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EXPLICIT SOLUTION OF DIRICHLET AND NEUMANN BVPS OF THE THEORY OF THERMOELASTICITY OF MICROSTRETCH MATERIALS WITH MICROTEMPERATURES AND MICRODILATATIONS FOR A BALL

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The paper deals with the Dirichlet and Neumann type BVPs of statics of the thermoelasticity theory for homogeneous isotropic microstretch elastic ball with microtemperatures and microdilatations. Explicit solutions of the BVPs in the form of absolutely and uniformly convergent series are constructed with the help of the general representation formulas for solutions of the corresponding system of differential equations describing the model.

1. Introduction

The mathematical model of a linear theory of thermodynamics for microstretch elastic solids with microtemperatures has been proposed by D. Ieşan in [5], who obtained the field equations of the linear theory of thermoelasticity with microtemperatures and studied the uniqueness theorem in the dynamical theory of anisotropic bodies and the continuous dependence of solutions upon initial data and body loads.

The Dirichlet and Neumann boundary value problems (BVP) for the systems of pseudo-oscillation and differential equations of statics for homogeneous isotropic elastic solids are investigated in [3, 4]. Applying the potential method and the theory of singular integral equations, the uniqueness and existence theorems of solutions to the Dirichlet and Neumann boundary value problems for general domains of arbitrary shape in appropriate function spaces are proved.

Note that in [4], the general representation formulas of solution of the homogeneous system of statics differential equations are constructed by means of harmonic and metaharmonic scalar functions. Namely, it is proved that the field vector can be expressed linearly by four harmonic and seven metaharmonic scalar functions. This representation formulas are very useful to investigate boundary value problems for domains with a concrete geometry, in particular, for domains bounded by spherical surfaces (see [1, 2, 6, 9–15] and references therein).

The present paper deals with the Dirichlet and Neumann type BVPs of statics of the thermoelasticity theory for homogeneous isotropic microstretch elastic ball with microtemperatures and microdilatations. With the help of the general representation formulas we construct explicit solutions of BVPs in the form of absolutely and uniformly convergent series.

2. Basic Differential Equations and Boundary Value Problems

The homogeneous system of differential equations of statics of the thermoelasticity theory of microstretch materials with microtemperatures and microdilatations in the case of isotropic homogeneous

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bodies has, according to [5], the form

$$\begin{aligned}(\mu+\varkappa)\Delta u+(\lambda+\mu)\operatorname{grad}\operatorname{div}u+\varkappa\operatorname{rot}\omega+\mu_0\operatorname{grad}v-\beta_0\operatorname{grad}\vartheta &=0,\\ \varkappa\operatorname{rot}u+(\gamma\Delta-2\varkappa)\omega+(\alpha+\beta)\operatorname{grad}\operatorname{div}\omega-\mu_1\operatorname{rot}w &=0,\\ (\varkappa_6\Delta-\varkappa_2)w+(\varkappa_4+\varkappa_5)\operatorname{grad}\operatorname{div}w-\varkappa_3\operatorname{grad}\vartheta &=0,\\ -\mu_0\operatorname{div}u-\mu_2\operatorname{div}w+(a_0\Delta-\eta)v+\beta_1\vartheta &=0,\\ \varkappa_1\operatorname{div}w+\varkappa_7\Delta\vartheta &=0,\end{aligned}\tag{2.1}$$

where α, β, γ, λ, μ, $\varkappa$, η, β_0, β_1, μ_0, μ_1, μ_2, a, b, a_0, $\varkappa_j$, $j=1,2,3,4,5,6,7$, are the real constants characterizing the mechanical and thermal properties of the body, Δ is the Laplace operator, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $\omega=(\omega_1,\omega_2,\omega_3)^\top$ is the microrotation vector, $w=(w_1,w_2,w_3)^\top$ is the microtemperature vector, v is the microdilatation function, ϑ is the temperature, measured from a fixed absolute temperature T_0 ($T_0>0$); the symbol $(\cdot)^\top$ denotes transposition operation.

We assume that the constitutive coefficients satisfy the following inequalities [5]

$$\begin{aligned}&a_0>0,\quad \mu>0,\quad 3\lambda+2\mu>0,\quad \varkappa>0,\quad (3\lambda+2\mu+\varkappa)\eta-3\mu_0^2\geq 0,\\ &\varkappa_6\pm\varkappa_5\geq 0,\quad 3\varkappa_4+\varkappa_5+\varkappa_6\geq 0,\quad \varkappa_7>0,\quad (\varkappa_1+\varkappa_3T_0)^2\leq 4T_0\varkappa_2\varkappa_7,\\ &\gamma+\beta\geq 0,\quad 3\alpha+\beta+\gamma\geq 0,\quad a_0(\gamma-\beta)-2b_0^2\geq 0.\end{aligned}\tag{2.2}$$

Let Ω^+ be a ball, whose boundary is a sphere $\partial\Omega^+$ of radius R and centered at the origin,

$$\Omega^+:=\{x:x\in\mathbb{R}^3,\ \ |x|<R\},\quad \partial\Omega^+=\{x:x\in\mathbb{R}^3,\ \ |x|=R\}.$$

Definition 2.1. A vector function $U=(u,\omega,w,v,\theta)^\top$ is said to be regular in a domain $\Omega^+\subset\mathbb{R}^3$, if $U\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})$.

