{s}_1$ and $\hat{\sigma}^{(2)}_j$, $\hat{\xi}^{(2)}$, $\hat{s}_2$ are the components of the equilibrated stress, the intrinsic equilibrated body force, the extrinsic equilibrated body force associated with the pore and fissure networks, respectively;

$$\hat{\xi}^{(1)} = -b_1 \hat{e}_{rr} - \alpha_1 \hat{\varphi}_1 - \alpha_3 \hat{\varphi}_2 + m_1 \hat{p}_1 + m_3 \hat{p}_2,$$
$$\hat{\xi}^{(2)} = -b_2 \hat{e}_{rr} - \alpha_3 \hat{\varphi}_1 - \alpha_2 \hat{\varphi}_2 + m_3 \hat{p}_1 + m_2 \hat{p}_2; \tag{2.5}$$

$\hat{e}_{lj}$ is the component of strain tensor given by

$$\hat{e}_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \tag{2.6}$$

λ and μ are the Lamé constants, β_1 and β_2 are the effective stress parameters, δ_{lj} is Kronecker's delta; $\hat{\mathbf{v}}^{(1)} = (\hat{v}^{(1)}_1, \hat{v}^{(1)}_2, \hat{v}^{(1)}_3)$ and $\hat{\mathbf{v}}^{(2)} = (\hat{v}^{(2)}_1, \hat{v}^{(2)}_2, \hat{v}^{(2)}_3)$ are the fluid flux vectors associated with the pore and fissure networks, respectively; γ_0 is the internal transport coefficient corresponding to a fluid transfer rate and respecting the intensity of the flow between pores and fissures, $\gamma_0 > 0$,

$$\hat{\zeta}_1 = \gamma_1 \hat{p}_1 + \gamma_3 \hat{p}_2 + m_1 \hat{\varphi}_1 + m_3 \hat{\varphi}_2,$$
$$\hat{\zeta}_2 = \gamma_3 \hat{p}_1 + \gamma_2 \hat{p}_2 + m_3 \hat{\varphi}_1 + m_2 \hat{\varphi}_2, \tag{2.7}$$

μ' is the fluid viscosity, ρ_1, $\hat{\mathbf{s}}_3$ and ρ_2, $\hat{\mathbf{s}}_4$ are the density of fluid and the external force (such as gravity) for the pore phase, respectively; ∇ is the gradient operator; the values b_l, m_j, a_j, α_j, γ_j, k'_j ($l = 1, 2$, $j = 1, 2, 3$) are the constitutive coefficients.

Substituting equations (2.1), (2.3) and (2.5)–(2.7) into (2.2) and (2.4), we obtain the following system of equations in the coupled linear quasi-static theory of elastic double-porosity materials expressed in terms of the displacement vector $\hat{\mathbf{u}}$, the changes of the volume fractions $\hat{\varphi}_1, \hat{\varphi}_2$ and the changes of the fluid pressures $\hat{p}_1, \hat{p}_2$:

$$\begin{gathered}
\mu \Delta \hat{\mathbf{u}} + (\lambda + \mu)\nabla \operatorname{div} \hat{\mathbf{u}} + b_\alpha \nabla \hat{\varphi}_\alpha - \beta_\alpha \nabla \hat{p}_\alpha = -\rho \hat{\mathbf{F}}', \\
(a_1 \Delta - \alpha_1)\hat{\varphi}_1 + (a_3 \Delta - \alpha_3)\hat{\varphi}_2 - b_1 \operatorname{div} \hat{\mathbf{u}} + m_1 \hat{p}_1 + m_3 \hat{p}_2 = -\rho \hat{s}_1, \\
(a_3 \Delta - \alpha_3)\hat{\varphi}_1 + (a_2 \Delta - \alpha_2)\hat{\varphi}_2 - b_2 \operatorname{div} \hat{\mathbf{u}} + m_3 \hat{p}_1 + m_2 \hat{p}_2 = -\rho \hat{s}_2, \\
(k_1 \Delta - \gamma_0)\hat{p}_1 + (k_3 \Delta + \gamma_0)\hat{p}_2 + \gamma_1 \dot{\hat{p}}_1 + \gamma_3 \dot{\hat{p}}_2 - \beta_1 \operatorname{div} \dot{\hat{\mathbf{u}}} \\
+ m_1 \dot{\hat{\varphi}}_1 + m_3 \dot{\hat{\varphi}}_2 = -\rho_1 \operatorname{div} \hat{\mathbf{s}}_3, \\
(k_3 \Delta + \gamma_0)\hat{p}_1 + (k_2 \Delta - \gamma_0)\hat{p}_2 + \gamma_3 \dot{\hat{p}}_1 + \gamma_2 \dot{\hat{p}}_2 - \beta_2 \operatorname{div} \dot{\hat{\mathbf{u}}} \\
+ m_3 \dot{\hat{\varphi}}_1 + m_2 \dot{\hat{\varphi}}_2 = -\rho_2 \operatorname{div} \hat{\mathbf{s}}_4,
\end{gathered} \tag{2.8}$$

where Δ is the Laplacian operator and $k_l = \frac{k'_l}{\mu'}$ ($l = 1, 2, 3$).

If we assume that $\hat{u}_j$, $\hat{F}'_j$, $\hat{\varphi}_l$, $\hat{p}_l$, $\hat{s}_l$ and $\hat{\mathbf{s}}_{l+2}$ $(l = 1,2,\ j = 1,2,3)$ are postulated to have a harmonic time variation

$$\big\{\hat{u}_j, \hat{F}'_j, \hat{\varphi}_l, \hat{p}_l,\ \hat{s}_l, \hat{\mathbf{s}}_{l+2}\big\}(\mathbf{x},t) = \mathrm{Re}\big[\big\{u_j, F'_j, \varphi_l, p_l, s_l, \mathbf{s}_{l+2}\big\}(\mathbf{x})\, e^{-i\omega t}\big],$$

then from (2.8), in the theory under consideration, we obtain the following system of equations of steady vibrations:

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} + b_\alpha\nabla\varphi_\alpha - \beta_\alpha\nabla p_\alpha = -\rho\mathbf{F}',\\
&(a_1\Delta-\alpha_1)\varphi_1 + (a_3\Delta-\alpha_3)\varphi_2 - b_1\,\mathrm{div}\,\mathbf{u} + m_1p_1 + m_3p_2 = -\rho s_1,\\
&(a_3\Delta-\alpha_3)\varphi_1 + (a_2\Delta-\alpha_2)\varphi_2 - b_2\,\mathrm{div}\,\mathbf{u} + m_3p_1 + m_2p_2 = -\rho s_2,\\
&(k_1\Delta+\gamma'_1)p_1 + (k_3\Delta+\gamma'_3)p_2 + \beta'_1\mathrm{div}\mathbf{u} + m'_1\varphi_1 + m'_3\varphi_2 = -\rho_1\mathrm{div}\,\mathbf{s}_3,\\
&(k_3\Delta+\gamma'_3)p_1 + (k_2\Delta+\gamma'_2)p_2 + \beta'_2\mathrm{div}\mathbf{u} + m'_3\varphi_1 + m'_2\varphi_2 = -\rho_2\mathrm{div}\,\mathbf{s}_4,
\end{aligned} \tag{2.9}$$

where $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{F}' = (F'_1,F'_2,F'_3)$, ω is the oscillation frequency, $\omega>0$, $\beta'_l = i\omega\beta_l$, $m'_j = i\omega m_j$, $\gamma'_l = i\omega\gamma_l - \gamma_0$, $\gamma'_3 = i\omega\gamma_3 + \gamma_0$ $(l = 1,2,\ j = 1,2,3)$.

For our further considerations, we will need the following second order matrix differential operator with the constant coefficients:

$$\begin{gathered}
\mathbf{M}(\mathbf{D_x}) = (M_{lj}(\mathbf{D_x}))_{7\times 7}, \qquad M_{lj} = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l\partial x_j},\\
M_{l;r+3} = -M_{r+3;l} = b_r\frac{\partial}{\partial x_l}, \qquad M_{l;r+5} = -\beta_r\frac{\partial}{\partial x_l}, \qquad M_{44} = a_1\Delta - \alpha_1,\\
M_{45} = M_{54} = a_3\Delta-\alpha_3, \quad M_{55} = a_2\Delta - \alpha_2, \qquad M_{46} = m_1,\\
M_{47} = M_{56} = m_3, \quad M_{57} = m_2, \qquad M_{r+5;l} = \beta'_r\frac{\partial}{\partial x_l}, \qquad M_{64} = m'_1,\\
M_{65} = M_{74} = m'_3, \qquad M_{75} = m'_2, \qquad M_{66} = k_1\Delta + \gamma'_1,\\
M_{67} = M_{76} = k_3\Delta + \gamma'_3, \qquad M_{77} = k_2\Delta + \gamma'_2,\\
\mathbf{D_x} = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big), \qquad l,j = 1,2,3, \qquad r = 1,2.
\end{gathered}$$

It is easily seen that system (2.9) can be rewritten in the following form:

$$\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{2.10}$$

where $\mathbf{U} = (\mathbf{u},\varphi_1,\varphi_2,p_1,p_2)$ and $\mathbf{F} = (-\rho\mathbf{F}', -\rho s_1, -\rho s_2, -\rho_1\mathrm{div}\,\mathbf{s}_3, -\rho_2\mathrm{div}\,\mathbf{s}_4)$ are the seven-component vector functions, $\mathbf{x}\in\mathbb{R}^3$.

