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DYNAMICAL THERMOSTABILITY OF ORTHOTROPIC SHELLS OF REVOLUTION WITH AN ELASTIC FILLER

SERGO KUKUDZHANOV

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. Dynamical thermostability of closed orthotropic shells of revolution, close by their form to cylindrical ones, with an elastic filler and under the action of meridional stresses, external pressure and temperature, is investigated. We consider the shells of middle length whose midsurface generatrix is a parabolic function. The shells of positive and negative Gaussian curvature are also considered. Formulas for finding the lowest frequencies, critical loadings and boundaries of domains of dynamical instability depending on orthotropy parameters, Gaussian curvature, initial stressed state, temperature and amplitude of deviation from cylindrical form, are obtained.

In the present paper, we consider dynamical thermostability of closed orthotropic shells of revolution, close by their form to cylindrical ones with an elastic filler and under the action of meridional stresses, uniformly distributed over the shell ends, external pressure and temperature. We consider a light filler for which the effect of tangential stresses on the contact surface and inertia forces may be neglected. Temperature is uniformly distributed in the shell body. The shell is assumed to be thin and elastic. An elastic filler is modeled by a Winkler base, its extension due to heating is not taken into account. We investigate the shells of middle length in which the shape of generatrix of the median surface is a parabolic function. The shells of positive and negative Gaussian curvature are also considered. The boundary conditions at the wall-ends correspond to a free support allowing some radial displacement in the initial state.

When solving the questions under consideration, the focus was on identifying the most dangerous area of dynamical instability and the lowest frequencies, practically the most important. Formulas and universal curves of dependence of the lowest frequency, the shape of wave formation and boundaries of regions of dynamical instability on the Gaussian curvature, orthotropy parameters, temperature and amplitude of shell deviation from the cylinder, are obtained in dimensionless form. It is shown that the elastic orthotropy parameters affect significantly the lowest frequency and the principal area of dynamical instability. The degree of influence of orthotropy parameters under a separate and joint action of the above-mentioned external forces both on the lowest frequencies and on the boundaries of the region of dynamical instability, is shown.

1. We consider the shells whose middle surface is formed by the rotation of a quadratic parabola around the z-axis of the rectangular system of coordinates x, y, z with the origin in the midsegment of the rotation axis. It is assumed that the cross-section radius R of a middle surface is determined by the equality $R = r + \delta_0\left[1 - \xi^2(r/\ell)^2\right]$, where r is the end-wall section radius, δ_0 is a maximal deviation from cylindrical form (for $\delta_0 > 0$, the shell is convex and for $\delta_0 < 0$, it is concave), $L = 2\ell$ is the shell length, $\xi = z/r$.

We consider the shells of middle length [8] and it is assumed that

$$\left(\delta_0/r\right)^2 \ll 1, \quad \left(\delta_0/\ell\right)^2 \ll 1. \tag{1.1}$$

As the basic equations of oscillations we take the equations of the theory of shallow shells [7]. For orthotropic shells of middle length, the forms of oscillations, corresponding to the lowest frequencies,

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vary weakly in a longitudinal direction in comparison with the circumferential, therefore the relation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = w, \psi), \tag{1.2}$$

is valid; here, w and ψ are, respectively, the functions of radial displacement and stress. As a result, the system of equations for the shells under consideration reduces to the following equation [3, 6]:

$$\begin{aligned} &\varepsilon \frac{\partial^8 w}{\partial \varphi^8} + \frac{E_1}{E_2}\Big(\frac{\partial^4 w}{\partial \xi^4} + 4\,\delta \frac{\partial^4 w}{\partial \xi^2\,\partial \varphi^2} + 4\,\delta^2 \frac{\partial^4 w}{\partial \varphi^4}\Big) - t_1^0 \frac{\partial^6 w}{\partial \xi^2\,\partial \varphi^6} \\ &- t_2^0 \frac{\partial^6 w}{\partial \varphi^6} - 2\,s^0 \frac{\partial^6 w}{\partial \xi\,\partial \varphi^5} - \gamma \frac{\partial^4 w}{\partial \varphi^4} - \frac{\rho\, r^2}{E_2}\,\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial \varphi^4}\Big) = 0, \\ &\varepsilon = h^2/12\,r^2(1 - v_1 v_2), \quad \delta = \delta_0\, r/\ell^2, \quad t_i = T_i^0/E_2 h \quad (i = 1, 2), \\ &s^0 = S^0/E_2 h, \quad \gamma = \beta r^2/E_2 h, \end{aligned} \tag{1.3}$$

E_1, E_2, v_1, v_2 are, respectively, the elastic moduli and the Poisson coefficients in the axial and circumferential directions ($E_1 v_2 = E_2 v_1$); T_1^0, T_2^0 are meridional and circumferential normal forces of the initial state; S^0 is the shearing stress of the initial state; h is a shell thickness; ρ is density of the shell material; β is the “bed” coefficient of an elastic filler (characterizing a filler rigidity); φ is an angular coordinate; t is time.

The initial state is assumed to be momentless. On the basis of the corresponding solution and taking into account the reaction of the filler and also inequalities (1.1), we obtain the following approximate expressions:

$$\begin{aligned} &T_1^0 = P_1\Big[1 + \frac{\delta_0}{2}\Big(\xi^2 (r/\ell)^2 - 1\Big)\Big] + q\delta_0\big[\xi^2 (r/\ell)^2 - 1\big], \\ &T_2^0 = -2\,P_1 \delta_0 \frac{r}{\ell^2} - qr + \beta_0 r w_0, \quad S^0 = 0, \end{aligned} \tag{1.4}$$

where w_0 and β_0 are, respectively, the deflection and “bed” coefficient of the filler in the initial state, P_1 is meridional stress, q is external pressure.

