Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 3, 423–429

ON A CONSISTENT ESTIMATOR OF A USEFUL SIGNAL IN ORNSTEIN–UHLENBECK STOCHASTIC MODEL IN $\mathbb{C}[-l, l[$

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. A transmittion process of a useful signal is considered in Ornstein–Uhlenbeck stochastic model in $\mathbb{C}[-l, l[$ defined by the stochastic differential equation

$$d\Psi(t,x,\omega) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}\Psi(t,x,\omega)dt + \sigma dW(t,\omega)$$

with initial the condition

$$\Psi(0,x,\omega) = \Psi_0(0) \in FD^{(0)}[-l,l[,$$

where $m \geq 1$, $(A_n)_{0\leq n\leq 2m} \in \mathbb{R}^+ \times \mathbb{R}^{2m-1}$, $((t,x,\omega) \in [0,+\infty[\times[-l,l[\times\Omega)$, $\sigma \in \mathbb{R}^+$, $\mathbb{C}[-l,l[$ is a Banach space of all real-valued bounded continuous functions on $[-l,l[$, $FD^{(0)}[-l,l[\subset \mathbb{C}[-l,l[$ is a class of all real-valued bounded continuous functions on $[-l,l[$ whose Fourier series converges to itself everywhere on $[-l,l[$, $(W(t,\omega))$ is a Wiener process and $\Psi_0(x)$ is a useful signal.

Using a sequence of transformed signals $(Z_k)_{k\in N} = (\Psi(t_0,x,\omega_k))_{k\in N}$ at moment $t_0 > 0$, the consistent and infinite-sample consistent estimates of the useful signal Ψ_0 is constructed under the assumption that parameters $(A_n)_{0\leq n\leq 2m}$ and σ are known.

1. Introduction

Suppose that Θ is a vector subspace of the Banach space $\mathbb{C}[-l,l[$ equipped with a usual norm, where $\mathbb{C}[-l,l[$ denotes the class of all bounded continuous functions on $[-l,l[$.

In the information transmitting theory, we consider the Ornstein–Uhlenbeck stochastic system

$$\begin{aligned}\xi(t,x,\omega)I_{[-l,l[}(x) &= e^{t\sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}}\theta(x)I_{[-l,l[}(x)\\ &+\sigma\int_0^t e^{(t-\tau)\sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \times I_{[-l,l[}(x)dW(\tau,\omega),\end{aligned} \tag{1.1}$$

where $(A_n)_{0\leq n\leq 2m} \in \mathbb{R}^+\times\mathbb{R}^{2m-1}$ $(m \geq 1)$, $\theta \in \Theta$ is a useful signal, $(W(t,\cdot))_{t\geq 0}$ is the Winner processes (the so-called "white noises") defined on the probability space (Ω,F,P), $(\xi(t,\cdot,\omega))$ (equivalently, $\xi(t,x,\omega)I_{[-l,l[}(x))$ is a transformed signal for $(t,\omega) \in [0,+\infty[\times\Omega$, $I_{[-l,l[}$ denotes the indicator function of the interval $[-l,l[$.

Let μ be a Borel probability measure on $\mathbb{C}^{[0,+\infty[}[-l,l[$ defined by generalized "white noise"

$$\left(\sigma\int_0^t e^{(t-\tau)\sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \times I_{[-l,l[}(x)dW(\tau,\omega)\right)_{t\geq 0}.$$

2020 *Mathematics Subject Classification.* 60G10, 60G35, 62F10.

Key words and phrases. Ornstein-Uhlenbeck process; Useful signal; Consistent and infinite-sample consistent estimators.

Then we have

$$(\forall X)\big(X \in \mathbb{B}\big(\mathbb{C}^{[0,+\infty[}[-l,l[\big) \longrightarrow \mu(X)$$
$$= P\Bigg(\Bigg\{\omega:\ \omega \in \Omega \& \Bigg(\sigma \int\limits_0^t e^{(t-\tau)\sum\limits_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \times I_{[-l,l[}(x) dW(\tau,\omega)\Bigg)_{t\geq 0} \in X\Bigg\}\Bigg)\Bigg),$$

where $\mathbb{B}\left(\mathbb{C}^{[0,+\infty[}[-l,l[\right)$ is the Borel σ-algebra of subsets of the space $\mathbb{C}^{[0,+\infty[}[-l,l[$.