In what follows, we assume that the following inequalities:

$$\begin{gathered}
\mu>0, \qquad 3\lambda+2\mu>0, \qquad a_1>0, \qquad a_1a_2 - a_3^2>0, \qquad (3\lambda+2\mu)\alpha_1 > 3b_1^2,\\
\alpha_1\alpha_2 - \alpha_3^2 > 0, \qquad \gamma_1>0, \qquad \gamma_1\gamma_2 - \gamma_3^2>0, \qquad k_1>0, \qquad k_1k_2 - k_3^2 > 0,\\
\frac{1}{3}(3\lambda+2\mu)(\alpha_1\alpha_2-\alpha_3^2) > \alpha_1b_2^2 - 2\alpha_3b_1b_2 + \alpha_2b_1^2
\end{gathered} \tag{2.11}$$

are fulfilled.

3. Boundary Value Problems

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S\in C^{1,\nu}$, $0<\nu\le 1$, $\overline{\Omega^+} = \Omega^+\cup S$, $\Omega^- = \mathbb{R}^3\backslash\overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^-\cup S$; $\mathbf{n}(\mathbf{z})$ is the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$.

Definition 1. Vector function $\mathbf{U} = (U_1,U_2,\dots,U_7)$ is called *regular* in Ω^- (or in Ω^+) if

(i)

$$U_l\in C^2(\Omega^-)\cap C^1(\overline{\Omega^-}) \qquad (\text{or } \ U_l\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+}));$$

(ii)

$$U_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad U_{l,j}(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \tag{3.1}$$

for $|\mathbf{x}| \gg 1$, where $l = 1, 2, \ldots, 7$ and $j = 1, 2, 3$.

In the sequel, we use the matrix differential operator

$$\mathbf{R}(\mathbf{D_x}, \mathbf{n}) = (R_{lj}(\mathbf{D_x}, \mathbf{n}))_{7\times 7},$$

where

$$\begin{gathered}
R_{lj}(\mathbf{D_x}, \mathbf{n}) = \mu\delta_{lj}\frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}, \qquad R_{lr}(\mathbf{D_x}, \mathbf{n}) = b_{r-3}\, n_l, \\
R_{l;r+2}(\mathbf{D_x}, \mathbf{n}) = -\beta_{r-3}\, n_l, \qquad R_{44}(\mathbf{D_x}, \mathbf{n}) = a_1 \frac{\partial}{\partial \mathbf{n}}, \\
R_{45}(\mathbf{D_x}, \mathbf{n}) = R_{54}(\mathbf{D_x}, \mathbf{n}) = a_3 \frac{\partial}{\partial \mathbf{n}}, \qquad R_{55}(\mathbf{D_x}, \mathbf{n}) = a_2 \frac{\partial}{\partial \mathbf{n}}, \\
R_{66}(\mathbf{D_x}, \mathbf{n}) = k_1 \frac{\partial}{\partial \mathbf{n}}, \qquad R_{67}(\mathbf{D_x}, \mathbf{n}) = R_{76}(\mathbf{D_x}, \mathbf{n}) = k_3 \frac{\partial}{\partial \mathbf{n}}, \\
R_{77}(\mathbf{D_x}, \mathbf{n}) = k_2 \frac{\partial}{\partial \mathbf{n}}, \qquad R_{sj}(\mathbf{D_x}, \mathbf{n}) = R_{r;m+2}(\mathbf{D_x}, \mathbf{n}) = R_{r+2;m}(\mathbf{D_x}, \mathbf{n}) = 0, \\
l, j = 1, 2, 3, \qquad r, m = 4, 5 \qquad s = 4, 5, 6, 7
\end{gathered} \tag{3.2}$$

and $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs in the coupled linear quasi-static theory of elasticity for materials with double porosity are formulated as follows.

Find a regular (classical) solution to system (2.10) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{3.3}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{3.4}$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are the prescribed seven-component vector functions.

Find a regular (classical) solution to system (2.10) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{3.5}$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{3.6}$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed seven-component vector functions and $\operatorname{supp}\mathbf{F}$ is a finite doma in in Ω^-.

Our goal is to prove the existence and uniqueness of classical solutions of the basic BVPs of steady vibrations $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ by using the potential method. Indeed, to prove the uniqueness theorems of classical solutions, we need Green's first identity. Moreover, the proof of the existence theorems requires the fundamental solution of the system (2.9) and the basic properties of the surface and volume potentials. In view of these results, we are able to reduce the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ to the equivalent singular integral equations for which Fredholm's theorems will be valid.

4. Uniqueness Theorems

In this section, Green's first identity of the coupled linear quasi-static theory of elasticity for materials with double porosity is obtained and the uniqueness theorems for the regular (classical) solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ are proved.

In what follows, the scalar product of two vectors $\mathbf{U} = (U_1, U_2, \ldots, U_7)$ and $\mathbf{U}' = (U'_1, U'_2, \ldots, U'_7)$ is denoted by $\mathbf{U} \cdot \mathbf{U}' = \sum_{j=1}^{7} U_j \overline{U'_j}$, where $\overline{U'_j}$ is the complex conjugate of U'_j.

In the sequel, we use the matrix differential operators:

1)

$$\begin{aligned}
&\mathbf{M}^{(0)}(\mathbf{D_x}) = \big(M_{lj}^{(0)}(\mathbf{D_x})\big)_{3\times 3}, && M_{lj}^{(0)}(\mathbf{D_x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},\\
&\mathbf{M}^{(1)}(\mathbf{D_x}) = \big(M_{lr}^{(1)}(\mathbf{D_x})\big)_{3\times 7}, && M_{lr}^{(1)}(\mathbf{D_x}) = M_{lr}(\mathbf{D_x}),\\
&\mathbf{M}^{(m)}(\mathbf{D_x}) = \big(M_{1r}^{(m)}(\mathbf{D_x})\big)_{1\times 7}, && M_{1r}^{(m)}(\mathbf{D_x}) = M_{m+2;r}(\mathbf{D_x}),\\
&\mathbf{M}^{(m+2)}(\mathbf{D_x}) = \big(M_{1r}^{(m+2)}(\mathbf{D_x})\big)_{1\times 7}, && M_{1r}^{(m+2)}(\mathbf{D_x}) = M_{m+4;r}(\mathbf{D_x});
\end{aligned}$$

2)

$$\begin{aligned}
&\mathbf{R}^{(0)}(\mathbf{D_x},\mathbf{n}) = \big(R_{lj}^{(0)}(\mathbf{D_x},\mathbf{n})\big)_{3\times 3}, && R_{lj}^{(0)}(\mathbf{D_x},\mathbf{n}) = R_{lj}(\mathbf{D_x},\mathbf{n}),\\
&\mathbf{R}^{(1)}(\mathbf{D_x},\mathbf{n}) = \big(R_{lr}^{(1)}(\mathbf{D_x},\mathbf{n})\big)_{3\times 7}, && R_{lr}^{(1)}(\mathbf{D_x},\mathbf{n}) = R_{lr}(\mathbf{D_x},\mathbf{n}),
\end{aligned}$$

where $l,j=1,2,3$, $m=2,3$ and $r=1,2,\dots,7$.

We introduce the following notation:

$$\begin{aligned}
W^{(0)}(\mathbf{u},\mathbf{u}') =& \frac{1}{3}(3\lambda+2\mu)\,\mathrm{div}\mathbf{u}\,\mathrm{div}\overline{\mathbf{u}'} + \frac{\mu}{2}\sum_{l,j=1;l\neq j}^{3}(u_{l,j}+u_{j,l})(\overline{u'_{l,j}}+\overline{u'_{j,l}})\\
&+\frac{\mu}{3}\sum_{l,j=1}^{3}\Big(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\Big)\Big(\frac{\partial \overline{u'_l}}{\partial x_l}-\frac{\partial \overline{u'_j}}{\partial x_j}\Big),\\
W^{(1)}(\mathbf{U},\mathbf{u}') =& W^{(0)}(\mathbf{u},\mathbf{u}') + (b_\alpha\,\varphi_\alpha - \beta_\alpha p_\alpha)\,\mathrm{div}\overline{\mathbf{u}'},\\
W^{(2)}(\mathbf{U},\varphi_1') =& (a_1\nabla\varphi_1 + a_3\nabla\varphi_2)\cdot\nabla\varphi_1'\\
&+(b_1\,\mathrm{div}\mathbf{u} + \alpha_1\varphi_1 + \alpha_3\varphi_2 - m_1p_1 - m_3p_2)\overline{\varphi_1'},\\
W^{(3)}(\mathbf{U},\varphi_2') =& (a_3\nabla\varphi_1 + a_2\nabla\varphi_2)\cdot\nabla\varphi_2'\\
&+(b_2\,\mathrm{div}\mathbf{u} + \alpha_3\varphi_1 + \alpha_2\varphi_2 - m_3p_1 - m_2p_2)\overline{\varphi_2'},\\
W^{(4)}(\mathbf{U},p_1') =& (k_1\nabla p_1 + k_3\nabla p_2)\cdot\nabla p_1'\\
&-(\beta_1'\,\mathrm{div}\mathbf{u} + m_1'\varphi_1 + m_3'\varphi_2 + \gamma_1'p_1 + \gamma_3'p_2)\overline{p_1'},\\
W^{(5)}(\mathbf{U},p_2') =& (k_3\nabla p_1 + k_2\nabla p_2)\cdot\nabla p_2'\\
&-(\beta_2'\,\mathrm{div}\mathbf{u} + m_3'\varphi_1 + m_2'\varphi_2 + \gamma_3'p_1 + \gamma_2'p_2)\overline{p_2'}.
\end{aligned}\tag{4.1}$$

The following Lemmas will be useful to study the uniqueness of classical solutions to the BVPs.