In view of (1.2), we get

$$\Big|\xi^2 (r/\ell)^2 - 1\Big| \frac{\partial^2 w}{\partial \xi^2} \ll 2 (r/\ell)^2 \frac{\partial^2 w}{\partial \varphi^2}, \quad \frac{\delta_0}{r}\Big|\xi^2 (r/\ell)^2 - 1\Big| \frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}.$$

Therefore expressions (1.4) after substitution into equation (1.3) can be simplified. Thus they take the form

$$T_1^0 = P_1, \quad T_2^0 = -2\,P_1 \delta_0 r/\ell^2 - qr + w_0 \beta_0 r, \quad T_i = \sigma_i^0 h \quad (i = 1, 2). \tag{1.5}$$

In view of the fact that in the initial state, the shell deformation in circumferential direction ε_φ^0 is defined by the equalities

$$\varepsilon_\varphi^0 = \frac{\sigma_2^0 - v_2 \sigma_1^0}{E_2} + \alpha_2 T, \quad \varepsilon_\varphi^0 = -\frac{w_0}{r},$$

where α_2 is the coefficient of linear extension in the circumferential direction and T is temperature, we obtain

$$w_0 = \big(-\sigma_2^0 + v_2 \sigma_1^0\big)\frac{r}{E_2} - \alpha_2 T r. \tag{1.6}$$

Substituting expression (1.6) into (1.5), we get

$$\frac{T_2^0}{E_2 h} = \frac{\sigma_2^0}{E_2} = -\frac{qr}{E_2 h} - 2\,\frac{P_1}{E_2 h}\delta + v_2 \frac{\sigma_1^0}{E_2}\,\frac{\beta_0 r}{E_2 h} - \alpha_2 T \frac{\beta_0 r^2}{E_2 h} - \frac{\sigma_2^0}{E_2}\,\frac{\beta_0 r^2}{E_2 h}.$$

Introducing the notations

$$\begin{aligned} &E_1 = e_1 E, \quad E_2 = e_2 E, \\ &\frac{qr}{Eh} = \overline{q}, \quad \frac{P_1}{Eh} = -p, \quad \frac{\beta_0 r^2}{Eh} = \gamma_0, \quad 1 + \gamma_0 e_2^{-1} = g. \end{aligned}$$

expressions (1.5) take the form

$$-\frac{\sigma_1^0}{E_2} = -e_2^{-1}p, \quad -\frac{\sigma_1^0}{E_2} = \big(\overline{q} - 2\,p\delta + v_1 p\gamma_0 + \alpha_2 T\gamma_0\big)e_2^{-1}g^{-1}. \tag{1.7}$$

Note that due to the nearness of R to r, in the expressions for stresses (1.7) we admit $R \approx r$. Consequently, equation (1.3) takes the form

$$\varepsilon\frac{\partial^8 w}{\partial\varphi^8} + \frac{e_1}{e_2}\Big[\frac{\partial^4 w}{\partial\xi^4} + 4\,\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} + 4\big(\delta^2 + \gamma/4e_1\big)\frac{\partial^4 w}{\partial\varphi^4}\Big]$$
$$+\big(\overline{q} - 2p\delta + v_1 p\gamma_0 + \alpha_2 T\gamma_0\big)e_2^{-1}g^{-1}\frac{\partial^6 w}{\partial\varphi^6} + p\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4}e_2^{-1} + \frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big)e_2^{-1} = 0. \tag{1.8}$$

Let us consider first harmonic oscillations. The given boundary conditions of free support and equation (1.8) are satisfied by the solution

$$w = A_{mn}\,\cos\lambda_m\xi\,\sin n\varphi\,\cos\omega_{mn}t, \quad \lambda_m = m\pi r/2\ell \tag{1.9}$$
$$(m = 2i+1, \quad i = 0,1,2,\dots).$$

Substituting expression (1.9) into (1.8), to determine eigenfrequencies, we obtain the following equalities (in the sequel, for ω_{mn}, the indices m and n well be omitted)

$$\omega^2 = \frac{E}{\rho r^2}\Big[e_2\varepsilon n^4 + e_1\big(\lambda_m^4 n^{-4} + 4\,\delta\lambda_m^2 n^{-2} + 4(\delta^2 + \gamma/4e_1)\big) - p(\lambda_m^2 - 2\,\widetilde{\delta}n^2) - (\overline{q} + \alpha_2 T\gamma_0)g^{-1}n^2\Big].$$

Introduce the notations

$$\widetilde{\delta} = (\delta - 0,5\,v_2\gamma_0)g^{-1}, \quad \overline{\delta}^2 = \delta^2 + \gamma/4e_1, \quad \widetilde{q} = (\overline{q} + \alpha_2 T\gamma_0)g^{-1}.$$

Then

$$\omega^2 = \frac{E}{\rho r^2}\Big[e_2\varepsilon n^4 + e_1\big(\lambda_m^4 n^{-4} + 4\,\delta\lambda_m^2 n^{-2} + 4\,\overline{\delta}^2\big) - p\big(\lambda_m^2 - 2\,\widetilde{\delta}n^2\big) - \widetilde{q}n^2\Big]. \tag{1.10}$$

It is not difficult to see that for $p = 0$, $\delta > 0$ to the lowest frequency there corresponds the value $m = 1$. It can also be shown that this condition holds for $\delta < 0$ by taking into account inequalities (1.1) and the fact that $\omega^2 > 0$. Thus let us first consider the forms of oscillations under which we have only one half-wave ($m = 1$) over the shell length, while in the circumferential direction we have n waves. For the compression, we have $p > 0$, and for the tension, $p < 0$; q is a normal pressure which is assumed to be positive if it is exterior.