Let λ be a Borel probability measure on $\mathbb{C}^{[0,+\infty[}[-l,l[$ defined by the transformed signal $\xi(t,x,\omega)$, that is,

$$(\forall X)\big(X \in \mathbb{B}\big(\mathbb{C}^{[0,+\infty[}[-l,l[\big) \longrightarrow \mu(X) = P\big(\{\omega:\ \omega \in \Omega \& \left(\xi(t,x,\omega)\right)_{t\geq 0} \in X\}\big)\big).$$

In the information transmitting theory, the general decision is that the Borel probability measure λ defined by the transformed signal coincides with $e^{t\sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}}\theta$ shift μ_θ of the measure μ for some $\theta_0 \in \Theta$, provided that

$$(\exists\theta_0)\big(\theta_0 \in \Theta \longrightarrow (\forall X)\big(X \in \mathbb{B}\big(\mathbb{C}^{[0,+\infty[}[-l,l[\big) \longrightarrow \lambda(X) = \mu_\theta(X)\big)\big),$$

where $\mu_\theta(\cdot) = \mu\big(\,\cdot + \big(e^{t\sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}}\theta\big)_{t\geq 0}\big)$.

Here, we consider a particular case of the above model when a vector space of useful signals Θ coincides with $FD^0[-l,l[$, where $FD^0[-l,l[\subset \mathbb{C}[-l,l[$ denotes a vector space of all bounded continuous real-valued functions on $[-l,l[$ whose Fourier series converges to itself everywhere on $[-l,l[$.

Definition 1.1 (see [2])**.** A triplet

$$\big(\big(\mathbb{C}^{[0,+\infty[}[-l,l[\big)^{\mathbb{N}},\ \mathbb{B}\big(\big(\mathbb{C}^{[0,+\infty[}[-l,l[\big)^{\mathbb{N}}\big),\ \mu_\theta^{\mathbb{N}}\big)_{\theta\in\Theta}$$

is called a statistical structure described the stochastic system (1.1).

Definition 1.2 (see [2])**.** The Borel measurable function $T_n:\ \left(\mathbb{C}^{[0,+\infty[}[-l,l[\right)^n \longrightarrow \Theta\ \ (n \in N)$ is called a consistent estimate of a parameter θ for the family $\left(\mu_\theta^{\mathbb{N}}\right)_{\theta\in\Theta}$ if the condition

$$\mu_\theta^{\mathbb{N}}\Big(\Big\{(x_k)_{k\in\mathbb{N}}:\ (x_k)_{k\in\mathbb{N}} \in \left(\mathbb{C}^{[0,+\infty[}[-l,l[\right)^{\mathbb{N}} \& \lim_{n\to\infty} \|T_n(x_1,\dots,x_n) - \theta\| = 0\Big\}\Big) = 1$$

holds for each $\theta \in \Theta$, where $\|\cdot\|$ is a usual norm in $\mathbb{C}[-l,l[$.

Definition 1.3 (see [2])**.** The Brel measurable function $T:\ \left(\mathbb{C}^{[0,+\infty[}[-l,l[\right)^{\mathbb{N}} \longrightarrow \Theta$ is called an infinite-sample consistent estimate of a parameter θ for the family $\left(\mu_\theta^{\mathbb{N}}\right)_{\theta\in\Theta}$ if the condition

$$\mu_\theta^{\mathbb{N}}\Big(\Big\{(x_k)_{k\in\mathbb{N}}:\ (x_k)_{k\in\mathbb{N}} \in \left(\mathbb{C}^{[0,+\infty[}[-l,l[\right)^{\mathbb{N}} \& T\left((x_k)_{k\in\mathbb{N}}\right) = \theta\Big\}\Big) = 1$$

holds for each $\theta \in \Theta$.