Lemma 1. *If* $\mathbf{U}=(\mathbf{u},\varphi_1,\varphi_2,p_1,p_2)$ *is a regular vector in* Ω^+, $u_j',\varphi_1',\varphi_2',p_1',p_2'\in C^1(\Omega^+)\cap C(\overline{\Omega^+})$, $j=1,2,3$, *then*

$$\begin{aligned}
&\int_{\Omega^+}\Big[\mathbf{M}^{(1)}(\mathbf{D_x})\,\mathbf{U}\cdot\mathbf{u}' + W^{(1)}(\mathbf{U},\mathbf{u}')\Big]\,d\mathbf{x} = \int_S \mathbf{R}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\,d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\Big[\mathbf{M}^{(2)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_1'} + W^{(2)}(\mathbf{U},\varphi_1')\Big]\,d\mathbf{x} = \int_S\Big(a_1\frac{\partial\varphi_1}{\partial\mathbf{n}} + a_3\frac{\partial\varphi_2}{\partial\mathbf{n}}\Big)\,\overline{\varphi_1'}d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\Big[\mathbf{M}^{(3)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_2'} + W^{(3)}(\mathbf{U},\varphi_2')\Big]\,d\mathbf{x} = \int_S\Big(a_3\frac{\partial\varphi_1}{\partial\mathbf{n}} + a_2\frac{\partial\varphi_2}{\partial\mathbf{n}}\Big)\,\overline{\varphi_2'}d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\Big[\mathbf{M}^{(4)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_1'} + W^{(4)}(\mathbf{U},p_1')\Big]\,d\mathbf{x} = \int_S\Big(k_1\frac{\partial p_1}{\partial\mathbf{n}} + k_3\frac{\partial p_1}{\partial\mathbf{n}}\Big)\,\overline{p_1'}d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\Big[\mathbf{M}^{(5)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_2'} + W^{(5)}(\mathbf{U},p_2')\Big]\,d\mathbf{x} = \int_S\Big(k_3\frac{\partial p_2}{\partial\mathbf{n}} + k_2\frac{\partial p_2}{\partial\mathbf{n}}\Big)\,\overline{p_2'}d_{\mathbf{z}}S,
\end{aligned}\tag{4.2}$$

where $\mathbf{u}' = (u_1', u_2', u_3')$ *and* $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2')$.

Proof. On the basis of Green's first identity of the classical theory of elasticity (see, e.g., Kupradze et al. [16])

$$\int_{\Omega^+} \Big[\mathbf{M}^{(0)}(\mathbf{D_x})\,\mathbf{u}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(0)}(\mathbf{u},\mathbf{u}')\Big]\,d\mathbf{x} = \int_S \mathbf{R}^{(0)}(\mathbf{D_z},\mathbf{n})\mathbf{u}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,$$

we obtain the first relation of (4.2).

On the other hand, the divergence theorem leads to the following identity

$$\int_{\Omega^+} \Big[\Delta\varphi_l(\mathbf{x})\,\overline{\varphi_j'(\mathbf{x})} + \nabla\varphi_l(\mathbf{x})\cdot\nabla\varphi_j'(\mathbf{x})\Big]\,d\mathbf{x} = \int_S \frac{\partial\varphi_l(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\,\overline{\varphi_j'(\mathbf{z})}\,d_{\mathbf{z}}S. \tag{4.3}$$

Now, in view of the relations (4.1), from (4.3), we can derive the last four relations of (4.2). □

Lemma 1 and the condition at infinity (3.1) lead to the following result.

Lemma 2. *If* $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$ *and* $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2')$ *are regular vectors in* Ω^-*, then*

$$\begin{aligned}
&\int_{\Omega^-} \Big[\mathbf{M}^{(1)}(\mathbf{D_x})\,\mathbf{U}\cdot\mathbf{u}' + W^{(1)}(\mathbf{U},\mathbf{u}')\Big]\,d\mathbf{x} = -\int_S \mathbf{R}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\,d_{\mathbf{z}}S,\\
&\int_{\Omega^-} \Big[\mathbf{M}^{(2)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_1'} + W^{(2)}(\mathbf{U},\varphi_1')\Big]\,d\mathbf{x} = -\int_S \Big(a_1\frac{\partial\varphi_1}{\partial\mathbf{n}} + a_3\frac{\partial\varphi_2}{\partial\mathbf{n}}\Big)\,\overline{\varphi_1'}d_{\mathbf{z}}S,\\
&\int_{\Omega^-} \Big[\mathbf{M}^{(3)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_2'} + W^{(3)}(\mathbf{U},\varphi_2')\Big]\,d\mathbf{x} = -\int_S \Big(a_3\frac{\partial\varphi_1}{\partial\mathbf{n}} + a_2\frac{\partial\varphi_2}{\partial\mathbf{n}}\Big)\,\overline{\varphi_2'}d_{\mathbf{z}}S,\\
&\int_{\Omega^-} \Big[\mathbf{M}^{(4)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_1'} + W^{(4)}(\mathbf{U},p_1')\Big]\,d\mathbf{x} = -\int_S \Big(k_1\frac{\partial p_1}{\partial\mathbf{n}} + k_3\frac{\partial p_1}{\partial\mathbf{n}}\Big)\,\overline{p_1'}d_{\mathbf{z}}S,\\
&\int_{\Omega^-} \Big[\mathbf{M}^{(5)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_2'} + W^{(5)}(\mathbf{U},p_2')\Big]\,d\mathbf{x} = -\int_S \Big(k_3\frac{\partial p_2}{\partial\mathbf{n}} + k_2\frac{\partial p_2}{\partial\mathbf{n}}\Big)\,\overline{p_2'}d_{\mathbf{z}}S.
\end{aligned} \tag{4.4}$$

Obviously, on the basis of Lemmas 1 and 2 it follows the following consequences.

Theorem 1. *If* $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$ *is a regular vector in* Ω^+, $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2') \in C^1(\Omega^+) \cap C(\overline{\Omega^+})$, *then*

$$\int_{\Omega^+} [\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} = \int_S \mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{4.5}$$

where

$$W(\mathbf{U},\mathbf{U}') = W^{(1)}(\mathbf{U},\mathbf{u}') + W^{(2)}(\mathbf{U},\varphi_1') + W^{(3)}(\mathbf{U},\varphi_2') + W^{(4)}(\mathbf{U},p_1') + W^{(5)}(\mathbf{U},p_2').$$

Theorem 2. *If* $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$ *and* $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2')$ *are regular vectors in* Ω^-*, then*

$$\int_{\Omega^-} [\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} = -\int_S \mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S. \tag{4.6}$$

Formulas (4.5) and (4.6) are Green's first identities in the coupled linear quasi-static theory of elastic double-porosity materials for domains Ω^+ and Ω^-, respectively.

It is easy to verify that (4.1) yields

$$\begin{aligned}
W^{(1)}(\mathbf{U},\mathbf{u}) =& \frac{1}{3}(3\lambda+2\mu)\,|\mathrm{div}\mathbf{u}|^2 + W_0(\mathbf{u}) + (b_\alpha\,\varphi_\alpha - \beta_\alpha p_\alpha)\,\mathrm{div}\overline{\mathbf{u}},\\
W^{(2)}(\mathbf{U},\varphi_1) =& (a_1\nabla\varphi_1 + a_3\nabla\varphi_2)\cdot\nabla\varphi_1\\
&+ (b_1\,\mathrm{div}\mathbf{u} + \alpha_1\varphi_1 + \alpha_3\varphi_2 - m_1p_1 - m_3p_2)\overline{\varphi_1},
\end{aligned}$$

$$
\begin{aligned}
W^{(3)}(\mathbf{U},\varphi_2) =&(a_3\nabla\varphi_1+a_2\nabla\varphi_2)\cdot\nabla\varphi_2\\
&+(b_2\,\mathrm{div}\mathbf{u}+\alpha_3\varphi_1+\alpha_2\varphi_2-m_3p_1-m_2p_2)\overline{\varphi_2},\\
W^{(4)}(\mathbf{U},p_1) =&(k_1\nabla p_1+k_3\nabla p_2)\cdot\nabla p_1\\
&-(\beta_1'\,\mathrm{div}\mathbf{u}+m_1'\varphi_1+m_3'\varphi_2+\gamma_1'p_1+\gamma_3'p_2)\overline{p_1},\\
W^{(5)}(\mathbf{U},p_2) =&(k_3\nabla p_1+k_2\nabla p_2)\cdot\nabla p_2\\
&-(\beta_2'\,\mathrm{div}\mathbf{u}+m_3'\varphi_1+m_2'\varphi_2+\gamma_3'p_1+\gamma_2'p_2)\overline{p_2},
\end{aligned}
\tag{4.7}
$$

where

$$
W_0(\mathbf{u})=\frac{\mu}{2}\sum_{l,j=1;\,l\neq j}^{3}\Big|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\Big|^2+\frac{\mu}{3}\sum_{l,j=1}^{3}\Big|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\Big|^2. \tag{4.8}
$$

We are now in a position to study the uniqueness of regular solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$. We have the following results.