Let expression (1.10) be dimensionless. Towards this end, we introduce the following dimensionless values:

$$\begin{aligned}
&\Theta = (e_1/e_2)^{1/4}N, \quad N = n^2/n_0^2, \quad \overline{P} = P/\sqrt{e_1e_2}, \quad P = p/p_*,\\
&\widetilde{Q} = \frac{\widetilde{q}}{\overline{q}_{0*}}, \quad \widetilde{q} = (\overline{q} + \alpha_2 T\gamma_0)g^{-1},\\
&n_0^2 = \lambda_1\varepsilon^{1/4}, \quad p_* = 2\varepsilon^{1/2}, \quad \overline{q}_{0*} = 0,855(1 - v_1v_2)^{-3/4}\Big(\frac{h}{r}\Big)^{3/2}\frac{r}{L},\\
&\delta_*^v = (e_1/e_2)^{1/4}\delta_*, \quad \delta_* = \delta\varepsilon_*^{-1/2},\\
&\widetilde{\delta}_*^v = (e_1/e_2)^{1/4}\Big(\delta - \frac{1}{2}v_2\gamma_0\Big)\varepsilon_*^{-1/2}g^{-1},\\
&\overline{\delta}_*^{v^2} = (e_1/e_2)^{1/2}\Big(\delta_*^2 + \frac{\gamma_*}{4e_1}\Big) = \delta_*^{v^2} + (e_1e_2)^{-1/2}\frac{\gamma_*}{4}, \quad \gamma_* = \gamma\varepsilon_*^{-1},\\
&\omega_*^2 = 2\lambda_1^2\varepsilon^{1/2}\frac{E}{\rho r^2}, \quad \varepsilon_* = (1 - v_1v_2)^{-1/2}\frac{h}{r}\Big(\frac{r}{L}\Big)^2,
\end{aligned} \tag{1.11}$$

where p_*, $\overline{q}_{o*}$ and ω_* are, respectively, compression critical loading, critical pressure and the lowest frequency for a cylindrical isotropic shell of middle length [2, 8]. Thus equality (1.10) can be written in the following dimensionless form:

$$\begin{aligned}\omega^2(\Theta)/\omega_*^2 = 0,5\sqrt{e_1e_2}\Big[\Theta^2 + \Theta^{-2} + 2,37\,\delta_*^v\Theta^{-1} + 1,404\,\overline{\delta}_*^{v^2}\\ - 1,755\,e_1^{-1/4}e_2^{-3/4}\Theta\widetilde{Q} - 2\,\overline{P}\big(1 - 1,185\,\widetilde{\delta}_*^v\Theta\big)\Big].\end{aligned} \tag{1.12}$$

The lowest frequency (for $\omega^2(\Theta) > 0$) is determined by means of the condition $[\omega^2(\Theta)]' = 0$. As a result, we get

$$0,8775\,e_1^{-1/4}e_2^{-3/4}\widetilde{Q} - 1,185\,\widetilde{\delta}_*^v\overline{P} = \Theta - 1,185\,\delta_*^v\Theta^{-2} - \Theta^{-3}, \tag{1.13}$$

or

$$\Theta^4 - \big(0,8775\,e_1^{-1/4}e_2^{-3/4}\widetilde{Q} - 1,185\,\widetilde{\delta}_*^v\overline{P}\big)\Theta^3 - 1,185\,\delta_*^v\Theta - 1 = 0. \tag{1.13$'$}$$

For $\widetilde{Q} = \overline{P} = 0$, this implies that

$$\Theta^4 - 1,185\,\delta_*^v\Theta - 1 = 0.$$

This equation has been considered in [3] for an isotropic shell. Investigating the roots of this equation analogous to [3], we obtain

$$\begin{aligned}\Theta &= \sqrt{1 - 0,0876\,\delta_*^2(e_1/e_2)^{1/2}} + 0,2962\,\delta_*(e_1/e_2)^{1/4} \quad (\delta_* > 0),\\ \Theta &= \sqrt{1 - 0,0876\,\delta_*^2(e_1/e_2)^{1/2}} - 0,2962\,\delta_*(e_1/e_2)^{1/2} \quad (\delta_* < 0).\end{aligned} \tag{1.14}$$

In particular, for $\delta_* = 0$, we obtain the well-known formula for a cylindrical orthotropic shell of middle length $(n^2 = (e_1/e_2)^{1/4}\lambda_1\varepsilon^{-1/4})$ [6].

By Θ_0 we denote the value of Θ, which can be defined by (1.14). The obtained in such a way Θ_0 (for fixed e_1, e_2, δ_*) and substituting it into (1.12) (for $\widetilde{Q} = \overline{P} = 0$), we obtain the lowest frequency $\omega(\Theta_0)$. For clarity, we pass to the value $N = \Theta(e_1/e_2)^{1/4}$.