Problem $(I)^+$ (Dirichlet problem). Find a regular vector $U=(u,\omega,w,v,\vartheta)^\top$ satisfying the system of differential equations (2.1) in Ω^+ and the boundary condition

$$\{U(z)\}^+=F(z),\quad z\in\partial\Omega^+.\tag{2.3}$$

Problem $(II)^+$ (Neumann problem). Find a regular vector $U=(u,\omega,w,v,\vartheta)^\top$ satisfying the system of differential equations (2.1) in Ω^+ and the boundary condition

$$\{P(\partial,n)U(z)\}^+=f(z),\ \ z\in\partial\Omega^+,\tag{2.4}$$

where the boundary operator $P(\partial,n)$ has the form [5]

$$P(\partial,n):=\begin{bmatrix}P^{(1)}(\partial,n) & P^{(2)}(\partial,n) & [0]_{3\times3} & \mu_0 n & -\beta_0 n\\ [0]_{3\times3} & P^{(3)}(\partial,n) & P^{(4)}(\partial,n) & -b_0S^\top(\partial,n) & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times3} & P^{(5)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & b_0S(\partial,n) & -\mu_2 n^\top & a_0\partial_n & 0\\ [0]_{1\times3} & [0]_{1\times3} & \varkappa_1 n^\top & 0 & \varkappa_7\partial_n\end{bmatrix}_{11\times11},\tag{2.5}$$

$$P^{(l)}(\partial,n)=\Big[P^{(l)}_{kj}(\partial,n)\Big]_{3\times3},\quad l=1,2,3,4,5,$$

$$P^{(1)}_{kj}(\partial,n)=(\mu+\varkappa)\delta_{kj}\partial_n+\lambda n_k\partial_j+\mu n_j\partial_k,\qquad P^{(2)}_{kj}(\partial,n)=\varkappa\sum_{p=1}^3\varepsilon_{pjk}n_p,$$

$$P^{(3)}_{kj}(\partial,n)=\gamma\delta_{kj}\partial_n+\alpha n_k\partial_j+\beta n_j\partial_k,\qquad P^{(4)}_{kj}(\partial,n)=\mu_1\sum_{p=1}^3\varepsilon_{kjp}n_p,$$

$$P^{(5)}_{kj}(\partial,n)=\varkappa_6\delta_{kj}\partial_n+\varkappa_4 n_k\partial_j+\varkappa_5 n_j\partial_k,\qquad S(\partial,n)=(\partial S_1,\partial S_2,\partial S_3),$$

$$\partial S_1=n_2\partial_3-n_3\partial_2,\qquad \partial S_2=n_3\partial_1-n_1\partial_3,\qquad \partial S_3=n_1\partial_2-n_2\partial_1,$$

ε_{kjp} is the permutation (Levi–Civita) symbol, $\partial_k = \partial/\partial x_k$, $n = (n_1, n_2, n_3)^\top$ is the outward unit normal to $\partial\Omega^+$, $\partial_n = \partial/\partial n$ is the normal derivative.

The generalized thermostress vector $P(\partial, n)\, U(z)$ has the form

$$P(\partial,n)U = \big(T^{(1)}(\partial,n)U,\ T^{(2)}(\partial,n)U,\ T^{(3)}(\partial,n)U,\ T^{(4)}(\partial,n)U,\ T^{(5)}(\partial,n)U\big)^\top,$$

where

$$\begin{aligned}
T^{(1)}(\partial,n)U =&(2\mu+\varkappa)\frac{\partial u}{\partial n} + \lambda n \operatorname{div} u + \mu[n\times \operatorname{rot} u] + \varkappa[n\times\omega] + (\mu_0 v - \beta_0\vartheta)n,\\
T^{(2)}(\partial,n)U =&(\beta+\gamma)\frac{\partial \omega}{\partial n} + \alpha\, n \operatorname{div}\omega + \beta[n\times \operatorname{rot}\omega] - \mu_1[n\times w] - b_0[n\times \operatorname{grad} v],\\
T^{(3)}(\partial,n)U =&(\varkappa_5+\varkappa_6)\frac{\partial w}{\partial n} + \varkappa_4 n \operatorname{div} w + \varkappa_5[n\times \operatorname{rot} w],\\
T^{(4)}(\partial,n)U =&a_0\frac{\partial v}{\partial n} - \mu_2\, n\cdot w + b_0 n\cdot \operatorname{rot}\omega,\\
T^{(5)}(\partial,n)U =&\varkappa_1\, n\cdot w + \varkappa_7\frac{\partial\vartheta}{\partial n};
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
F =&(F^{(1)}, F^{(2)}, F^{(3)}, F_4, F_5)^T,\quad F^{(i)} = (F_1^{(i)}, F_2^{(i)}, F_3^{(i)})^T,\quad i=1,2,3,\\
f =&(f^{(1)}, f^{(2)}, f^{(3)}, f_4, f_5)^T,\quad f^{(i)} = (f_1^{(i)}, f_2^{(i)}, f_3^{(i)})^T,\quad i=1,2,3;
\end{aligned}$$

the vectors $f^{(i)}$, $F^{(i)}$, $i = 1,2,3$, and the functions f_i, F_i, $i = 4,5$ are given on the boundary $\partial\Omega^+$; we recall that the central dot denotes the real scalar product $a\cdot b = \sum_{k=1}^N a_k b_k$ for $a, b \in \mathbb{R}^N$, and $c \times d$ denotes the cross product of two vectors $c, d \in \mathbb{R}^3$.

The following uniqueness theorems hold [4].