Theorem 3. *The internal BVP $(I)^{+}_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution.*

Proof. Suppose that there are two regular solutions of problem $(I)^{+}_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^{+}_{\mathbf{0},\mathbf{0}}$. Hence $\mathbf{U}$ is a regular solution of the homogeneous equation

$$
\mathbf{M}(\mathbf{D_x})\mathbf{U}(\mathbf{x})=\mathbf{0} \tag{4.9}
$$

for $\mathbf{x}\in\Omega^+$, satisfying the homogeneous boundary condition

$$
\{\mathbf{U}(\mathbf{z})\}^+=\mathbf{0}\qquad\text{for}\quad \mathbf{z}\in S. \tag{4.10}
$$

Clearly, by virtue of (4.9) and (4.10), from (4.4) it follows that

$$
\begin{gathered}
\int\limits_{\Omega^+}W^{(1)}(\mathbf{U},\mathbf{u})d\mathbf{x}=0,\qquad \int\limits_{\Omega^+}W^{(l+1)}(\mathbf{U},\varphi_l)d\mathbf{x}=0,\\
\int\limits_{\Omega^+}W^{(l+1)}(\mathbf{U},p_l)d\mathbf{x}=0,\qquad l=1,2.
\end{gathered}
\tag{4.11}
$$

In view of relations (4.7), we can easily verify that

$$
\begin{aligned}
\mathrm{Re}W^{(1)}(\mathbf{U},\mathbf{u}) =&\frac{1}{3}(3\lambda+2\mu)\,|\mathrm{div}\mathbf{u}|^2+W_0(\mathbf{u})+b_\alpha\mathrm{Re}(\varphi_\alpha\mathrm{div}\overline{\mathbf{u}})-\beta_\alpha\mathrm{Re}(p_\alpha\mathrm{div}\overline{\mathbf{u}}),\\
\mathrm{Im}[W^{(2)}(\mathbf{U},\varphi_1)+W^{(3)}(\mathbf{U},\varphi_2)] =& a_1|\nabla\varphi_1|^2+2a_3\mathrm{Re}(\nabla\varphi_1\cdot\nabla\varphi_2)+a_2|\nabla\varphi_2|^2\\
&+\alpha_1|\varphi_1|^2+2\alpha_3\mathrm{Re}(\varphi_1\overline{\varphi_2})+\alpha_2|\varphi_2|^2+b_\alpha\mathrm{Re}(\varphi_\alpha\mathrm{div}\overline{\mathbf{u}})\\
&-\left[m_1\mathrm{Re}(\varphi_1\overline{p_1})+m_3\mathrm{Re}(\varphi_1\overline{p_2}+\varphi_2\overline{p_1})+m_2\mathrm{Re}(\varphi_2\overline{p_2})\right],\\
\mathrm{Im}[W^{(4)}(\mathbf{U},p_1)+W^{(5)}(\mathbf{U},p_2)] =& -\omega\beta_\alpha\mathrm{Re}(p_\alpha\mathrm{div}\overline{\mathbf{u}})\\
&-\omega\left[m_1\mathrm{Re}(\varphi_1\overline{p_1})+m_3\mathrm{Re}(\varphi_1\overline{p_2}+\varphi_2\overline{p_1})+m_2\mathrm{Re}(\varphi_2\overline{p_2})\right]\\
&-\omega[\gamma_1|p_1|^2+2\gamma_3\mathrm{Re}(p_1\overline{p_2})+\gamma_2|p_2|^2]
\end{aligned}
$$

and, consequently, we can write

$$
\begin{gathered}
\mathrm{Re}W^{(1)}(\mathbf{U},\mathbf{u})+\mathrm{Im}[W^{(2)}(\mathbf{U},\varphi_1)+W^{(3)}(\mathbf{U},\varphi_2)]\\
-\frac{1}{\omega}\left\{\mathrm{Im}[W^{(4)}(\mathbf{U},p_1)+W^{(5)}(\mathbf{U},p_2)]\right\}=W_0(\mathbf{u})\\
+\Big[\frac{1}{3}(3\lambda+2\mu)\,|\mathrm{div}\mathbf{u}|^2+2b_\alpha\mathrm{Re}(\varphi_\alpha\mathrm{div}\overline{\mathbf{u}})+\alpha_1|\varphi_1|^2+2\alpha_3\mathrm{Re}(\varphi_1\overline{\varphi_2})+\alpha_2|\varphi_2|^2\Big]\\
+\left[a_1|\nabla\varphi_1|^2+2a_3\mathrm{Re}(\nabla\varphi_1\cdot\nabla\varphi_2)+a_2|\nabla\varphi_2|^2\right]\\
+[\gamma_1|p_1|^2+2\gamma_3\mathrm{Re}(p_1\overline{p_2})+\gamma_2|p_2|^2].
\end{gathered}
\tag{4.12}
$$

On the basis of the assumption (2.11) and relation (4.8), from (4.12), we have

$$
\begin{gathered}
W_0(\mathbf{u}) \geq 0, \qquad a_1|\nabla\varphi_1|^2 + 2a_3\mathrm{Re}(\nabla\varphi_1\cdot\nabla\varphi_2) + a_2|\nabla\varphi_2|^2 \geq 0,\\
\frac{1}{3}(3\lambda+2\mu)\,|\mathrm{div}\mathbf{u}|^2 + 2b_\alpha\mathrm{Re}(\varphi_\alpha\mathrm{div}\overline{\mathbf{u}}) + \alpha_1|\varphi_1|^2 + 2\alpha_3\mathrm{Re}(\varphi_1\overline{\varphi_2}) + \alpha_2|\varphi_2|^2 \geq 0,\\
\gamma_1|p_1|^2 + 2\gamma_3\mathrm{Re}(p_1\overline{p_2}) + \gamma_2|p_2|^2 \geq 0.
\end{gathered} \tag{4.13}
$$

Obviously, from (4.13) we deduce that

$$
\mathrm{Re}W^{(1)}(\mathbf{U},\mathbf{u}) + \mathrm{Im}[W^{(2)}(\mathbf{U},\varphi_1) + W^{(3)}(\mathbf{U},\varphi_2)] - \frac{1}{\omega}\left\{\mathrm{Im}[W^{(4)}(\mathbf{U},p_1) + W^{(5)}(\mathbf{U},p_2)]\right\} \geq 0. \tag{4.14}
$$

On the other hand, by virtue of (4.14), from (4.11), it follows that

$$
\begin{gathered}
W_0(\mathbf{u}) = 0, \qquad a_1|\nabla\varphi_1|^2 + 2a_3\mathrm{Re}(\nabla\varphi_1\cdot\nabla\varphi_2) + a_2|\nabla\varphi_2|^2 = 0,\\
\frac{1}{3}(3\lambda+2\mu)\,|\mathrm{div}\mathbf{u}|^2 + 2b_\alpha\mathrm{Re}(\varphi_\alpha\mathrm{div}\overline{\mathbf{u}}) + \alpha_1|\varphi_1|^2 + 2\alpha_3\mathrm{Re}(\varphi_1\overline{\varphi_2}) + \alpha_2|\varphi_2|^2 = 0,\\
\gamma_1|p_1|^2 + 2\gamma_3\mathrm{Re}(p_1\overline{p_2}) + \gamma_2|p_2|^2 = 0.
\end{gathered} \tag{4.15}
$$

Now, using the assumption (2.11), from the third and fourth relations of (4.15), we obtain

$$
\mathrm{div}\mathbf{u}(\mathbf{x}) = 0, \qquad \varphi_l(\mathbf{x}) = p_l(\mathbf{x}) = 0, \qquad l = 1,2 \tag{4.16}
$$

for $\mathbf{x}\in\Omega^+$. Combining the first relations of (4.15) and (4.16), we deduce that $\mathbf{u}$ is a rigid displacement vector of the following form:

$$
\mathbf{u}(\mathbf{x}) = \tilde{\mathbf{a}} + \tilde{\mathbf{b}}\times\mathbf{x}, \tag{4.17}
$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors, $\tilde{\mathbf{b}}\times\mathbf{x}$ is the vector product of the vectors $\tilde{\mathbf{b}}$ and $\mathbf{x}$.

Finally, in view of the homogeneous boundary condition (4.10), from (4.17), we get $\mathbf{u}(\mathbf{x})\equiv\mathbf{0}$ for $\mathbf{x}\in\Omega^+$. Thus, $\mathbf{U}(\mathbf{x})\equiv\mathbf{0}$ for $\mathbf{x}\in\Omega^+$, and we have the desired result. □

Quite similarly, the following result is proved.

Theorem 4. *Two regular solutions of the internal BVP $(II)^+_{\mathbf{F},\mathbf{f}}$ may differ only for an additive vector $\mathbf{U} = (\mathbf{u},\varphi_1,\varphi_2,p_1,p_2)$, where φ_l and p_l $(l = 1,2)$ satisfy conditions* (4.16)*, the vector $\mathbf{u}$ is a rigid displacement vector of the form* (4.17) *for $\mathbf{x}\in\Omega^+$, where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors.*

Theorem 5. *The external BVP $(K)^-_{\mathbf{F},\mathbf{f}}$ has one regular solution, where $K = I, II$.*

Proof. Suppose that there are two regular solutions of problem $(K)^-_{\mathbf{F},\mathbf{f}}$, $K = I, II$. Then their difference $\mathbf{U}$ is a regular solution of the external homogeneous BVP $(K)^+_{\mathbf{0},\mathbf{0}}$. Hence $\mathbf{U}$ is a regular solution of the homogeneous equation (4.9) for $\mathbf{x}\in\Omega^-$, satisfying the homogeneous boundary conditions

$$
\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{4.18}
$$

for $K = I$ and

$$
\{\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{4.19}
$$

for $K = II$.

Clearly, by virtue of (4.7), (4.18), (4.19), from (4.4), we obtain

$$
\begin{gathered}
\int\limits_{\Omega^-} W^{(1)}(\mathbf{U},\mathbf{u})d\mathbf{x} = 0, \qquad \int\limits_{\Omega^-} W^{(l+1)}(\mathbf{U},\varphi_l)d\mathbf{x} = 0,\\
\int\limits_{\Omega^-} W^{(l+2)}(\mathbf{U},p_l)d\mathbf{x} = 0, \qquad l = 1,2.
\end{gathered} \tag{4.20}
$$

In a similar manner as in Theorem 3, from (4.20), we obtain the relations

$$
\mathbf{u}(\mathbf{x}) = \tilde{\mathbf{a}} + \tilde{\mathbf{b}}\times\mathbf{x}, \qquad \varphi_l(\mathbf{x}) = p_l(\mathbf{x}) = 0, \qquad l = 1,2 \tag{4.21}
$$

for $\mathbf{x} \in \Omega^-$, where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors. In view of the condition at infinity (3.1), from (4.21), we get $\mathbf{u}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^-$. Thus we have the desired result. □

5. Fundamental Solution

In this section, the fundamental solution of the system of equations (2.9) is constructed explicitly and its basic properties are established.