In Figures 1 and 2 we can see the graphs of dependence of $N_0 = n^2/n_0^2$ and $\omega(N_0)/\omega_*$ on the parameter δ_* for the cases $e_1 = e_2 = 1(0)$; $e_1 = 1$, $e_2 = 2(1)$, $e_1 = 2$, $e_2 = 1(2)$, the curves are denoted by $N_{0(i)}u(i)$, $i = 0, 1, 2$. It is not difficult to see that for convex shells ($\delta > 0$) the role of an elastic parameter in the axial direction is greater than in circumferential one, while for concave shells ($\delta < 0$), we have an opposite phenomenon.

For $\omega = 0$, $P = 0$, equality (1.12) yields

$$1,755\,e_1^{-1/4}\,e_2^{-3/4}\,\widetilde{Q} = \Theta + \Theta^{-3} + 2,37\,\delta_*^v\,\Theta^{-2} + 1,404\,\overline{\delta}_*^{v^2}\,\Theta^{-1}. \tag{1.15}$$

The lowest value $\widetilde{Q} > 0$ depending on Θ is realised for $\widetilde{Q}'_\Theta = 0$. Thus we obtain

$$\Theta^4 - 1,404\,\overline{\delta}_*^{v^2}\,\Theta^2 - 4,74\,\delta_*^v\,\Theta - 3 = 0.$$

The positive root of this equation $\Theta = \Theta_*$ ($N = N_*$) corresponds to a number of waves in the circumferential direction under which critical loading is realized upon the loss of the stability $\widetilde{Q}_*$. This equation for the orthotropic shell, analogous to the equation for isotropic one, has been considered in [3], where the expression for a positive root is given explicitly. Generalizing this result to the orthotropic case, we present the corresponding curves of dependence of N_* on δ_* for the cases $i = 0, 1, 2$ considered above. In Figure 1, these curves are denoted respectively by $N_*(i)$. The graphs of dependence of $\widetilde{Q}_*$ on δ_* for these cases are given in Figure 3.

Owing to equality (1.13) (for $P = 0$), it is not difficult to constract the curves $N(\widetilde{Q})$ realizing the lowest frequency for different values e_1, e_2, δ_*, γ_*, T. Towards this end, we fix these parameters and having the values of Θ from the interval $\Theta_0 \le \Theta \le \Theta_*$ (see [3]), we find the value of $\widetilde{Q}$ by formula (1.13). Substituting these values into formula (1.12), we obtain (for the case $P = 0$) the corresponding value of the lowest frequency. In Figure 4, we can see the curves of dependence of the lowest frequency ω/ω_* on $\widetilde{Q}$ (when $\gamma = 0$) for $\delta_* = 0,4$ and $\delta_* = -0,4$, when $i = 0, 1, 2$. The curves are denoted by $(0)_+, (1)_+, (2)_+$ and $(0)_-, (1)_-, (2)_-$, respectively.

Let us now consider general cases $\overline{P} \neq 0$, $\widetilde{Q} \neq 0$. As it has been shown above, the frequency is defined by equality (1.12). For $\omega = 0$, in view of (1.12), we obtain

$$1,755\, e_1^{-1/4}\, e_2^{-3/4}\, \widetilde{Q} = \Theta + \Theta^{-3} + 2,37\, \delta_*^v\, \Theta^{-2} + 1,404\, \delta_*^{v^2}\, \Theta^{-1} - 2\, \overline{P}\big(\Theta^{-1} - 1,185\, \delta_*^v\big).$$

The lowest value $\widetilde{Q} > 0$ depending on Θ is realized for $\widetilde{Q}'_\Theta = 0$. Thus we get

$$\Theta^4 + c\,\Theta^2 + d\,\Theta + e = 0, \quad c = 2\,\overline{P} - 1,404\,\delta_*^{v^2}, \quad d = -4,74\,\delta_*^v, \quad e = -3.$$

The roots of the above equation coincide with the roots of the two quadratic equations which are given explicitly for an isotropic shell in [3]. Generalizing our investigation of these roots to an orthotropic case, we obtain

$$\Theta_{1,2} = \sqrt{\sqrt{3} + 0,234\big(\delta_*^{v^2} + 3/4e_1\,\gamma_*^v\big) - \overline{P}} \pm 0,684\,|\delta_*^v|,$$

where the index "1" corresponds to $\delta_* > 0$, and the index "2" corresponds to $\delta_* < 0$. Taking into account that Θ in an expanded form $\Theta = (e_1/e_2)^{1/4}\, n^2/\lambda_1\varepsilon^{-1/4}$, we arrive at

$$\begin{aligned} n_{1,2}^2 = (e_1/e_2)^{1/4}\big\{&\big(\sqrt{3} + 0,234(e_1/e_2)^{1/2}\varepsilon^{-1/2}\big[(\delta_0/\ell)^2 + 3/4\,\gamma/e_1\,(\ell/r^2\big] - \overline{P}\big)^{1/2} \\ &\pm 0,735(e_1/e_2)^{1/4}\,\varepsilon^{-1/4}\,|\delta_0|/\ell\big\}\lambda_1\,\varepsilon^{-1/4}. \end{aligned} \tag{1.16}$$

In particular, for $\delta_* = p = \gamma = 0$, we obtain the well-known formula for a critical wave number of the cylindrical shell of middle length $n^2 = (e_1/e_2)^{1/4}\lambda_1\,\varepsilon^{-1/4}\sqrt{3}$ [5]. It can be easily noticed that under the action of contractive forces a number of critical waves in circumferential direction decreases, while under the action of tensile forces this number of critical waves increases.