The main goal of the present paper is to construct consistent and infinite-sample estimators of the useful signal for the stochastic model (1.1) which is a particular case of the Ornstein–Uhlenbeck process in $\mathbb{C}[-l,l[$. Concerning the estimations of parameters for another versions of the Ornstein–Uhlenbeck processes the reader may consult with [1, 4, 5, 9].

The rest of the present paper is the following.

Section 2 contains some auxiliary notions and fact from the theories of ordinary and stochastic differential equations. In Section 3 we present our main results.

2. Materials and Methods

We begin this section with a short description of a certain result concerning a solution of some differential equations with initial value problem obtained in papers [3, 6]. Further, by using this approach and technique developed in [7], their some applications for a solution of the Ornstein–Uhlenbeck stochastic differential equation in $\mathbb{C}[-l,l[$ are obtained. At the end of this section, the well-known Kolmogorov's Strong Law of Large Numbers is presented.

Lemma 2.1 ([6, Corollary 2.1, p. 6])**.** *For $m \geq 1$, let us consider a linear partial differential equation*

$$\frac{\partial}{\partial t}\Psi(t,x) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}\Psi(t,x) \qquad ((t,x) \in [0,+\infty[\times[-l,l[) \tag{2.1}$$

with the initial condition

$$\Psi(0,x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}\Big(c_k\cos\Big(\frac{k\pi x}{l}\Big) + d_k\sin\Big(\frac{k\pi x}{l}\Big)\Big) \in FD^{(0)}[-l,l[. \tag{2.2}$$

If $\big(\frac{c_0}{2}, c_1, d_1, c_2, d_2, \ldots\big)$ is a sequence of real numbers such that a series $\Psi(t,x)$ defined by

$$\Psi(t,x) = \frac{e^{tA_0}c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t}\Big(\Big(c_k\cos(\omega_k t) + d_k\sin(\omega_k t)\Big)\cos\Big(\frac{k\pi x}{l}\Big)$$
$$+\Big(d_k\cos(\omega_k t) - c_k\sin(\omega_k t)\Big)\sin\Big(\frac{k\pi x}{l}\Big)\Big)$$

belongs to the class $FD^{(2m)}[-l,l[$, as a series of a variable x for all $t \geq 0$, and is differentiable term by term as a series of a variable t for all $x \in [-l,l[$, then Ψ is a solution of (2.1)–(2.2).

Using the approach developed in [7], we get the validity of the following assertion.

Lemma 2.2. *For $m \geq 1$, let us consider the Ornstein–Uhlenbeck process in $\mathbb{C}[-l,l[$ defined by the stochastic differential equation*

$$d\Psi(t,x,\omega) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}\Psi(t,x,\omega)dt$$
$$+\,\sigma dW(t,\omega)I_{[-l,l[}(x), \quad \big((t,x,\omega) \in [0,+\infty[\times[-l,l[\times\Omega\big) \tag{2.3}$$

with the initial condition

$$\Psi(0,x,\omega) = \Psi_0(x), \tag{2.4}$$

where $(A_n)_{0\leq n\leq 2m} \in \mathbb{R}^+ \times \mathbb{R}^{2m-1}$, $(W(t,\omega))_{t\geq 0}$ is a Wiener process and

$$\Psi(0,x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}\Big(c_k\cos\Big(\frac{k\pi x}{l}\Big) + d_k\sin\Big(\frac{k\pi x}{l}\Big)\Big) \in FD^{(0)}[-l,l[.$$