Definition 2. The fundamental solution of system (2.9) is the matrix $\mathbf{G}(\mathbf{x}) = (G_{lj}(\mathbf{x}))_{7\times 7}$ satisfying the following equation in the class of generalized functions

$$\mathbf{M}(\mathbf{D}_\mathbf{x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{7\times 7}$ is the unit matrix, $\mathbf{x} \in \mathbb{R}^3$.

We now construct the matrix $\mathbf{G}(\mathbf{x})$. Consider the system of nonhomogeneous equations

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\mathbf{u} - b_\alpha\nabla\varphi_\alpha + \beta'_\alpha\nabla p_\alpha = \boldsymbol{\mathcal{F}}',\\
&(a_1\Delta - \alpha_1)\varphi_1 + (a_3\Delta - \alpha_3)\varphi_2 + b_1\mathrm{div}\mathbf{u} + m'_1 p_1 + m'_3 p_2 = \mathcal{F}_4,\\
&(a_3\Delta - \alpha_3)\varphi_1 + (a_2\Delta - \alpha_2)\varphi_2 + b_2\mathrm{div}\mathbf{u} + m'_3 p_1 + m'_2 p_2 = \mathcal{F}_5,\\
&(k_1\Delta + \gamma'_1)p_1 + (k_3\Delta + \gamma'_3)p_2 - \beta_1\mathrm{div}\mathbf{u} + m_1\varphi_1 + m_3\varphi_2 = \mathcal{F}_6,\\
&(k_3\Delta + \gamma'_3)p_1 + (k_2\Delta + \gamma'_2)p_2 - \beta_2\mathrm{div}\mathbf{u} + m_3\varphi_1 + m_2\varphi_2 = \mathcal{F}_7.
\end{aligned} \tag{5.1}$$

where $\mathcal{F}_l$ $(l = 1, 2, \dots, 7)$ are smooth functions on $\mathbb{R}^3$, $\boldsymbol{\mathcal{F}}' = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. Obviously, system (5.1) can be written in the form

$$\mathbf{M}^\top(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x}) = \boldsymbol{\mathcal{F}}(\mathbf{x}), \tag{5.2}$$

where $\mathbf{M}^\top$ is the transpose of matrix $\mathbf{M}$, $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2)$, $\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}', \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7)$.

Applying the operator div to the first equation of (5.1), we obtain the following system:

$$\begin{aligned}
&\mu_0\Delta\mathrm{div}\mathbf{u} - b_\alpha\Delta\varphi_\alpha + \beta_\alpha\Delta p_\alpha = \mathrm{div}\boldsymbol{\mathcal{F}}',\\
&(a_1\Delta - \alpha_1)\varphi_1 + (a_3\Delta - \alpha_3)\varphi_2 + b_1\mathrm{div}\mathbf{u} + m'_1 p_1 + m'_3 p_2 = \mathcal{F}_4,\\
&(a_3\Delta - \alpha_3)\varphi_1 + (a_2\Delta - \alpha_2)\varphi_2 + b_2\mathrm{div}\mathbf{u} + m'_3 p_1 + m'_2 p_2 = \mathcal{F}_5,\\
&(k_1\Delta + \gamma'_1)p_1 + (k_3\Delta + \gamma'_3)p_2 - \beta_1\mathrm{div}\mathbf{u} + m_1\varphi_1 + m_3\varphi_2 = \mathcal{F}_6,\\
&(k_3\Delta + \gamma'_3)p_1 + (k_2\Delta + \gamma'_2)p_2 - \beta_2\mathrm{div}\mathbf{u} + m_3\varphi_1 + m_2\varphi_2 = \mathcal{F}_7,
\end{aligned} \tag{5.3}$$

where $\mu_0 = \lambda + 2\mu$. From (5.3), we have

$$\mathbf{A}(\Delta)\mathbf{V} = \boldsymbol{\Phi}, \tag{5.4}$$

where $\mathbf{V} = (\mathrm{div}\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2) = (V_1, V_2, \dots, V_5)$, $\boldsymbol{\Phi} = (\mathrm{div}\boldsymbol{\mathcal{F}}', \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7) = (\Phi_1, \Phi_2, \dots, \Phi_5)$ and

$$\mathbf{A}(\Delta) = (A_{lj}(\Delta))_{5\times 5} = \begin{pmatrix} \mu_0\Delta & -b_1\Delta & -b_2\Delta & \beta'_1\Delta & \beta'_2\Delta\\ b_1 & a_1\Delta - \alpha_1 & a_3\Delta - \alpha_3 & m'_1 & m'_3\\ b_2 & a_3\Delta - \alpha_3 & a_2\Delta - \alpha_2 & m'_3 & m'_2\\ -\beta_1 & m_1 & m_3 & k_1\Delta + \gamma'_1 & k_3\Delta + \gamma'_3\\ -\beta_2 & m_3 & m_2 & k_3\Delta + \gamma'_3 & k_2\Delta + \gamma'_2 \end{pmatrix}_{5\times 5}.$$

Let us introduce the notation

$$\Lambda_1(\Delta) = \frac{1}{a_0 k_0 \mu_0}\mathrm{det}\mathbf{A}(\Delta) = \Delta\prod_{j=1}^{4}(\Delta + \lambda_j^2), \tag{5.5}$$

where $a_0 = a_1a_2 - a_3^2$; $k_0 = k_1k_2 - k_3^2$; λ_1^2, λ_2^2, λ_3^2 and λ_4^2 are the roots of the following equation with respect to ξ

$$\det\begin{pmatrix} \mu_0 & -b_1 & -b_2 & \beta_1' & \beta_2' \\ b_1 & -a_1\xi-\alpha_1 & -a_3\xi-\alpha_3 & m_1' & m_3' \\ b_2 & -a_3\xi-\alpha_3 & -a_2\xi-\alpha_2 & m_3' & m_2' \\ -\beta_1 & m_1 & m_3 & -k_1\xi+\gamma_1' & -k_3\xi+\gamma_3' \\ -\beta_2 & m_3 & m_2 & -k_3\xi+\gamma_3' & -k_2\xi+\gamma_2' \end{pmatrix}_{5\times 5} = 0.$$

We assume that $\mathrm{Im}\lambda_l > 0$, $\lambda_l \neq \lambda_j$ for $l, j = 1, 2, 3, 4$ and $l \neq j$.

From equation (5.4), we deduce that

$$\Lambda_1(\Delta)\mathrm{div}\mathbf{u} = \Psi_1, \quad \Lambda_1(\Delta)\varphi_l = \Psi_{l+1}, \quad \Lambda_1(\Delta)p_l = \Psi_{l+3}, \quad l = 1, 2, \tag{5.6}$$

where

$$\Psi_m = \frac{1}{a_0k_0\mu_0}\sum_{j=1}^{5} A_{jm}^*\Phi_j, \qquad m = 1, 2, \ldots, 5 \tag{5.7}$$

and A_{jm}^* is the cofactor of the element A_{jm} of matrix $\mathbf{A}$.

Now, applying the operator $\Lambda_1(\Delta)$ to the first equation of system (5.1), by virtue of (5.6), it follows that

$$\Lambda_2(\Delta)\mathbf{u} = \tilde{\mathbf{\Psi}}, \tag{5.8}$$

where $\Lambda_2(\Delta) = \Delta\Lambda_1(\Delta)$ and

$$\tilde{\mathbf{\Psi}} = \frac{1}{\mu}\Lambda_1(\Delta)\boldsymbol{\mathcal{F}}' - \frac{1}{\mu}\nabla[(\lambda+\mu)\Psi_1 - b_\alpha\,\Psi_{\alpha+1} + \beta_\alpha'\,\Psi_{\alpha+3}]. \tag{5.9}$$

In view of relations (5.6) and (5.8), we can write

$$\mathbf{\Lambda}(\Delta)\mathbf{U} = \mathbf{\Psi}, \tag{5.10}$$

where $\mathbf{\Psi} = (\tilde{\mathbf{\Psi}}, \Psi_2, \Psi_3, \Psi_4, \Psi_5)$ is a seven-component vector function and

$$\begin{gathered} \mathbf{\Lambda} = (\Lambda_{lj})_{7\times 7}, \qquad \Lambda_{11} = \Lambda_{22} = \Lambda_{33} = \Lambda_2, \qquad \Lambda_{44} = \Lambda_{55} = \Lambda_{66} = \Lambda_{77} = \Lambda_1, \\ \Lambda_{lj} = 0, \qquad l \neq j, \qquad l, j = 1, 2, \ldots, 7. \end{gathered} \tag{5.11}$$

We introduce the notation

$$\begin{aligned} m_{l1}(\Delta) &= -\frac{1}{a_0k_0\mu\mu_0}[(\lambda+\mu)A_{l1}^*(\Delta) - b_\alpha A_{l;\alpha+1}^*(\Delta) + \beta_\alpha' A_{l;\alpha+3}^*(\Delta)], \\ m_{lj}(\Delta) &= \frac{1}{a_0k_0\mu_0}A_{lj}^*(\Delta), \qquad l = 1, 2, \ldots, 5, \qquad j = 2, 3, 4, 5. \end{aligned} \tag{5.12}$$

Taking into account (5.12), from (5.7) and (5.9), we obtain

$$\begin{aligned} \tilde{\mathbf{\Psi}} &= \frac{1}{\mu}\Lambda_1(\Delta)\boldsymbol{\mathcal{F}}' + m_{11}(\Delta)\nabla\mathrm{div}\boldsymbol{\mathcal{F}}' + \sum_{l=2}^{5} m_{l1}(\Delta)\nabla\mathcal{F}_{l+2}, \\ \Psi_j &= m_{1j}\mathrm{div}\boldsymbol{\mathcal{F}}' + \sum_{l=2}^{5} m_{lj}(\Delta)\mathcal{F}_{l+2}, \qquad j = 2, 3, 4, 5. \end{aligned} \tag{5.13}$$