Let us now consider equation (1.13′) and write it in the form

$$\begin{gathered} \Theta^4 + b\,\Theta^3 + d\,\Theta + e = 0, \quad b = 1,185\,\delta_*^v\,\overline{P} - 0,8775\,\overline{Q}, \\ d = -1,185\,\delta_*^v, \quad e = -1, \quad \overline{Q} = \widetilde{Q}\,e_1^{-1/4}\,e_2^{-3/4}. \end{gathered} \tag{1.17}$$

Conducting a study of roots of that equation, similar to that carried out in [3], we find that the positive roots of equation (1.17) take the form

$$\begin{aligned} \Theta_1 &= \Big[1 + 0,1755\,\widetilde{\delta}^{v^2}\,\overline{P}\,M_1\big(1 - \overline{P}^2\,M_1^2\big) - 0,0877\,\widetilde{\delta}_*^{v^2}\big(1 + 2\,\overline{P}\,M_1 - 2\,\overline{P}^2\,M_1^2\big)\Big]^{1/2} \\ &\quad + 0,2962\,\widetilde{\delta}_*^v(1 - \overline{P}\,M_1) \quad (\delta_* > 0), \\ \Theta_2 &= \Big[1 + 0,1755\,\widetilde{\delta}^{v^2}\,\overline{P}\,M_2\big(1 - \overline{P}^2\,M_2^2\big) - 0,0877\,\widetilde{\delta}_*^{v^2}\big(1 + 2\,\overline{P}\,M_2 - 2\,\overline{P}^2\,M_2\big)\Big]^{1/2} \\ &\quad - 0,2962\,\widetilde{\delta}_*^v(1 - \overline{P}\,M_2) \quad (\delta_* < 0), \\ M_1 &= 1 - 0,7405\,\overline{Q}/\delta_*^v\,\overline{P}, \qquad M_2 = 1 + 0,7405\,\overline{Q}/|\delta_*^v|\,\overline{P}. \end{aligned}$$

Substituting the obtained expression for Θ (for fixed $\overline{P}$, Q, γ^v) into formula (1.12), we obtain the lowest value for dimensionless frequency ω/ω_*.

Next, consider the value $m > 1$. Using notations (1.11), formula (1.10) can be represented as follows:

$$\begin{aligned} \omega^2/\omega_*^2 = 0,5\sqrt{e_1 e_2}\,m^2\Big[&\overline{\Theta}^2 + \overline{\Theta}^{-2} + 2,37\,\delta_*^v\,\overline{\Theta}^{-1}\,m^{-1} + 1,404\,\overline{\delta}_*^{v^2}\,m^{-2} \\ &-2\overline{P}\big(1 - 1,185\,\overline{\delta}_*\,\overline{\Theta}\,m^{-1}\big) - 1,755\,e_1^{-1/4}\,e_2^{-3/4}\,\overline{\Theta}\,\widetilde{Q}\,m^{-1}\Big], \quad \overline{\Theta} = \Theta/m. \end{aligned} \tag{1.18}$$

Let us consider the expression for finding critical loading $\widetilde{Q} > 0$. The right-hand side in (1.18) vanishes for

$$\begin{aligned} 1,755\,e_1^{-1/4}\,e_2^{-3/4}\,\widetilde{Q} = m\big(&\overline{\Theta} + \overline{\Theta}^{-3} - 2\,\overline{P}\,\Theta^{-1}\big) + 2,37\,\delta_*^v\,\Theta^{-2} \\ &+ 1,404\,\overline{\delta}_*^v\,\Theta^{-1}\,m^{-1} + 2,37\,\overline{P}\,\widetilde{\delta}_*^v. \end{aligned} \tag{1.19}$$

The value $\overline{\Theta}$ realizing the lowesr value $\widetilde{Q}$ (for fixed m) is determined by a positive root of the equation

$$\overline{\Theta}^4 + \big(2\overline{P} - 1,404\,\overline{\delta}_m^{v^2}\big)\overline{\Theta}^2 - 4,74\,\delta_m^v\,\overline{\Theta} - 3 = 0, \quad \delta_m^v = \delta_*^v/m.$$

Analogously to the above-said (replacing δ_*^v by δ_m^v), we have

$$\begin{aligned}\overline{\Theta} &= \big[\sqrt{3} + 0,234\big(\delta_*^{v^2} + 3/4\gamma_*^v\big)m^{-2} - \overline{P}\big]^{1/2} + 0,684\,\frac{\delta_*^v}{m} \quad (\delta_*^v > 0),\\ \overline{\Theta} &= \big[\sqrt{3} + 0,234\big(\delta_*^{v^2} + 3/4\gamma_*^v\big)m^{-2} - \overline{P}\big]^{1/2} - 0,684\,\frac{|\delta_*^v|}{m} \quad (\delta_*^v < 0).\end{aligned} \tag{1.20}$$

Substituting (1.20) into formula (1.19) (for fixed m, e_1, e_2, δ_*^v, γ_*^v, $\overline{P}$), we obtain the corresponding value $\widetilde{Q}_{kp}$.