Theorem 2.2. *The homogeneous Dirichlet boundary value problem* $(I)_0^+$ $(F = 0)$ *has only the trivial solution in the class of regular vector-functions.*

Theorem 2.3. *A solution* $U = (u, \omega, w, v, \vartheta)^\top$ *to the homogeneous Neumann boundary value problem* $(II)_0^+$ $(f = 0)$ *is defined modulo the vector*

$$U^{(0)}(x) = (1/3\, p'Cx + [a\times x] + b,\ a,\ 0,\ q'C,\ C)^T, \tag{2.7}$$

where a *and* b *are arbitrary three-dimensional real constant vectors,* C *is an arbitrary real constant, and*

$$p' = \frac{\beta_0\eta - \mu_0\beta_1}{\alpha_0'\eta - \mu_0^2},\quad q' = \frac{\alpha_0'\beta_1 - \mu_0\beta_0}{\alpha_0'\eta - \mu_0^2},\quad \alpha_0' = \frac{3\lambda + 2\mu + \varkappa}{3}. \tag{2.8}$$

3. Solution of The Boundary Value Problems

Here, we consider the Neumann boundary value problem $(II)^+$. We seek the solution by the formula [4]

$$\begin{aligned}
u(x) =&a_4 \operatorname{grad}\big(r^2\Phi_0(x)\big) + \operatorname{grad}\Phi_1(x) - a_5\operatorname{grad}\Big(r^2\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2(x)\Big)\\
&+ \operatorname{rot}\operatorname{rot}\big(xr^2\Phi_2(x)\big) + \frac{a_3}{\lambda_1^2}\operatorname{grad}\Phi_4(x) - \frac{\mu_0}{\lambda_0\lambda_4^2}\operatorname{grad}\Phi_5(x)\\
&+ \frac{\varkappa\mu_1}{\gamma(\mu+\varkappa)(\lambda_3^2-\lambda_5^2)}\operatorname{rot}\big(x\Phi_6(x)\big) - \frac{\varkappa\mu_1}{\gamma(\mu+\varkappa)(\lambda_3^2-\lambda_5^2)\lambda_3^2}\operatorname{rot}\operatorname{rot}\big(x\Phi_7(x)\big)\\
&- \frac{\varkappa}{(\mu+\varkappa)\lambda_5^2}\operatorname{rot}\operatorname{rot}\big(x\Phi_8(x)\big) - \frac{\varkappa}{(\mu+\varkappa)\lambda_5^2}\operatorname{rot}\big(x\Phi_9(x)\big),
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\omega(x) =& -\operatorname{rot}\Big(x\Big(2r\frac{\partial}{\partial r}+3\Big)\Phi_2(x)\Big) + \frac{1}{2}\operatorname{rot}\operatorname{rot}\big(x\Phi_3(x)\big) + \frac{\mu_1}{\gamma(\lambda_3^2-\lambda_5^2)}\operatorname{rot}\operatorname{rot}\big(x\Phi_6(x)\big) +\\
&+ \frac{\mu_1}{\gamma(\lambda_3^2-\lambda_5^2)}\operatorname{rot}\big(x\Phi_7(x)\big) + \operatorname{rot}\big(x\Phi_8(x)\big) - \frac{1}{\lambda_5^2}\operatorname{rot}\operatorname{rot}\big(x\Phi_9(x)\big) + \operatorname{grad}\Phi_{10}(x),
\end{aligned}$$

$$w(x) = -\frac{2\varkappa_3\lambda_0}{\varkappa_2}\operatorname{grad}\Big(\Big(2r\frac{\partial}{\partial r}+3\Big)\Phi_0(x)\Big) - \lambda_0\varkappa_7\operatorname{grad}\Phi_4(x) + \operatorname{rot}\big(x\Phi_6(x)\big)$$
$$- \frac{1}{\lambda_3^2}\operatorname{rot}\operatorname{rot}\big(x\Phi_7(x)\big),$$

$$v(x) = -\frac{2a_1}{\lambda_4^2}\Big(2r\frac{\partial}{\partial r}+3\Big)\Phi_0(x) + \frac{\mu_0(2\mu+\varkappa)}{\lambda_0\eta-\mu_0^2}\Big(2r\frac{\partial}{\partial r}+3\Big)\Big(r\frac{\partial}{\partial r}+1\Big)\Phi_2(x)$$
$$+ \frac{a_2}{\lambda_1^2-\lambda_4^2}\Phi_4(x) + \Phi_5(x),$$

$$\vartheta(x) = 2\lambda_0\Big(2r\frac{\partial}{\partial r}+3\Big)\Phi_0(x) + \varkappa_1\lambda_0\Phi_4(x),$$

where

$$\Delta\Phi_j(x)=0,\ \ j=0,1,2,3,\ \ (\Delta-\lambda_1^2)\Phi_4(x)=0,\ \ (\Delta-\lambda_4^2)\Phi_5(x)=0,\ \ (\Delta-\lambda_3^2)\Phi_j(x)=0,\ j=6,7,$$
$$(\Delta-\lambda_5^2)\Phi_j(x)=0,\ \ j=8,9,\quad (\Delta-\lambda_2^2)\Phi_{10}(x)=0,$$
$$\begin{aligned}
&\lambda_1^2=\frac{\varkappa_2\varkappa_7-\varkappa_1\varkappa_3}{l_0\varkappa_7}>0,\quad l_0=\varkappa_4+\varkappa_5+\varkappa_6>0,\quad \lambda_2^2=\frac{2\varkappa}{\alpha_0}>0,\quad \alpha_0=\alpha+\beta+\gamma>0,\\
&\lambda_3^2=\frac{\varkappa_2}{\varkappa_6}>0,\quad \lambda_4^2=\frac{\lambda_0\eta-\mu_0^2}{\lambda_0 a_0}>0,\quad \lambda_0=\lambda+2\mu+\varkappa>0,\quad \lambda_5^2=\frac{\varkappa(2\mu+\varkappa)}{\gamma(\mu+\varkappa)}>0,
\end{aligned} \tag{3.2}$$
$$a_1=\frac{\beta_0\mu_0-\lambda_0\beta_1}{a_0},\quad a_2=\varkappa_1 a_1-\frac{\varkappa_7\mu_2\lambda_0\lambda_1^2}{a_0},\quad a_3=\varkappa_1\beta_0-\frac{a_2\mu_0}{\lambda_0(\lambda_1^2-\lambda_4^2)},$$
$$a_4=\frac{\eta\beta_0-\mu_0\beta_1}{a_0\lambda_4^2},\quad a_5=\frac{\eta(2\mu+\varkappa)}{2(\lambda_0\eta-\mu_0^2)}.$$