Then from (5.13), we can derive

$$\mathbf{\Psi} = \mathbf{N}^\top(\mathbf{D_x})\boldsymbol{\mathcal{F}}, \tag{5.14}$$

where

$$\begin{gathered} \mathbf{N}(\mathbf{D_x}) = (N_{lj}(\mathbf{D_x}))_{7\times 7}, \qquad N_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\Lambda_1\delta_{lj} + m_{11}\frac{\partial^2}{\partial x_l\partial x_j}, \\ N_{l;r+2}(\mathbf{D_x}) = m_{1r}\frac{\partial}{\partial x_l}, \qquad N_{r+2;j}(\mathbf{D_x}) = m_{r1}\frac{\partial}{\partial x_j}, \\ N_{r+2;m+2}(\mathbf{D_x}) = m_{rm}(\Delta), \qquad r, m = 2, 3, 4, 5. \end{gathered} \tag{5.15}$$

Combining the relations (5.2) and (5.10) with (5.14), we may further conclude that $\mathbf{\Lambda U}=\mathbf{N}^\top\mathbf{M}^\top\mathbf{U}$. Obviously, from the last identity, we get

$$\mathbf{M}(\mathbf{D_x})\mathbf{N}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta). \tag{5.16}$$

Let

$$\begin{gathered}
\mathbf{\Upsilon}(\mathbf{x}) = (\Upsilon_{lj}(\mathbf{x}))_{7\times 7}\,, \\
\Upsilon_{11}(\mathbf{x}) = \Upsilon_{22}(\mathbf{x}) = \Upsilon_{33}(\mathbf{x}) = \sum_{r=0}^{4}\eta_{2r}\gamma^{(r)}(\mathbf{x}) + \eta_{10}\gamma_0'(\mathbf{x}), \\
\Upsilon_{44}(\mathbf{x}) = \Upsilon_{55}(\mathbf{x}) = \Upsilon_{66}(\mathbf{x}) = \Upsilon_{77}(\mathbf{x}) = \sum_{r=0}^{4}\eta_{1r}\gamma^{(r)}(\mathbf{x}), \\
\Upsilon_{lj}(\mathbf{x}) = 0, \qquad l\neq j, \qquad l,j=1,2,\dots,7,
\end{gathered} \tag{5.17}$$

where we have used the notations

$$\gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \qquad \gamma_0'(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \qquad \gamma^{(j)}(x) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi|\mathbf{x}|} \tag{5.18}$$

and

$$\begin{gathered}
\eta_{10} = \prod_{l=1}^{4}\lambda_l^{-2}, \qquad \eta_{1j} = \lambda_j^{-2}\prod_{l=1;l\neq j}^{4}(\lambda_j^2-\lambda_l^2)^{-1}, \\
\eta_{20} = \eta_{10}\sum_{l=1}^{4}\lambda_l^{-2}, \qquad \eta_{2j} = \lambda_j^{-4}\prod_{l=1;l\neq j}^{4}(\lambda_j^2-\lambda_l^2)^{-1}, \qquad j=1,2,3,4.
\end{gathered} \tag{5.19}$$

On the basis of (5.5), (5.11), (5.18) and (5.19), it is easy to prove that

$$\mathbf{\Lambda}(\Delta)\mathbf{\Upsilon}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{5.20}$$

i.e., $\mathbf{\Upsilon}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{\Lambda}(\Delta)$.

Now, we introduce the notation

$$\mathbf{G}(\mathbf{x}) = \mathbf{N}(\mathbf{D_x})\mathbf{\Upsilon}(\mathbf{x}). \tag{5.21}$$

By virtue of (5.16), (5.20) and (5.21), we have

$$\mathbf{M}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \mathbf{M}(\mathbf{D_x})\mathbf{N}(\mathbf{D_x})\mathbf{\Upsilon}(\mathbf{x}) = \mathbf{\Lambda}(\Delta)\mathbf{\Upsilon}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Consequently, $\mathbf{G}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{M}(\mathbf{D_x})$. We have thereby proved the following consequence.

Theorem 6. *The matrix* $\mathbf{G}(\mathbf{x}) = (G_{lj}(\mathbf{x}))_{7\times 7}$ *defined by* (5.21) *is the fundamental solution of system* (2.9)*, where* $\mathbf{N}(\mathbf{D_x})$ *and* $\mathbf{\Upsilon}(\mathbf{x})$ *are given by* (5.15) *and* (5.17)*, respectively.*

Note that the matrix $\mathbf{G}(\mathbf{x})$ is constructed explicitly by means of six elementary functions: $\gamma_0'(\mathbf{x})$, $\gamma^{(j)}(\mathbf{x})$ $(j=0,1,2,3,4)$.

Theorem 6 leads to the following basic properties of the matrix $\mathbf{G}(\mathbf{x})$.

Theorem 7. *Each column of the matrix* $\mathbf{G}(\mathbf{x})$ *is a solution of the homogeneous equation*

$$\mathbf{M}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \mathbf{0}$$

at every point $\mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}$.

Theorem 8. *The relations*

$$\begin{gathered}
G_{lj}(\mathbf{x}) = O\big(|\mathbf{x}|^{-1}\big), \qquad G_{rm}(\mathbf{x}) = O\big(|\mathbf{x}|^{-1}\big), \qquad G_{r+2;m+2}(\mathbf{x}) = O\big(|\mathbf{x}|^{-1}\big), \\
G_{ls}(\mathbf{x}) = O(1), \qquad G_{sl}(\mathbf{x}) = O(1), \qquad G_{r;m+2}(\mathbf{x}) = O(1), \\
G_{r+2;m}(\mathbf{x}) = O(1), \qquad l,j=1,2,3, \qquad r,m=4,5, \quad s=4,5,6,7
\end{gathered}$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

Theorem 9. *The matrix* $\mathbf{G}^{(0)}(\mathbf{x}) = \big(G^{(0)}_{lj}(\mathbf{x})\big)_{7\times 7}$ *defined by*

$$G^{(0)}_{lj}(\mathbf{x}) = -\frac{\lambda+3\mu}{8\pi\mu\mu_0}\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{\lambda+\mu}{8\pi\mu\mu_0}\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$G^{(0)}_{44}(\mathbf{x}) = \frac{a_2}{a_0}\gamma^{(0)}(\mathbf{x}), \qquad G^{(0)}_{45}(\mathbf{x}) = G^{(0)}_{54}(\mathbf{x}) = -\frac{a_3}{a_0}\gamma^{(0)}(\mathbf{x}),$$
$$G^{(0)}_{55}(\mathbf{x}) = \frac{a_1}{a_0}\gamma^{(0)}(\mathbf{x}), \qquad G^{(0)}_{66}(\mathbf{x}) = \frac{k_2}{k_0}\gamma^{(0)}(\mathbf{x}),$$
$$G^{(0)}_{67}(\mathbf{x}) = G^{(0)}_{76}(\mathbf{x}) = -\frac{k_3}{k_0}\gamma^{(0)}(\mathbf{x}), \qquad G^{(0)}_{77}(\mathbf{x}) = \frac{k_1}{k_0}\gamma^{(0)}(\mathbf{x}), \qquad l,j=1,2,3$$

is the fundamental solution of the system

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\mathbf{u} = \mathbf{0}, \qquad a_1\Delta\varphi_1 + a_3\Delta\varphi_2 = 0, \qquad a_3\Delta\varphi_1 + a_2\Delta\varphi_2 = 0,$$
$$k_1\Delta p_1 + k_3\Delta p_2 = 0, \qquad k_3\Delta p_1 + k_2\Delta p_2 = 0.$$

Theorem 10. *The relations*

$$G_{lj}(\mathbf{x}) - G^{(0)}_{lj}(\mathbf{x}) = \mathrm{const} + O\left(|\mathbf{x}|\right), \qquad l,j=1,2,\dots,7$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

Thus, on the basis of Theorems 8 and 10, the matrix $\mathbf{G}^{(0)}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{G}(\mathbf{x})$ in the neighborhood of the origin of $\mathbb{R}^3$.

6. Basic Properties of Potentials and Singular Integral Operators

In this section, the surface (single-layer and double-layer) and volume potentials are defined, the useful singular integral operators are introduced and the basic properties of these potentials and operators are established.

In the sequel, we use the following matrix differential operator

$$\begin{gathered}\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{x}},\mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D}_{\mathbf{x}},\mathbf{n}))_{7\times 7}, \qquad \tilde{R}_{lj}(\mathbf{D}_{\mathbf{x}},\mathbf{n}) = R_{lj}(\mathbf{D}_{\mathbf{x}},\mathbf{n}),\\ \tilde{R}_{l;r+5}(\mathbf{D}_{\mathbf{x}},\mathbf{n}) = -\beta'_r n_l, \qquad \tilde{R}_{ms}(\mathbf{D}_{\mathbf{x}},\mathbf{n}) = R_{ms}(\mathbf{D}_{\mathbf{x}},\mathbf{n}), \qquad l=1,2,3,\\ j=1,2,\dots,5, \qquad r=1,2, \qquad m=4,5,6,7, \qquad s=1,2,\dots,7,\end{gathered} \tag{6.1}$$

where $R_{lj}(\mathbf{D}_{\mathbf{x}},\mathbf{n})$ is given by (3.2).