Consider now expression (1.18). The lowest value ω^2 depending on $\overline{\Theta}$ (for fixed m) is determined from the condition $[\omega^2]'_{\overline{\Theta}} = 0$. Thus we obtain

$$\begin{aligned}&\overline{\Theta}^4 + b\,\overline{\Theta}^3 + d\,\overline{\Theta} + e = 0, \quad b = 1,185\,\overline{P}\,\widetilde{\delta}_m^v - 0,8775\,\overline{\Theta}_m, \quad d = -1,185\,\delta_m^v, \quad e = 1,\\ &\overline{Q}_m = \overline{Q}/m, \quad \delta_m^v = \delta_*^v/m, \quad \overline{Q} = e_1^{-1/4}\,e_2^{-3/4}\,\widetilde{Q}.\end{aligned}$$

This equation is the same as equation (1.17), where we have replace δ_*^v by δ_m^v, $\widetilde{\delta}_*^v$ by $\widetilde{\delta}_m^v$ and $\overline{Q}$ by $\overline{Q}_m$. So, we obtain

$$\begin{aligned}\overline{\Theta} &= \Big[1 + 1,765\,\widetilde{\delta}_m^{v^2}\,\overline{P}\,M_1(1 - \overline{P}^2\,M_1^2) - 0,08775\,\widetilde{\delta}_m^{v^2}(1 + 2\overline{P}M_1\\ &\quad - 2\,\overline{P}^2\,M_1^2)\Big]^{1/2} + 0,2962\,\delta_m^v(1 - \overline{P}\,M_1) \quad (\delta_* > 0),\\ \overline{\Theta} &= \Big[1 + 1,755\,\widetilde{\delta}_m^{v^2}\,\overline{P}\,M_2(1 - \overline{P}^2\,M_2^2) - 0,08775\,\widetilde{\delta}_m^{v^2}(1 + 2\overline{P}\,M_2\\ &\quad - 2\,\overline{P}^2 M_2)\Big]^{1/2} - 0,2962\,|\delta_m^v|(1 - \overline{P}\,M_2) \quad (\delta_* < 0),\end{aligned} \tag{1.21}$$

where $M_{1,2} = 1 \mp \big(0,7405\,\overline{Q}/|\delta_*^v|\overline{P}\big)$; the index "1" corresponds to $\delta_* > 0$ and the index "2" corresponds to $\delta_* < 0$.

Substituting (1.21) into formula (1.18) (for fixed m, e_1, e_2, δ_*^v, γ_*^v, $\overline{P}$, $\overline{Q}$), we obtain the values of the lowest frequecies for different fixed m.

The carried out calculations show that the orthotropy parameters e_1, e_2 affect significantly the lowest frequencies of the prestressed shell, whereas the influence of these parameters on the higher frequencies is practically insignificant. At the same time, the influence of meridional loading is significant both on the lowest and on the higher frequencies.

2. Let us now consider the case for

$$q_1 = q_0 + q_t\cos\Omega t, \quad P_1 = P_0 + P_t\cos\Omega t, \quad T_1 = T_0 + T_t\cos\Omega t.$$

A solution of equation (1.8) will be sought in the form

$$w = f_{mn}(t)\,\cos\lambda_m\xi\,\sin n\varphi, \quad \lambda_m = \frac{m\pi r}{2\,\ell}, \quad m = 2i + 1 \quad (i = 0, 1, 2, \dots).$$

Substituting this solution into (1.8) and requiring that the latter be satisfied for any ξ and φ, we have

$$\begin{aligned}\frac{d^2 f_{mn}}{dt^2} + \frac{E}{\rho r^2}\Big[&e_2\varepsilon n^4 + e_1\big(\lambda_m^4 n^{-4} + 4\,\delta\,\lambda_m^2 n^{-2} + 4(\delta^2 + \gamma/4e_1)\big)\\ &- p(t)(\lambda_m^2 - 2\,\widetilde{\delta}\,n^2) - \big(q_1(t) + \alpha_2\,T_1(t)\,\gamma_0\big)g^{-1}n^2\Big]f_{mn} = 0.\end{aligned} \tag{2.1}$$

Frequencies of eigenoscillations of orthotropic shell (for $q_1 = q_0$, $P_1 = P_0$, $T_1 = T_0$) are determined from equation (2.1) by putting $f_{mn} = C\sin\omega_{mn}t$, and they are expressed by formula (1.10). Since equation (2.1) is identical for all forms of oscillations, the indices m, n may be omitted.

Analogously to the above, we introduce dimensionless values (1.11) and $\overline{\Theta} = \Theta/m$. We write equation (2.1) in the form

$$\frac{d^2 f}{dt^2} + 0,5\, m^2\, \overline{\omega}_*^2 \Big\{ \overline{\Theta}^2 + \overline{\Theta}^{-2} + 2,37\, \delta_m^v \overline{\Theta}^{-1} + 1,404\, \overline{\delta}_m^{v^2} - 2(\overline{P}_0 + \overline{P}_t \cos\Omega t)(1 - 1,185\, \delta_m^v\, \overline{\Theta})$$
$$-1,755\big[(\overline{\Theta}_0 + \alpha_2\,\gamma\,\widetilde{T}_0) + (\overline{\Theta}_t + \alpha_2\,\gamma\,\widetilde{T}_t)\cos\Omega t\big] g^{-1}\overline{\Theta}\, m^{-1}\Big\} f = 0. \qquad (2.2)$$

where

$$\overline{Q}_i = e_1^{-1/4} e_2^{-3/4} Q_i, \quad \overline{P}_i = e_1^{-1/2} e_2^{-1/2} P_i,$$
$$\widetilde{T}_i = e_1^{-1/4} e_2^{-3/4}\, \overline{T}_i \quad (i = 0, t), \quad \overline{\omega}_*^2 = \sqrt{e_1 e_2 p}\,\omega_*^2,$$
$$Q_i = \overline{q}/\overline{q}_{0*}, \quad P_i = p_i/p_{0*}, \quad \overline{T}_i = T_i/\overline{q}_{0*}.$$