Remark 3.1. The vector $U=(u,\omega,w,v,\vartheta)^\top$ represented by (3.1) will be uniquely defined by the functions $\Phi_j(x)$, $j=0,1,2,\dots,10$, if in the interior domain Ω^+

$$\int\limits_{\Sigma(0,1)}\Phi_j(x)d\Sigma(0,1)=0,\ \ j=1,3,6,7,8,9, \tag{3.3}$$

which means that to the zero vector $U=(u,\omega,w,v,\vartheta)^\top$ there corresponds the zero vector $(\Phi_0,\Phi_1,\dots,\Phi_{10})^\top$, and vice versa; $\Sigma(0,1)$ is the unit sphere centered at the origin.

The functions $\Phi_j(x)$, $j=0,1,2,\dots,10$ are represented as the series

$$\begin{aligned}
\Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\frac{r}{R}\Big)^k Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)},\quad j=0,1,2,3,\\
\Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}g_k(\sigma r)Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)},\quad j=4,5,\dots,10;
\end{aligned} \tag{3.4}$$

here, $A_{mk}^{(j)}$, $j=0,1,\dots,10$ are the sought for constants and $Y_k^{(m)}(\vartheta,\varphi)$ are the spherical functions,

$$Y_k^{(m)}(\theta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\,\mathbf{P}_k^{(m)}(\cos\theta)e^{im\varphi};$$

$\mathbf{P}_k^{(m)}(\cos\theta)$ are the Legendre associated polynomials, r, θ, φ ($0\le r<+\infty$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$) are the spherical coordinates of $x\in\mathbb{R}^3$;

$$g_k(\sigma r)=\sqrt{\frac{R}{r}}\,\frac{I_{k+1/2}(\sigma r)}{I_{k+1/2}(\sigma R)},$$

where $\sigma=\lambda_1$, when $j=4$; $\sigma=\lambda_2$, when $j=10$; $\sigma=\lambda_3$, when $j=6,7$; $\sigma=\lambda_4$, when $j=5$; $\sigma=\lambda_5$, when $j=8,9$; $I_{k+1/2}$ is the Bessel function of a complex (pure imaginary) argument [7].

Note that the constants $A_{mk}^{(j)}$ in (3.4) are complex numbers. If we assume that

$$\begin{aligned} A_{0k}^{(j)} &= a_{0k}^{(j)}, \quad k \geq 0, \\ A_{mk}^{(j)} &= \frac{1}{2}\big(a_{mk}^{(j)} + i b_{mk}^{(j)}\big), \quad A_{-mk}^{(j)} = \frac{1}{2}(-1)^m \big(a_{mk}^{(j)} - i b_{mk}^{(j)}\big), \quad k \geq 0, \ \ m > 0, \end{aligned}$$

where $a_{0k}^{(j)}$, $a_{mk}^{(j)}$, $b_{mk}^{(j)}$ are the real constants, then $\Phi_j(x)$ will be real functions. In our analysis, it is convenient to use a complex version of the Fourier–Laplace series of the form (3.4).

Substituting $\Phi_j(x)$, $j = 1, 3, 6, 7, 8, 9$ from (3.4), into (3.3) and taking into account the equalities

$$\int\limits_{\Sigma(0,1)} Y_k^{(m)}(\theta, \varphi) ds = \begin{cases} 2\sqrt{\pi}, & \text{for } k = 0, m = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

we get $A_{00}^{(j)} = 0$, $j = 1, 3, 6, 7, 8, 9$.

Substituting $\Phi_j(x)$, $j = 0, 1, \dots, 10$ into (3.1) and using the relations [1]

$$\begin{aligned} &\operatorname{grad}\big[a(r) Y_k^{(m)}(\theta, \varphi)\big] = \frac{da(r)}{dr} X_{mk}(\theta, \varphi) + \frac{\sqrt{k(k+1)}}{r} a(r) Y_{mk}(\theta, \varphi), \\ &\operatorname{rot}\big[x a(r) Y_k^{(m)}(\theta, \varphi)\big] = \sqrt{k(k+1)}\, a(r) Z_{mk}(\theta, \varphi), \\ &\operatorname{rot}\operatorname{rot}\big[x a(r) Y_k^{(m)}(\theta, \varphi)\big] = \frac{k(k+1)}{r} a(r) X_{mk}(\theta, \varphi) + \sqrt{k(k+1)} \Big(\frac{d}{dr} + \frac{1}{r}\Big) a(r) Y_{mk}(\theta, \varphi) \end{aligned} \tag{3.6}$$