Let us now introduce the potential of a single-layer

$$\mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}) = \int_S \mathbf{G}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S;$$

the potential of a double-layer

$$\mathbf{P}^{(2)}(\mathbf{x},\mathbf{g}) = \int_S \left[\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{G}^\top(\mathbf{x}-\mathbf{y})\right]^\top \mathbf{g}(\mathbf{y})d_{\mathbf{y}}S;$$

and the potential of volume

$$\mathbf{P}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^\pm) = \int_{\Omega^\pm} \mathbf{G}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y})d\mathbf{y},$$

where $\mathbf{G}$ is the fundamental matrix of the operator $\mathbf{M}(\mathbf{D}_{\mathbf{x}})$ and defined by (5.21), the operator $\tilde{\mathbf{R}}$ is given by (6.1), $\mathbf{g}$ and $\boldsymbol{\phi}$ are seven-component vector functions.

It is not very difficult to prove the basic properties of these potentials. Namely, we can obtain the following consequences.

Theorem 11. *If* $S \in C^{r+1,\nu}$, $\mathbf{g} \in C^{r,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *and* r *is a non-negative integer, then:*
(i)
$$\mathbf{P}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\nu'}(\mathbb{R}^3) \cap C^{r+1,\nu'}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm});$$
(ii)
$$\mathbf{M}(\mathbf{D_x})\,\mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^{\pm};$$
(iii) $\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(1)}(\mathbf{z}, \mathbf{g})$ *is a singular integral for* $\mathbf{z} \in S$;
(iv)
$$\big\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(1)}(\mathbf{z}, \mathbf{g})\big\}^{\pm} = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(1)}(\mathbf{z}, \mathbf{g}), \tag{6.2}$$
for $\mathbf{z} \in S$;
(v)
$$\mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-1}), \quad \frac{\partial}{\partial x_l}\mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-2}) \ \text{for} \ |\mathbf{x}| \gg 1 \ \text{and} \ l = 1, 2, 3.$$

Theorem 12. *If* $S \in C^{r+1,\nu}$, $\mathbf{g} \in C^{r,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *then:*
(i)
$$\mathbf{P}^{(2)}(\cdot, \mathbf{g}) \in C^{r,\nu'}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm});$$
(ii)
$$\mathbf{M}(\mathbf{D_x})\,\mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^{\pm};$$
(iii) $\mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})$ *is a singular integral for* $\mathbf{z} \in S$;
(iv)
$$\big\{\mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})\big\}^{\pm} = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g}), \qquad \mathbf{z} \in S \tag{6.3}$$
for the non-negative integer r;
(v)
$$\mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-2}), \qquad \frac{\partial}{\partial x_l}\mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-3})$$
for $|\mathbf{x}| \gg 1$ *and* $l = 1, 2, 3$;
(vi)
$$\big\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})\big\}^{+} = \big\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})\big\}^{-}$$
for the natural number m *and* $\mathbf{z} \in S$.

Theorem 13. *If* $S \in C^{1,\nu}$, $\boldsymbol{\phi} \in C^{0,\nu'}(\Omega^{+})$, $0 < \nu' < \nu \leq 1$, *then:*
(i)
$$\mathbf{P}^{(3)}(\cdot, \boldsymbol{\phi}, \Omega^{+}) \in C^{1,\nu'}(\mathbb{R}^3) \cap C^{2}(\Omega^{+}) \cap C^{2,\nu'}(\overline{\Omega_0^{+}});$$
(ii)
$$\mathbf{M}(\mathbf{D_x})\,\mathbf{P}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^{+}) = \boldsymbol{\phi}(\mathbf{x}), \qquad \mathbf{x} \in \Omega^{+},$$
where Ω_0^{+} *is a domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{+}} \subset \Omega^{+}$.

Theorem 14. *If* $S \in C^{1,\nu}$, $\mathrm{supp}\boldsymbol{\phi} = \Omega \subset \Omega^{-}$, $\boldsymbol{\phi} \in C^{0,\nu'}(\Omega^{-})$, $0 < \nu' < \nu \leq 1$, *then:*
(i)
$$\mathbf{P}^{(3)}(\cdot, \boldsymbol{\phi}, \Omega^{-}) \in C^{1,\nu'}(\mathbb{R}^3) \cap C^{2}(\Omega^{-}) \cap C^{2,\nu'}(\bar{\Omega}_0^{-});$$
(ii)
$$\mathbf{M}(\mathbf{D_x})\,\mathbf{P}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^{-}) = \boldsymbol{\phi}(\mathbf{x}), \qquad \mathbf{x} \in \Omega^{-},$$
where Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{-}} \subset \Omega^{-}$.

Now, we introduce the following integral operators:

$$\begin{aligned}
\mathcal{H}^{(1)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{H}^{(2)}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{H}^{(3)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{H}^{(4)}\mathbf{g}(\mathbf{z}) &\equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{H}_{\varsigma}\mathbf{g}(\mathbf{z}) &\equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \varsigma\,\mathbf{Z}^{(2)}(\mathbf{z},\mathbf{g}), \qquad \mathbf{z}\in S,
\end{aligned} \tag{6.4}$$

where ς is an arbitrary complex number. On the basis of Theorems 11 and 12, we can prove that $\mathcal{H}^{(l)}$ $(l=1,2,3,4)$ and $\mathcal{H}_{\varsigma}$ are the singular integral operators.

On the other hand, if $\mathbf{\Gamma}^{(r)} = (\Gamma^{(r)}_{lj})_{7\times 7}$ is the symbol of the operator $\mathcal{H}^{(r)}$ $(r=1,2,3,4)$, then from (6.4), we have

$$\begin{gathered}
\det \mathbf{\Gamma}^{(1)} = \det \mathbf{\Gamma}^{(2)} = -\det \mathbf{\Gamma}^{(3)} = -\det \mathbf{\Gamma}^{(4)}\\
= \Big(-\frac{1}{2}\Big)^7\Big(1-\frac{\mu^2}{(\lambda+2\mu)^2}\Big) = -\frac{(\lambda+\mu)(\lambda+3\mu)}{128(\lambda+2\mu)^2} < 0,
\end{gathered}$$

i.e., the operator $\mathcal{H}^{(r)}$ is of normal type, where $r=1,2,3,4$.

Moreover, let $\mathbf{\Gamma}_{\varsigma}$ and $\operatorname{ind}\mathcal{H}_{\varsigma}$ be the symbol and the index of the operator $\mathcal{H}_{\varsigma}$, respectively. It may be easily shown that

$$\det \mathbf{\Gamma}_{\varsigma} = -\frac{(\lambda+2\mu)^2-\mu^2\varsigma^2}{128(\lambda+2\mu)^2}$$

and $\det\mathbf{\Gamma}_{\varsigma}$ vanishes only at two points ς_1 and ς_2 of the complex plane. By virtue of (6.4) and $\det\mathbf{\Gamma}_1 = \det\mathbf{\Gamma}^{(1)}$, we get $\varsigma_j \neq 1$ $(j=1,2)$ and

$$\operatorname{ind}\mathcal{H}_1 = \operatorname{ind}\mathcal{H}^{(1)} = \operatorname{ind}\mathcal{H}_0 = 0.$$

Similarly, we obtain

$$\operatorname{ind}\mathcal{H}^{(2)} = -\operatorname{ind}\mathcal{H}^{(1)} = 0, \qquad \operatorname{ind}\mathcal{H}^{(3)} = -\operatorname{ind}\mathcal{H}^{(4)} = 0.$$

Thus, the singular integral operator $\mathcal{H}^{(r)}$ $(r=1,2,3,4)$ is of normal type with an index equal to zero and, consequently, Fredholm's theorems are valid for $\mathcal{H}^{(r)}$.

For the definitions of a normal type singular integral operator, the symbol and the index of the 2D singular integral operators see, e.g., Kupradze et al. [16].

7. Existence Theorems

In this section, applying the potential method and the theory of singular integral equations, the existence of classical solutions of the internal and external basic BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$ are proved, where $K = I, II$.

Taking into account Theorems 13 and 14, we deduce that the volume potential $\mathbf{P}^{(3)}(\mathbf{x},\mathbf{F},\Omega^{\pm})$ is a particular solution of the nonhomogeneous equation (2.9), where $\mathbf{F}\in C^{0,\nu'}(\Omega^{\pm})$, $0<\nu'\le 1$; $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-. Bearing this in view, we prove the existence theorems of a regular (classical) solution of problems $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$, where $K = I, II$.

Problem $(I)^+_{\mathbf{0},\mathbf{f}}$. We are looking for a regular solution to this problem in the form of the double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(2)}(\mathbf{x},\mathbf{g}) \qquad \text{for} \qquad \mathbf{x}\in\Omega^+, \tag{7.1}$$

where $\mathbf{g}$ is the required seven-component vector function.

In view of Theorem 12, the vector function $\mathbf{U}$ is a solution of the following homogeneous equation:

$$\mathbf{M}(\mathbf{D}_{\mathbf{x}})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{7.2}$$

for $\mathbf{x} \in \Omega^+$. By virtue of the boundary condition (3.3) and using (6.3), from (7.1), we obtain the singular integral equation

$$\mathcal{H}^{(1)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \tag{7.3}$$

for determining the unknown vector function $\mathbf{g}$, where $\mathbf{z} \in S$. We prove that equation (7.3) is always solvable for an arbitrary vector $\mathbf{f}$.

Obviously, the homogeneous adjoint integral equation of (7.3) has the following form:

$$\mathcal{H}^{(2)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S, \tag{7.4}$$

where $\mathbf{h}$ is the required seven-component vector function. Now, we prove that (7.4) has only the trivial solution.

Let $\mathbf{h}_0$ be a solution of the homogeneous equation (7.4). On the basis of Theorem 11 and equation (7.4), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of the external homogeneous BVP $(II)^-_{\mathbf{0},\mathbf{0}}$. By virtue of Theorem 5, the problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, i.e.,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{7.5}$$

In addition, by Theorem 11 and (7.5), we get

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S.$$

Consequently, the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the internal homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$ and using Theorem 3, it follows that

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \qquad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{7.6}$$

In view of relations (7.5), (7.6) and identity (6.2), we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S.$$

Thus, the homogeneous equation (7.4) has only the trivial solution. On the basis of Fredholm's theorem, the nonhomogeneous integral equation (7.3) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following result.