Further, we introduce the notation

$$\widetilde{Q}_i = e_1^{-1/4} e_2^{-3/4}\big(Q_i + \alpha_2 \gamma\, \overline{T}_i\big) g^{-1}.$$

and reduce equation (2.2) to the standard form of the Mathieu equation

$$\frac{d^2 f}{dt^2} + \omega^2(\overline{\Theta})\big[1 - 2\,\mu(\overline{\Theta})\,\cos\Omega t\big] = 0, \qquad (2.3)$$

$$\omega^2(\overline{\Theta}) = \omega_0^2(\overline{\Theta})\big[1 - M_0(\overline{\Theta}\big], \quad \omega_0^2(\overline{\Theta}) = 0,5\,\omega_*^2 m^2 D(\overline{\Theta}), \qquad (2.4)$$

$$\omega_*^2 = 2\,\lambda_1 \varepsilon^{1/2}\sqrt{e_1 e_2}\,\frac{E}{\rho r^2},$$

$$D(\overline{\Theta}) = \Theta^2 + \overline{\Theta}^{-2} + 2,37\,\delta_m^v \overline{\Theta}^{-1} + 1,404\,\overline{\delta}_m^{v^2}, \quad \overline{\delta}_m^{v^2} = \frac{\overline{\delta}_x^{v^2}}{m^2} = \Big(\delta_m^{v^2} + \sqrt{e_1 e_2}\,\frac{\gamma_*}{4}\Big) m^{-2},$$

$$\mu(\overline{\Theta}) = \frac{M_t(\overline{\Theta})}{2[1 - M_0]}, \quad M_0(\overline{\Theta}) = \frac{P_0}{P(\overline{\Theta})} + \frac{\overline{Q}_0}{\widetilde{Q}(\overline{\Theta})}, \quad M_t(\overline{\Theta}) = \frac{P_t}{P(\overline{\Theta})} + \frac{\overline{Q}_t}{\widetilde{Q}(\overline{\Theta})}, \qquad (2.5)$$

$$P(\overline{\Theta}) = \frac{D(\overline{\Theta})}{2(1 - 1,185\,\delta_m^v \overline{\Theta})}, \quad \widetilde{Q}(\overline{\Theta}) = \frac{D(\overline{\Theta})}{1,755\, e_1^{-1/4} e_2^{-3/4} m^{-1}\overline{\Theta}}. \qquad (2.6)$$

If $\frac{\overline{P}(t)}{\widetilde{Q}(t)} = \chi$, then

$$\frac{\overline{P}_i}{\widetilde{Q}_i} = \chi, \quad \widetilde{Q}_i = e_1^{-1/4}\, e_2^{-3/4}\big(Q_i + \alpha_2\,\gamma\, T_i\big) g^{-1}, \quad i = 0, t$$

and we get

$$M_0 = \frac{\widetilde{Q}_0}{\widetilde{Q}_c}, \quad M_t = \frac{\widetilde{Q}_t}{\widetilde{Q}_c}, \quad \widetilde{Q}_c = \frac{D(\overline{\Theta})}{2\chi(1 - 1,185\,\widetilde{\delta}_m^v \overline{\Theta}) + 1,755\, e_1^{-1/4} e_2^{-3/4} m^{-1}\overline{\Theta}}.$$

The value μ is usually called an excitation coefficient. The solution of equation (2.3) has been studied in a cycle of works, where it was mentioned that for the certain relations between μ, Ω, ω and $t \to \infty$ a solution of equation (2.3) will infinitely grow in areas of instability. Generalizing the results of [1] to the shell under consideration, we will give below the following formulas. To show the influence of orthotropy parameters and temperature on the location of domains of dynamical instability, we first of all consider the case for $P_t \to 0$ ($\mu \to 0$). In this connection, we find that these domains are located in the vicinity of frequencies

$$\Omega_* = 2\omega(\overline{\Theta})/k.$$

Depending on the number k, we distinguish the first, second, third and so on domains of dynamical instability. The domain of instability ($k = 1$) lying in the vicinity of $\Omega_* = 2\omega(\overline{\Theta})$, when $\omega(\overline{\Theta})$ takes the lowest value, is the most dangerous and hence is of the greatest practical value. This domain is called a principal domain of dynamical instability.

For P_t, other than zero, the boundaries of the principal domain of instabiliti takes the following form:

$$\Omega_* = 2\,\omega(\overline{\Theta})\sqrt{1 \pm \mu(\overline{\Theta})}.$$

Taking into account resistance forces, proportional to the first derivative of displacement in time (with damping coefficient ε), the formula for determining the boundaries of the principal domain of instability takes the form

$$\Omega_* = 2\,\omega(\overline{\Theta})\sqrt{1 \pm \sqrt{\mu^2(\overline{\Theta}) - (\Delta/\pi)^2}}, \quad \Delta = 2\,\pi\varepsilon/\omega(\overline{\Theta}), \tag{2.7}$$

where the terms involving higher degrees Δ/π are rejected, taking into account the fact that the damping factor Δ is usually very small compared with unity.