we obtain

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(u_{mk}^{(1)}(r) X_{mk}(\theta, \varphi) + \sqrt{k(k+1)}\big[v_{mk}^{(1)}(r) Y_{mk}(\theta, \varphi) + w_{mk}^{(1)}(r) Z_{mk}(\theta, \varphi)\big]\Big), \\ \omega(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(u_{mk}^{(2)}(r) X_{mk}(\theta, \varphi) + \sqrt{k(k+1)}\big[v_{mk}^{(2)}(r) Y_{mk}(\theta, \varphi) + w_{mk}^{(2)}(r) Z_{mk}(\theta, \varphi)\big]\Big), \\ w(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(u_{mk}^{(3)}(r) X_{mk}(\theta, \varphi) + \sqrt{k(k+1)}\big[v_{mk}^{(3)}(r) Y_{mk}(\theta, \varphi) + w_{mk}^{(3)}(r) Z_{mk}(\theta, \varphi)\big]\Big), \\ v(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} u_{mk}^{(4)}(r) Y_k^{(m)}(\theta, \varphi), \\ \vartheta(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} u_{(mk)}^{(5)}(r) Y_k^{(m)}(\theta, \varphi), \end{aligned} \tag{3.7}$$

where [1, 2]

$$\begin{aligned} X_{mk}(\theta, \varphi) &= e_r Y_k^{(m)}(\theta, \varphi), \quad k \geq 0, \\ Y_{mk}(\theta, \varphi) &= \frac{1}{\sqrt{k(k+1)}} \Big(e_\theta \frac{\partial}{\partial \theta} + \frac{e_\varphi}{\sin\theta} \frac{\partial}{\partial \varphi}\Big) Y_k^{(m)}(\theta, \varphi), \quad k \geq 1, \\ Z_{mk}(\theta, \varphi) &= \frac{1}{\sqrt{k(k+1)}} \Big(\frac{e_\theta}{\sin\theta} \frac{\partial}{\partial \varphi} - e_\varphi \frac{\partial}{\partial \theta}\Big) Y_k^{(m)}(\theta, \varphi), \quad k \geq 1, \ \ |m| \leq k; \end{aligned} \tag{3.8}$$

e_r, e_θ, and e_φ are the unit vectors:

$$\begin{aligned} e_r &= (\cos\varphi \sin\theta, \ \sin\varphi \sin\theta, \ \cos\theta)^\top, \\ e_\theta &= (\cos\varphi \cos\theta, \ \sin\varphi \cos\theta, \ -\sin\theta)^\top, \\ e_\varphi &= (-\sin\varphi, \ \cos\varphi, \ 0)^\top. \end{aligned}$$

The system of vectors $\{X_{mk}(\theta, \varphi), Y_{mk}(\theta, \varphi), Z_{mk}(\theta, \varphi)\}$, $|m| \leq k$, $k = \overline{1, \infty}$ is orthogonal and complete in $L_2(\Sigma_1)$, where Σ_1 is the unit sphere.

Note that in formula (3.7) and in the sequel $\sqrt{k(k+1)}\,Y_{mk}(\theta,\varphi)=0$, $\sqrt{k(k+1)}\,Z_{mk}(\theta,\varphi)=0$, when $k=0$.

The coefficients $u^{(j)}_{mk}(r)$, $v^{(j)}_{mk}(r)$, $w^{(j)}_{mk}(r)$, $j=1,2,3$, and $u^{(j)}_{mk}(r)$, $j=4,5$, in (3.7) have the form