If $\big(\frac{c_0}{2}, c_1, d_1, c_2, d_2, \ldots\big)$ is a sequence of real numbers such that a series

$$\frac{e^{tA_0}c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t}\Big(\,(c_k\cos(\omega_k t) + d_k\sin(\omega_k t))\cos\Big(\frac{k\pi x}{l}\Big)$$
$$+\,(d_k\cos(\omega_k t) - c_k\sin(\omega_k t))\sin\Big(\frac{k\pi x}{l}\Big)\Big) + \sigma\int_0^t e^{(t-\tau)A_0}dW(\tau,\omega)I_{[-l.l[}(x)$$

belongs to the class $FD^{(2m)}[-l,l[$, as a series of a variable x for all $t \geq 0$, $\omega \in \Omega$, and is differentiable term by term as a series of a variable t for all $x \in [-l,l[$, $\omega \in \Omega$, then the solution of (2.3)–(2.4) *is given by*

$$d\Psi(t,x,\omega) = e^{t\sum_{n=0}^{2m} A_n\frac{\partial^n}{\partial x^n}}\left(\Psi(0,x,\omega)\right) + \sigma\int_0^t e^{(t-s)\sum_{n=0}^{2m} A_n\frac{\partial^n}{\partial x^n}}dW(s,\omega).$$

Proof. Putting $\mathbb{A} = \sum_{n=0}^{2m} A_n\frac{\partial^n}{\partial x^n}$ and $f\left(t,\Psi(t,x,\omega)\right) = e^{-t\mathbb{A}}\Psi(t,x,\omega)$, we get

$$df\left(t,\Psi(t,x,\omega)\right) = -\mathbb{A}e^{-t\mathbb{A}}\Psi(t,x,\omega)dt + e^{-t\mathbb{A}}d\Psi(t,x,\omega)$$
$$= -\mathbb{A}e^{-t\mathbb{A}}\Psi(t,x,\omega)dt + e^{-t\mathbb{A}}\left(\mathbb{A}\Psi(t,x,\omega) + \sigma dW(t,\omega)\right) = \sigma e^{-t\mathbb{A}}dW(t,\omega).$$

By integration of both sides, we get

$$f\left(t,\Psi(t,x,\omega)\right)-f\left(0,\Psi(0,x,\omega)\right)=\sigma\int\limits_0^t e^{-\tau\mathbb{A}}dW(\tau,\omega)$$

which implies

$$e^{-t\mathbb{A}}\Psi(t,x,\omega)-e^{-0\mathbb{A}}\Psi(0,x,\omega)=\sigma\int\limits_0^t e^{-\tau\mathbb{A}}dW(\tau,\omega).$$

Now, we get

$$e^{t\mathbb{A}}\left(e^{-t\mathbb{A}}\Psi(t,x,\omega)\right)-e^{t\mathbb{A}}\left(e^{-0\mathbb{A}}\Psi(0,x,\omega)\right)=e^{t\mathbb{A}}\Big(\sigma\int\limits_0^t e^{-\tau\mathbb{A}}dW(\tau,\omega)\Big),$$

which is equivalent to the equality

$$\Psi(t,x,\omega)=e^{t\mathbb{A}}\left(\Psi(0,x,\omega)\right)+\int\limits_0^t e^{(t-\tau)\mathbb{A}}I_{[-l,l[}(x)dW(\tau,\omega).$$ □

Remark 2.3. Under the condition of Lemma 2.2, we have

$$\Psi(t,x,\omega)=\frac{e^{tA_0}c_0}{2}+\sum_{k=1}^{\infty}e^{\sigma_k t}\Big(\left(c_k\cos(\omega_k t)+d_k\sin(\omega_k t)\right)\cos\Big(\frac{k\pi x}{l}\Big)$$
$$+\left(d_k\cos(\omega_k t)-c_k\sin(\omega_k t)\right)\sin\Big(\frac{k\pi x}{l}\Big)\Big)+\sigma\int\limits_0^t e^{(t-\tau)A_0}I_{[-l.l[}(x)dW(\tau,\omega).$$