The values of $\omega(\overline{\Theta})$, $P(\overline{\Theta})$, $\mu(\overline{\Theta})$ are determined by virtue of formulas (2.4), (2.5), (2.6), where m and $\overline{\Theta}$ correspond to the lowest value of $\omega(\overline{\Theta})$. For $m = 1$, owing to formula (1.18), we have $\overline{\Theta} = (e_2/e_1)^{1/4}N$. It follows from formula (2.7) that the minimal value of the excitation coefficient (critical) for which undamped oscillations are still possible, is determined by the equality

$$\mu_{*1} = \Delta/\pi.$$

For the boundary of the second domain of instability ($k = 2$), the formula

$$\Omega_* = \omega(\overline{\Theta})\sqrt{1 + \mu^2(\overline{\Theta}) \pm \sqrt{\mu^4(\overline{\Theta}) - (\Delta/\pi)^2\big[1 - \mu^2(\overline{\Theta})\big]}}$$

holds.

In the given case, the critical value of the excitation coefficient is determined approximately by the equality $\mu_{*2}(\overline{\Theta}) = (\Delta/\pi)^{1/2}$. Analogously, generalizing the results obtained in [1], we can present likewise formulas for the third domain of dynamical instability which is practically rarely realized.

Relying on the formulas given above, it is not difficult to determine the intervals of change of exciting frequencies (depending on e_1, e_2, δ_*, P_0, P_t, Q_0, Q_t, T_0, T_t) falling into the domains of dynamical instability. The formulas obtained above for the problems under consideration make it quite easy to determine how significantly the parameters of orthotropy, temperature and acting loading can affect the boundaries of domains of dynamical instability.

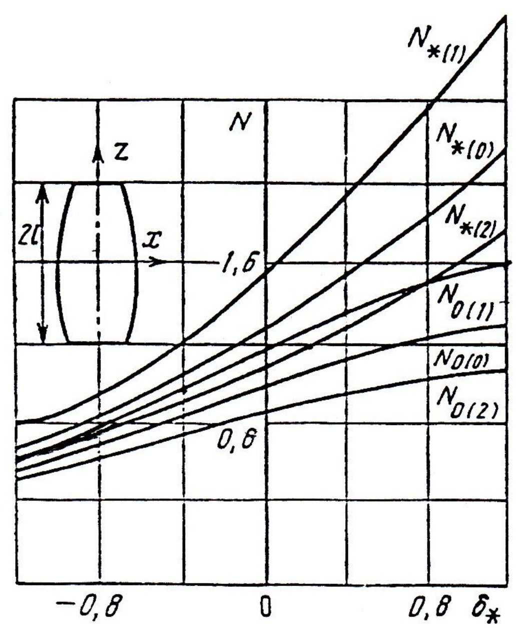

Figure 1

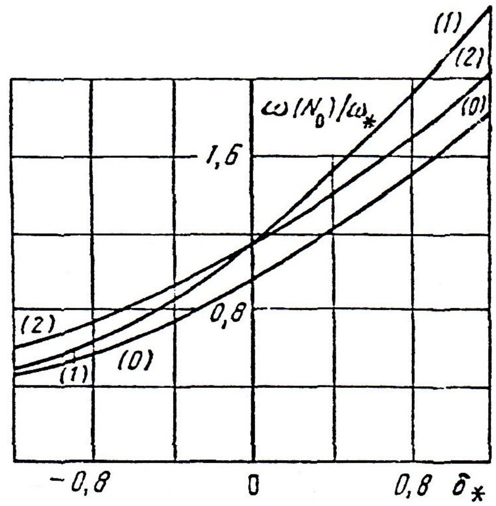

FIGURE 2

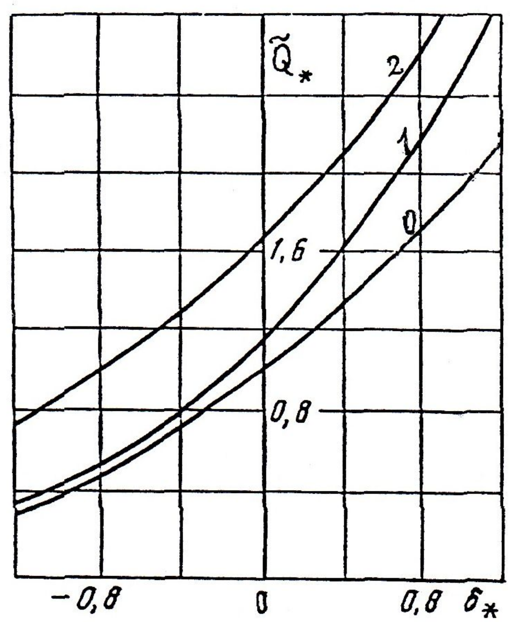

FIGURE 3

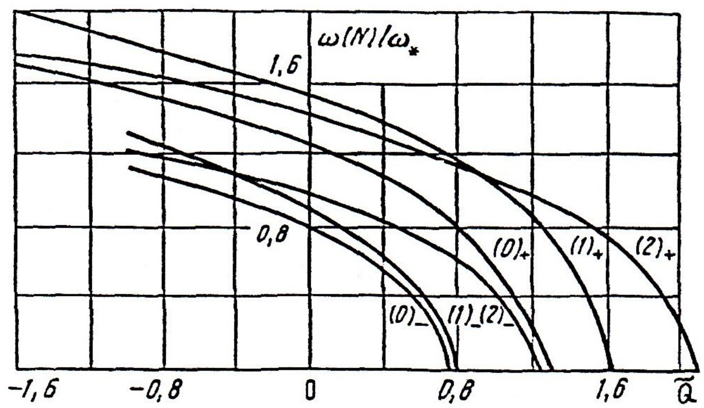

FIGURE 4

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A. Razmadze Mathematical Institute of Iv. Javakhishvili, Tbilisi State University 2, Merab Aleksidze Lane, Tbilisi 0193, Georgia