$$
\begin{aligned}
u^{(1)}_{mk}(r)=&a_4R(k+2)\Big(\frac{r}{R}\Big)^{k+1}A^{(0)}_{mk}+\frac{k}{R}\Big(\frac{r}{R}\Big)^{k-1}A^{(1)}_{mk}+(k-(k+2)a_5)(k+1)R\Big(\frac{r}{R}\Big)^{k+1}A^{(2)}_{mk}\\
&+\frac{a_3}{\lambda_1^2}\frac{d}{dr}g_k(\lambda_1r)A^{(4)}_{mk}-\frac{\mu_0}{\lambda_0\lambda_4^2}\frac{d}{dr}g_k(\lambda_4r)A^{(5)}_{mk}-\frac{\varkappa\mu_1k(k+1)}{\gamma(\mu+\varkappa)(\lambda_3^2-\lambda_5^2)\lambda_3^2}\frac{1}{r}g_k(\lambda_3r)A^{(7)}_{mk}\\
&-\frac{\varkappa k(k+1)}{(\mu+\varkappa)\lambda_5^2}\frac{1}{r}g_k(\lambda_5r)A^{(8)}_{mk},\\
v^{(1)}_{mk}(r)=&a_4R\Big(\frac{r}{R}\Big)^{k+1}A^{(0)}_{mk}+\frac{1}{R}\Big(\frac{r}{R}\Big)^{k-1}A^{(1)}_{mk}\\
&+(k+3-(k+1)a_5)R\Big(\frac{r}{R}\Big)^{k+1}A^{(2)}_{mk}+\frac{a_3}{\lambda_1^2}\frac{1}{r}g_k(\lambda_1r)A^{(4)}_{mk}\\
&-\frac{\mu_0}{\lambda_0\lambda_4^2}\frac{1}{r}g_k(\lambda_4r)A^{(5)}_{mk}-\frac{\varkappa\mu_1}{\gamma(\mu+\varkappa)(\lambda_3^2-\lambda_5^2)\lambda_3^2}\Big(\frac{d}{dr}+\frac{1}{r}\Big)g_k(\lambda_3r)A^{(7)}_{mk}\\
&-\frac{\varkappa}{(\mu+\varkappa)\lambda_5^2}\Big(\frac{d}{dr}+\frac{1}{r}\Big)g_k(\lambda_5r)A^{(8)}_{mk},\\
w^{(1)}_{mk}(r)=&\Big(\frac{r}{R}\Big)^{k}A^{(3)}_{mk}+\frac{\varkappa\mu_1}{\gamma(\mu+\varkappa)(\lambda_3^2-\lambda_5^2)}g_k(\lambda_3r)A^{(6)}_{mk}-\frac{\varkappa}{(\mu+\varkappa)\lambda_5^2}g_k(\lambda_5r)A^{(9)}_{mk},\\
u^{(2)}_{mk}(r)=&\frac{k(k+1)}{2R}\Big(\frac{r}{R}\Big)^{k-1}A^{(3)}_{mk}+\frac{\mu_1k(k+1)}{\gamma(\lambda_3^2-\lambda_5^2)}\frac{1}{r}g_k(\lambda_3r)A^{(6)}_{mk}\\
&-\frac{k(k+1)}{\lambda_3^2}\frac{1}{r}g_k(\lambda_5r)A^{(9)}_{mk}+\frac{d}{dr}g_k(\lambda_2r)A^{(10)}_{mk},\\
v^{(2)}_{mk}(r)=&\frac{k+1}{2R}\Big(\frac{r}{R}\Big)^{k-1}A^{(3)}_{mk}+\frac{\mu_1}{\gamma(\lambda_3^2-\lambda_5^2)}\Big(\frac{d}{dr}+\frac{1}{r}\Big)g_k(\lambda_3r)A^{(6)}_{mk}\\
&-\frac{1}{\lambda_3^2}\Big(\frac{d}{dr}+\frac{1}{r}\Big)g_k(\lambda_5r)A^{(9)}_{mk}+\frac{1}{r}g_k(\lambda_2r)A^{(10)}_{mk},\\
w^{(2)}_{mk}(r)=&-(2k+3)\Big(\frac{r}{R}\Big)^{k}A^{(2)}_{mk}+\frac{\mu_1}{\gamma(\lambda_3^2-\lambda_5^2)}g_k(\lambda_3r)A^{(7)}_{mk}+g_k(\lambda_5r)A^{(8)}_{mk},\qquad(3.9)\\
u^{(3)}_{mk}(r)=&-\frac{2\varkappa_3\lambda_0k(2k+3)}{\varkappa_2R}\Big(\frac{r}{R}\Big)^{k-1}A^{(0)}_{mk}-\lambda_0\varkappa_7\frac{d}{dr}g_k(\lambda_1r)A^{(4)}_{mk}-\frac{k(k+1)}{\lambda_3^2}\frac{1}{r}g_k(\lambda_3r)A^{(7)}_{mk},\\
v^{(3)}_{mk}(r)=&-\frac{2\varkappa_3\lambda_0(2k+3)}{\varkappa_2R}\Big(\frac{r}{R}\Big)^{k-1}A^{(0)}_{mk}-\lambda_0\varkappa_7\frac{1}{r}g_k(\lambda_1r)A^{(4)}_{mk}-\frac{1}{\lambda_3^2}\Big(\frac{d}{dr}+\frac{1}{r}\Big)g_k(\lambda_3r)A^{(7)}_{mk},\\
w^{(3)}_{mk}(r)=&\;g_k(\lambda_3r)A^{(6)}_{mk},\\
u^{(4)}_{mk}(r)=&-\frac{2a_1(2k+3)}{\lambda_4^2}\Big(\frac{r}{R}\Big)^{k}A^{(0)}_{mk}+\frac{\mu_0(2\mu+\varkappa)}{\lambda_0\eta-\mu_0^2}(k+1)(2k+3)\Big(\frac{r}{R}\Big)^{k}A^{(2)}_{mk}\\
&+\frac{a_2}{\lambda_1^2-\lambda_4^2}g_k(\lambda_1r)A^{(4)}_{mk}+g_k(\lambda_4r)A^{(5)}_{mk},\\
u^{(5)}_{mk}(r)=&2\lambda_0(2k+3)\Big(\frac{r}{R}\Big)^{k}A^{(0)}_{mk}+\varkappa_1\lambda_0g_k(\lambda_1r)A^{(4)}_{mk}.
\end{aligned}
$$

After the substitution of the vectors $u(x)$, $\omega(x)$, $w(x)$ and the functions $v(x)$, $\vartheta(x)$ from (3.7) into (2.6) and using the following equalities [1]:

$$
\begin{aligned}
&e_r\times X_{mk}(\theta,\varphi)=0,\ \ e_r\times Y_{mk}(\theta,\varphi)=-Z_{mk}(\theta,\varphi),\ \ e_r\times Z_{mk}(\theta,\varphi)=Y_{mk}(\theta,\varphi),\\
&e_r\cdot X_{mk}(\theta,\varphi)=Y^{(m)}_k(\theta,\varphi),\ \ e_r\cdot Y_{mk}(\theta,\varphi)=0,\ \ e_r\cdot Z_{mk}(\theta,\varphi)=0,\\
&\operatorname{div}\big[a(r)X_{mk}(\theta,\varphi)\big]=\Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)Y^{(m)}_k(\theta,\varphi),
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{div}\big[a(r)Y_{mk}(\theta,\varphi)\big] = -\frac{\sqrt{k(k+1)}}{r}a(r)Y_k^{(m)}(\theta,\varphi),\\
&\operatorname{div}\big[a(r)Z_{mk}(\theta,\varphi)\big] = 0,\\
&\operatorname{rot}\big[a(r)X_{mk}(\theta,\varphi)\big] = \frac{\sqrt{k(k+1)}}{r}a(r)Z_{mk}(\theta,\varphi),\\
&\operatorname{rot}\big[a(r)Y_{mk}(\theta,\varphi)\big] = -\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Z_{mk}(\theta,\varphi),\\
&\operatorname{rot}\big[a(r)Z_{mk}(\theta,\varphi)\big] = \frac{\sqrt{k(k+1)}}{r}a(r)X_{mk}(\theta,\varphi)+\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\theta,\varphi),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
T^{(j)}(\partial,n)U(x) =&\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(a_{mk}^{(j)}(r)X_{mk}(\theta,\varphi)+\sqrt{k(k+1)}\big[b_{mk}^{(j)}(r)Y_{mk}(\theta,\varphi)\\
&+c_{mk}^{(j)}(r)Z_{mk}(\theta,\varphi)\big]\Big),\quad j=1,2,3,\\
T^{(j)}(\partial,n)U(x) =&\sum_{k=0}^{\infty}\sum_{m=-k}^{k}a_{mk}^{(j)}(r)Y_k^{m}(\theta,\varphi),\quad j=4,5,
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
a_{mk}^{(1)}(r) &= \Big(\lambda_0\frac{d}{dr}+\frac{2\lambda}{r}\Big)u_{mk}^{(1)}(r)-\frac{\lambda k(k+1)}{r}v_{mk}^{(1)}(r)+\mu_0u_{mk}^{(4)}(r)-\beta_0u_{mk}^{(5)}(r),\\
b_{mk}^{(1)}(r) &= \frac{\mu}{r}u_{mk}^{(1)}(r)+\Big((\mu+\varkappa)\frac{d}{dr}-\frac{\mu}{r}\Big)v_{mk}^{(1)}(r)+\varkappa w_{mk}^{(2)}(r),\\
c_{mk}^{(1)}(r) &= \Big((\mu+\varkappa)\frac{d}{dr}-\frac{\mu}{r}\Big)w_{mk}^{(1)}(r)-\varkappa v_{mk}^{(2)}(r),\\
a_{mk}^{(2)}(r) &= \Big(\alpha_0\frac{d}{dr}+\frac{2\alpha}{r}\Big)u_{mk}^{(2)}(r)-\frac{\alpha k(k+1)}{r}v_{mk}^{(2)}(r),\\
b_{mk}^{(2)}(r) &= \frac{\beta}{r}u_{mk}^{(2)}(r)+\Big(\gamma\frac{d}{dr}-\frac{\beta}{r}\Big)v_{mk}^{(2)}(r)-\mu_1w_{mk}^{(2)}(r),\\
c_{mk}^{(2)}(r) &= \Big(\gamma\frac{d}{dr}-\frac{\beta}{r}\Big)w_{mk}^{(2)}(r)+\mu_1v_{mk}^{(2)}(r)+\frac{b_0}{r}u_{mk}^{(4)}(r),\\
a_{mk}^{(3)}(r) &= \Big(l_0\frac{d}{dr}+\frac{2\varkappa_4}{r}\Big)u_{mk}^{(3)}(r)-\frac{\varkappa_4k(k+1)}{r}v_{mk}^{(3)}(r),\\
b_{mk}^{(3)}(r) &= \frac{\varkappa_5}{r}u_{mk}^{(3)}(r)+\Big(\varkappa_6\frac{d}{dr}-\frac{\varkappa_5}{r}\Big)v_{mk}^{(3)}(r),\\
c_{mk}^{(3)}(r) &= \Big(\varkappa_6\frac{d}{dr}-\frac{\varkappa_5}{r}\Big)w_{mk}^{(3)}(r),\\
a_{mk}^{(4)}(r) &= a_0\frac{d}{dr}u_{mk}^{(4)}(r)-\mu_2u_{mk}^{(3)}(r)+\frac{b_0k(k+1)}{r}w_{mk}^{(2)}(r),\\
a_{mk}^{(5)}(r) &= \varkappa_7\frac{d}{dr}u_{mk}^{(5)}(r)+\varkappa_1u_{mk}^{(3)}(r).
\end{aligned}
\tag{3.11}
$$

Here, the functions $u_{mk}^{(j)},\ j=1,2,\dots,5$ and $v_{mk}^{(j)},\ w_{mk}^{(j)},\ j=1,2,3$, are given by (3.9).

Represent the boundary data as the Fourier - Laplace series in the system of vectors (3.8),

$$
\begin{aligned}
&f^{(j)}(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\alpha_{mk}^{(j)}X_{mk}(\theta,\varphi)+\sqrt{k(k+1)}\big[\beta_{mk}^{(j)}Y_{mk}(\theta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\theta,\varphi)\big]\Big),\ j=1,2,3,\\
&f_j(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}^{(j)}Y_k^{(m)}(\theta,\varphi),\ j=4,5,
\end{aligned}
\tag{3.12}
$$

where

$$\alpha_{mk}^{(j)}\ \ j=1,2,\dots,5,\quad \sqrt{k(k+1)}\,\beta_{mk}^{(j)},\quad \sqrt{k(k+1)}\,\gamma_{mk}^{(j)},\quad j=1,2,3,$$

are the Fourier coefficients.