Theorem 15. *If* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *then a regular solution of the internal BVP* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by double-layer potential* (7.1), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.3) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$. Now, we seek for a regular solution to this problem in the form of the single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{h}) \qquad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{7.7}$$

where $\mathbf{h}$ is the required seven-component vector function. Clearly, by Theorem 11, the vector function $\mathbf{U}$ is a solution of (7.2) for $\mathbf{x} \in \Omega^-$. By virtue of the boundary condition (3.6) and using (6.2), to determine the unknown vector $\mathbf{h}$, we obtain from (7.7) the following singular integral equation:

$$\mathcal{H}^{(2)}\,\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.8}$$

In Theorem 15, we have proved that the corresponding homogeneous equation (7.4) has only the trivial solution. Hence by Fredholm's theorem, (7.8) is always solvable. We have the following consequence.

Theorem 16. *If* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *then a regular solution of the external BVP* $(II)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by the single-layer potential* (7.7), *where* $\mathbf{h}$ *is a solution of the singular integral equation* (7.8) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$. We seek for a regular solution to this problem in the sum of the single-and double-layer potentials

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) + \mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) \qquad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{7.9}$$

where $\mathbf{g}$ is the required seven-component vector function.

Obviously, by Theorems 11 and 12, the vector function $\mathbf{U}$ is a regular solution of (7.2) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition (3.5) and using (6.3), to determine the unknown vector $\mathbf{g}$, we obtain from (7.9) the following singular integral equation:

$$\mathcal{H}^{(5)}\,\mathbf{g}(\mathbf{z}) \equiv \mathcal{H}^{(3)}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}) = \mathbf{f}(\mathbf{z}) \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.10}$$

We prove that equation (7.10) is always solvable for an arbitrary vector $\mathbf{f}$. It can be easily verified that the singular integral operator $\mathcal{H}^{(5)}$ is of the normal type and $\operatorname{ind}\mathcal{H}^{(5)} = \operatorname{ind}\mathcal{H}^{(3)} = 0$.

Now, we prove that the homogeneous equation

$$\mathcal{H}^{(5)}\,\mathbf{g}_0(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S \tag{7.11}$$

has only a trivial solution. Let $\mathbf{g}_0$ be a solution of the homogeneous equation (7.11). Then the vector

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}_0) + \mathbf{P}^{(2)}(\mathbf{x},\mathbf{g}_0) \qquad \text{for} \quad \mathbf{x} \in \Omega^- \tag{7.12}$$

is a regular solution of the external BVP $(I)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 5, we have (7.5).

Moreover, by identities (6.2) and (6.3), from (7.12), we get

$$\begin{aligned} \{\mathbf{V}(\mathbf{z})\}^+ - \{\mathbf{V}(\mathbf{z})\}^- &= \mathbf{g}_0(\mathbf{z}), \\ \{\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^+ - \{\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^- &= -\mathbf{g}_0(\mathbf{z}) \qquad \text{for} \quad \mathbf{z} \in S. \end{aligned} \tag{7.13}$$

In view of (7.5), from (7.13), it follows that

$$\{\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{V}(\mathbf{z}) + \mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.14}$$

Obviously, the vector $\mathbf{V}$ is a solution of equation (7.2) in Ω^+ satisfying the boundary condition (7.14). Now, applying identity (4.5) for the vector $\mathbf{V}$, we obtain

$$\{\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.15}$$

Finally, by virtue of (7.5) and (7.15), from the first equation of (7.13), we get $\mathbf{g}_0(\mathbf{z}) \equiv \mathbf{0}$ for $\mathbf{z} \in S$.

Thus, the homogeneous equation (7.11) has only the trivial solution and therefore on the basis of Fredholm's theorem, the integral equation (7.10) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following

Theorem 17. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then a regular solution of the external BVP $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the sum of double- and single-layer potentials* (7.9), *where $\mathbf{g}$ is a solution of the singular integral equation* (7.10) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$. Finally, we are looking for a regular solution to this problem in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}) \qquad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{7.16}$$

where $\mathbf{g}$ is the required seven-component vector function.

In view of Theorem 11, the vector function $\mathbf{U}$ is a solution of the homogeneous equation (7.2). Then, taking into account identity (6.3) and the boundary condition (3.4), to determine the unknown vector $\mathbf{g}$, we obtain from (7.16), the following singular integral equation:

$$\mathcal{H}^{(4)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.17}$$

To investigate the solvability of equation (7.17), we consider the homogeneous equation

$$\mathcal{H}^{(4)}\,\mathbf{g}(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.18}$$

Clearly, the adjoint homogeneous integral equation of (7.18) has the form

$$\mathcal{H}^{(3)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S. \tag{7.19}$$

In our further analysis, we will need the following consequence.

Lemma 3. *The homogeneous equations* (7.18) *and* (7.19) *have six linearly independent solutions each, and they constitute the complete systems of solutions.*

Lemma 3 can be proved similarly to the corresponding result in the quasi-static theory of elasticity for single-porosity materials (see [19]).

Introduce now the seven-component vector functions $\boldsymbol{\vartheta}^{(j)}(\mathbf{x})$ $(j = 1, 2, \ldots, 6)$ by

$$\begin{aligned} &\boldsymbol{\vartheta}^{(1)}(\mathbf{x}) = (1, 0, 0, 0, 0, 0, 0), && \boldsymbol{\vartheta}^{(2)}(\mathbf{x}) = (0, 1, 0, 0, 0, 0, 0),\\ &\boldsymbol{\vartheta}^{(3)}(\mathbf{x}) = (0, 0, 1, 0, 0, 0, 0), && \boldsymbol{\vartheta}^{(4)}(\mathbf{x}) = (0, -x_3, x_2, 0, 0, 0, 0),\\ &\boldsymbol{\vartheta}^{(5)}(\mathbf{x}) = (x_3, 0, -x_1, 0, 0, 0, 0), && \boldsymbol{\vartheta}^{(6)}(\mathbf{x}) = (-x_2, x_1, 0, 0, 0, 0, 0). \end{aligned} \tag{7.20}$$

Obviously, $\{\boldsymbol{\vartheta}^{(j)}(\mathbf{x})\}_{j=1}^{6}$ is the system of linearly independent vectors. Moreover, by Theorem 4, each vector $\boldsymbol{\vartheta}^{(j)}(\mathbf{x})$ is a regular solution of the internal homogeneous BVP $(II)^{+}_{\mathbf{0},\mathbf{0}}$ and the homogeneous singular integral equation (7.19), i.e., we have

$$\mathbf{M}(\mathbf{D_x})\,\boldsymbol{\vartheta}^{(j)}(\mathbf{x}) = \mathbf{0} \qquad \text{for} \quad \mathbf{x} \in \Omega^{+},$$

$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n})\,\boldsymbol{\vartheta}^{(j)}(\mathbf{z})\}^{+} = \mathbf{0}, \qquad \mathcal{H}^{(4)}\,\boldsymbol{\vartheta}^{(j)}(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S$$

and $j = 1, 2, \ldots, 6$. Hence $\{\boldsymbol{\vartheta}^{(j)}(\mathbf{x})\}_{j=1}^{6}$ is a complete system of linearly independent solutions of equation (7.19).

Applying Fredholm's theorem, the necessary and sufficient condition for (7.17) to be solvable has the form

$$\int_S \mathbf{f}(\mathbf{z}) \cdot \boldsymbol{\vartheta}^{(j)}(\mathbf{z}) d_{\mathbf{z}}S = 0, \quad j = 1, 2, \ldots, 6, \tag{7.21}$$

where $\boldsymbol{\vartheta}^{(j)}$ is determined by (7.20).

On the other hand, if $\mathbf{f} = (f_1, f_2, \ldots, f_7)$ and $\mathbf{f}^{(0)} = (f_1, f_2, f_3)$, then by virtue of (7.20), condition (7.21) can be rewritten as

$$\int_S \mathbf{f}^{(0)}(\mathbf{z}) d_{\mathbf{z}}S = \mathbf{0}, \qquad \int_S \mathbf{z} \times \mathbf{f}^{(0)}(\mathbf{z}) d_{\mathbf{z}}S = \mathbf{0}. \tag{7.22}$$

We have thereby proved the following result.

Theorem 18. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then problem $(II)^{+}_{\mathbf{0},\mathbf{f}}$ is solvable if and only if conditions* (7.22) *are fulfilled. The solution of this problem is represented by a potential of single-layer* (7.16) *and is determined to within an additive vector of $\tilde{\mathbf{U}} = (\tilde{\mathbf{u}}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{p}_1, \tilde{p}_2)$, where $\mathbf{g}$ is a solution of the singular integral equation* (7.17) *and*

$$\tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{a}} + \tilde{\mathbf{b}} \times \mathbf{x}, \qquad \tilde{\varphi}_l(\mathbf{x}) = \tilde{p}_l(\mathbf{x}) \equiv 0, \qquad l = 1, 2$$

for $\mathbf{x} \in \Omega^{+}$, $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors.

8. Concluding Remarks

1. In this paper the basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of elasticity for materials with double porosity are investigated and the following results are obtained:

i) On the basis of Green's identity the uniqueness theorems for classical solutions of the above mentioned BVPs are proved;

ii) The fundamental solution of the system of steady vibration equations is constructed explicitly by means of six elementary functions;

iii) The basic properties of the surface (single-layer and double-layer) and volume potentials are established;

iv) Then some useful singular integral operators are constructed for which Fredholm's theorems are valid;

v) Finally, the existence theorems for classical solutions of the BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

2. On the basis of results of this paper are possible to investigate the BVPs in the coupled linear quasi-static theory of thermoelasticity for materials with double porosity by using the potential method and the theory of singular integral equations.

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