Lemma 2.4. *Under the conditions of Lemma* 2.2*, the following conditions:*

(i) $$E\Psi(t,x,\cdot)=\frac{e^{tA_0}c_0}{2}+\sum_{k=1}^{\infty}e^{\sigma_k t}\Big(\left(c_k\cos(\omega_k t)+d_k\sin(\omega_k t)\right)\cos\Big(\frac{k\pi x}{l}\Big)$$
$$+\left(d_k\cos(\omega_k t)-c_k\sin(\omega_k t)\right)\sin\Big(\frac{k\pi x}{l}\Big)\Big);$$

(ii) $$\operatorname{cov}\big(\Psi(s,x,\cdot),\Psi(t,x,\cdot)\big)=\frac{\sigma^2}{2A_0}\big(e^{-A_0(t-s)}-e^{-A_0(t+s)}\big);$$

(iii) $$\operatorname{var}\left(\Psi(s,x,\cdot)\right)=\frac{\sigma^2}{2A_0}\left(1-e^{-2A_0 s}\right).$$

are valid.

Proof. The validity of item (i) is obvious. In order to prove the validity of items (ii)–(iii), we can use the Ito isometry to calculate the covariance function by virtue of

$$\operatorname{cov}\left(\Psi(s,x,\cdot),\Psi(t,x,\cdot)\right)=E[(\Psi(s,x,\cdot)-E[\Psi(s,x,\cdot)])(\Psi(t,x,\cdot)-E[\Psi(t,x,\cdot)])]$$
$$=E\Big[\int\limits_0^s\sigma e^{A_0(u-s)}dW(u,\omega)\int\limits_0^t\sigma e^{A_0(\nu-t)}dW(\nu,\omega)\Big]$$
$$=\sigma^2e^{-A_0(s+t)}E\Big[\int\limits_0^s\sigma e^{A_0u}dW(u,\omega)\int\limits_0^t\sigma e^{A_0\nu}dW(\nu,\omega)\Big]$$
$$=\frac{\sigma^2}{2A_0}e^{-A_0(s+t)}\big(e^{2A_0\min(s,t)}-1\big).$$

Thus if $s < t$ (so that $\min(s,t) = s$), then we have

$$\operatorname{cov}\left(\Psi(s,x,\cdot),\Psi(t,x,\cdot)\right) = \frac{\sigma^2}{2A_0}\left(e^{-A_0(t-s)} - e^{-A_0(t+s)}\right).$$

Similarly, if $s = t$ (so that $\min(s,t) = s$), then we have

$$\operatorname{var}\left(\Psi(s,x,\cdot)\right) = \frac{\sigma^2}{2A_0}\left(1 - e^{-2A_0 s}\right). \qquad \square$$

In the next section, we will need the well-known fact from the probability theory (see [8, p. 390]).

Lemma 2.5 (Kolmogorov's strong law of large numbers)**.** *Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables defined on the probability space (Ω, F, P). If these random variables have a finite expectation m (i.e., $E(X_1) = E(X_2) = \cdots = m < \infty$), then the following condition*

$$P\left(\left\{\omega : \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} X_k(\omega) = m\right\}\right) = 1$$

holds true.

3. Results

In this section, by the use of Kolmogorov's Strong Law of Large Numbers, we construct consistent and infinite-sample consistent estimators of a useful signal which is transmitted by the Ornstein–Uhlenbeck stochastic system (1.1).

Theorem 3.1. *Let us consider $\mathbb{C}[-l,l[$-valued stochastic process $\left(\xi(t,x,\omega)I_{[-l,l[}(x)\right)_{t\geq 0}$ defined by*

$$\xi(t,x,\omega)I_{[-l,l[}(x) = e^{t\mathbb{A}}\left(\theta(x)I_{[-l,l[}(x)\right) + \sigma\int_0^t e^{(t-\tau)\mathbb{A}} I_{[-l,l[}(x)dW(\tau,\omega),$$

where $\theta \in FD^{(0)}$ and $\mathbb{A} = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}$. Assume that all conditions of Lemma 2.4 are satisfied. For a fixed $t = t_0 > 0$, we denote by μ_θ a probability measure in $\mathbb{C}[-l,l[$ defined by the random element $\Xi(t_0,\omega)$, where

$$\Xi(t_0,\omega) = e^{t_0\mathbb{A}}\left(\theta(x)I_{[-l,l[}(x)\right) + \sigma\int_0^{t_0} e^{(t_0-s)\mathbb{A}} I_{[-l,l[}(x)dW(s,\omega),$$

for each elementary ω event.

For each $(Z_k)_{k\in\mathbb{N}} \in \left(\mathbb{C}[-l,l[\right)^{\mathbb{N}}$, we put

$$T_n\left((Z_k)_{k\in\mathbb{N}}\right) = e^{-t_0\mathbb{A}}\left(\frac{\sum_{k=1}^{n} Z_k}{n}\right).$$

Then T_n is a consistent estimate of a useful signal θ, provided that

$$\mu_\theta^{\mathbb{N}}\left\{(Z_k)_{k\in\mathbb{N}} : \lim_{n\to\infty}\left\|T_n\left((Z_k)_{k\in\mathbb{N}}\right) - \theta\right\| = 0\right\} = 1,$$

for each $\theta \in FD^{(0)}[-l,l[$.

Proof. For each $X \in \mathbb{B}\left(\mathbb{C}[-l,l[\right)$, we have

$$\mu_\theta(X) = P\left\{\omega : \Xi(t_0,\omega) \in X\right\}$$

$$= P\left\{\omega : e^{t_0\mathbb{A}}(\theta) + \sigma\int_0^{t_0} e^{(t_0-\tau)\mathbb{A}} I_{[-l,l[}dW(\tau,\omega) \in X\right\}$$

$$= P\Big\{\omega :\ \sigma \int_0^{t_0} e^{(t_0-\tau)A_0} dW(\tau,\omega) I_{[-l,l[} \in \big(X - e^{t_0\mathbb{A}}(\theta)\big) \cap \big\{\alpha I_{[-l,l[} :\ \alpha \in \mathbb{R}\big\}\Big\}.$$

We also have

$$\mu_\theta^{\mathbb{N}} \Big\{(Z_k)_{k\in\mathbb{N}} :\ \lim_{n\to\infty} \big\|T_n\left((Z_k)_{k\in\mathbb{N}}\right) - \theta\big\| = 0\Big\}$$
$$= \mu_\theta^{\mathbb{N}} \Bigg\{(Z_k)_{k\in\mathbb{N}} :\ \lim_{n\to\infty} \Bigg\| e^{-t_0\mathbb{A}} \left(\frac{\sum_{k=1}^n Z_k}{n}\right) - \theta \Bigg\| = 0\Bigg\}$$
$$= \mu_\theta^{\mathbb{N}} \Bigg\{(Z_k)_{k\in\mathbb{N}} :\ \lim_{n\to\infty} \Bigg\| \frac{\sum_{k=1}^n Z_k}{n} - e^{t_0\mathbb{A}}(\theta) \Bigg\| = 0\Bigg\}$$
$$= \mu_\theta^{\mathbb{N}} \Bigg\{(Z_k)_{k\in\mathbb{N}} :\ Z_k \in \big\{\alpha_k I_{[-l,l[} :\ \alpha_k \in \mathbb{R}\big\} + e^{t_0\mathbb{A}}(\theta) \& \lim_{n\to\infty} \Bigg\| \frac{\sum_{k=1}^n Z_k}{n} - e^{t\mathbb{A}}(\theta) \Bigg\| = 0\Bigg\}$$
$$= \mu_\theta^{\mathbb{N}} \Bigg\{(Z_k)_{k\in\mathbb{N}} :\ (\exists (\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty) \left(Z_k = \beta_k I_{[-l,l[} + e^{t_0\mathbb{A}}(\theta) \& \lim_{n\to\infty} \Bigg\| \frac{\sum_{k=1}^n Z_k}{n} - e^{t_0\mathbb{A}}(\theta) \Bigg\| = 0\right)\Bigg\}$$
$$= \mu_\theta^{\mathbb{N}} \Bigg\{(Z_k)_{k\in\mathbb{N}} :\ (\exists (\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty) \left(Z_k = \beta_k I_{[-l,l[} + e^{t_0\mathbb{A}}(\theta) \& \lim_{n\to\infty} \Bigg\| \frac{\sum_{k=1}^n \beta_k}{n} I_{[-l,l[} \Bigg\| = 0\right)\Bigg\}$$
$$\mu_\theta^{\mathbb{N}} \Bigg\{(Z_k)_{k\in\mathbb{N}} :\ (\exists (\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty) \left(Z_k = \beta_k I_{[-l,l[} + e^{t_0\mathbb{A}}(\theta) \& \lim_{n\to\infty} \Bigg| \frac{\sum_{k=1}^n \beta_k}{n} \Bigg| = 0\right)\Bigg\}$$
$$= \gamma(0,s)^{\mathbb{N}} \Bigg\{\exists (\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty :\ \lim_{n\to\infty} \Bigg| \frac{\sum_{k=1}^n \beta_k}{n} \Bigg| = 0\Bigg\} = 1,$$

where $\gamma(0,s)$ denotes the Gaussian measure in $\mathbb{R}$ with the mean 0 and the variance $s^2 = \frac{\sigma^2}{2A_0} \times \left(1 - 2^{-2A_0 t_0}\right)$. The validity of the last equality is a direct consequence of Lemmas 2.4 and 2.5. □

Theorem 3.2 (Continue)**.** *Let* $\theta^* \in FD^{(0)}$. *For* $(Z_k)_{k\in\mathbb{N}} \in (\mathbb{C}[-l,l[)^{\mathbb{N}}$, *we put* $T\left((Z_k)_{k\in\mathbb{N}}\right) = \lim_{n\to\infty} T_n\left((Z_k)_{k\in\mathbb{N}}\right)$, *if the sequence* $\left(T_n\left((Z_k)_{k\in\mathbb{N}}\right)\right)_{n\to\infty}$ *is convergent and this limit belongs to the class* $FD^{(0)}$, *and* $T\left((Z_k)_{k\in\mathbb{N}}\right) = \theta^*$, *otherwise. Then* $T :\ (\mathbb{C}[-l,l[)^{\mathbb{N}} \longrightarrow FD^{(0)}$ *is an infinite-sample consistent estimate of a useful signal* $\theta \in FD^{(0)}$ *with respect to a family* $\mu_\theta^{\mathbb{N}}$, *provided that the condition*

$$\mu_\theta^{\mathbb{N}}\Big(\Big\{(Z_k)_{k\in\mathbb{N}} :\ (Z_k)_{k\in\mathbb{N}} \in \big(\mathbb{C}^{[0,+\infty[}[-l,l[\big)^{\mathbb{N}} \& T\left((Z_k)_{k\in\mathbb{N}}\right) = \theta\Big\}\Big) = 1$$

holds for each $\theta \in FD^{(0)}[-l,l[$.

Proof. For $\theta \in FD^{(0)}[-l,l[$, by the use of the result of Theorem 3.1 we get

$$\mu_\theta^{\mathbb{N}}\Big(\Big\{(Z_k)_{k\in\mathbb{N}} :\ (Z_k)_{k\in\mathbb{N}} \in \big(\mathbb{C}^{[0,+\infty[}[-l,l[\big)^{\mathbb{N}} \& T\left((Z_k)_{k\in\mathbb{N}}\right) = \theta\Big\}\Big)$$
$$\geq \mu_\theta^{\mathbb{N}}\Big(\Big\{(Z_k)_{k\in\mathbb{N}} :\ (Z_k)_{k\in\mathbb{N}} \in \big(\mathbb{C}^{[0,+\infty[}[-l,l[\big)^{\mathbb{N}} \& \lim_{n\to\infty} T_n\left\{(Z_k)_{k\in\mathbb{N}}\right) = \theta\Big\}\Big) = 1.$$ □

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(Received 18.07.2022)

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