Necessary and sufficient conditions for Problem $(II)^+$ to be solvable read as [4]

$$\int\limits_{\partial\Omega} f^{(1)}(z)ds=0,\qquad \int\limits_{\partial\Omega}\big(z\times f^{(1)}(z)+f^{(2)}(z)\big)ds=0,\qquad \int\limits_{\partial\Omega} f_5(z)ds=0. \tag{3.13}$$

Substituting $f^{(j)}$, $j=1,2$, and f_5 into (3.13), and taking into consideration (3.5) and the equalities

$$\begin{aligned}
&\int\limits_{\partial\Omega} X_{mk}(\theta,\varphi)ds=\begin{cases}\sqrt{\frac{2\pi}{3}}\big(\delta_{-1m}e_1-\delta_{1m}e_2+\sqrt{2}\delta_{0m}e_3\big)R^2, \text{ for } \ k=1,\ m=0,\pm1,\\ 0,\ \text{ otherwise},\end{cases}\\
&\int\limits_{\partial\Omega} Y_{mk}(\theta,\varphi)ds=\begin{cases}2\sqrt{\frac{\pi}{3}}\big[(\delta_{-1m}-\delta_{1m})e_1-i\big(\delta_{-1m}+\delta_{1m})e_2+\sqrt{2}\delta_{0m}e_3\big)R^2,\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } k=1,\ m=0,\pm1,\\ 0,\ \text{ otherwise},\end{cases}\\
&\int\limits_{\partial\Omega} Z_{mk}(\theta,\varphi)ds=0,\ \text{ for all } \ k \ \text{ and } \ m,
\end{aligned} \tag{3.14}$$

where δ_{kj} is the Kronecker symbol,

$$e_1=(1,-i,0)^\top,\ e_2=(1,i,0)^\top,\ e_3=(0,0,1)^\top, \tag{3.15}$$

we get

$$\begin{aligned}
&\alpha_{m1}^{(1)}+2\beta_{m1}^{(1)}=0,\\
&\alpha_{m1}^{(2)}+2\beta_{m1}^{(2)}+2R\gamma_{m1}^{(1)}=0,\ \ m=0,\pm1,\\
&\alpha_{00}^{(5)}=0.
\end{aligned} \tag{3.16}$$

On the other hand, if in both parts of equalities (3.10) we pass to the limit as $x\to z\in\partial\Omega$, take into account the boundary condition (2.4) and formulas (3.12), for the unknown constants $A_{mk}^{(j)}$, $j=0,1,2,\dots,10$, we obtain the system of linear algebraic equations:

a) if $k=0$, $m=0$, then $A_{00}^{(j)}$, $j=0,2,4,5,10$, are defined from the following system:

$$a_{00}^{(1)}(R)=\alpha_{00}^{(1)},\quad a_{00}^{(2)}(R)=\alpha_{00}^{(2)},\quad a_{00}^{(3)}(R)=\alpha_{00}^{(3)},\quad a_{00}^{(4)}(R)=\alpha_{00}^{(4)},\quad a_{00}^{(5)}(R)=\alpha_{00}^{(5)}; \tag{3.17}$$

b) if $k\geq 1$, $|m|\leq k$, then $A_{mk}^{(j)}$, $j=0,1,2,\dots,10$, are defined from the following system:

$$a_{mk}^{(j)}(R)=\alpha_{mk}^{(j)},\ j=1,2,3,4,5,\quad b_{mk}^{(j)}(R)=\beta_{mk}^{(j)},\quad c_{mk}^{(j)}(R)=\gamma_{mk}^{(j)},\ j=1,2,3. \tag{3.18}$$

The system of equations (3.17) leads to the corresponding system of algebraic linear equations with respect to unknown constants $A_{00}^{(j)}$, $j=0,2,4,5,10$, via the relations (3.9) and (3.11).

Note that the fifth equation of system (3.17) is fulfilled identically, because $a_{00}^{(5)}(R)=\alpha_{00}^{(5)}=0$. The second equation of (3.17) contains only one unknown constant $A_{00}^{(10)}$; the third equation of (3.17) contains only one unknown constant $A_{00}^{(4)}$; the fourth equation of (3.17) contains only two unknown constants $A_{00}^{(4)}$ and $A_{00}^{(5)}$; the first equation of (3.17) contains the constants $A_{00}^{(0)}$, $A_{0\,0}^{(2)}$, $A_{0\,0}^{(4)}$ and $A_{0\,0}^{(5)}$. Theorem 2.3 and Remark 3.1 imply that system (3.17) is solvable and only one unknown constant remains arbitrary. Let $A_{00}^{(0)}$ be this arbitrary constant.

When $k=1$, $|m|\leq 1$, system of equations (3.18) leads to the corresponding system of eleven algebraic linear equations with respect to eleven unknown constants $A_{m1}^{(j)}$, $j=0,1,2,\dots,10$. Taking into account (3.16) and performing equivalent transformations, we find that we have actually only nine independent linear equations with respect to eleven unknown constants $A_{m1}^{(j)}$, $j=0,1,2,\dots,10$. Therefore, for $k=1$, $|m|\leq 1$, two constants remain undefined. For definiteness, let $A_{m1}^{(1)}$ and $A_{m1}^{(3)}$ be these arbitrary constants.

When $k > 1$, $|m| \leq k$, the system of equations (3.18) leads to the corresponding system of eleven algebraic linear equations with respect to eleven unknown constants $A^{(j)}_{m1}$, $j = 0, 1, 2, \dots, 10$. Theorem 2.3 and Remark 3.1 imply that the system (3.18) when $k > 1$, $|m| \leq k$, is uniquely solvable.

Therefore, from the systems of equations (3.17) and (3.18), we can define all unknown coefficients except $A^{(0)}_{00}$, $A^{(1)}_{m1}$ and $A^{(3)}_{m1}$, $m = 0, \pm 1$.

This is natural and reflects the fact that the solution is defined modulo a rigid displacement vector.

Let us show that the undefined coefficients $A^{(0)}_{00}$, $A^{(1)}_{m1}$ and $A^{(3)}_{m1}$, $m = 0, \pm 1$, give the summands corresponding to the rigid displacement vector. We separate the terms containing these undefined coefficients and rewrite the vector $U = (u, \omega, w, v, \theta)^\top$ represented by relations (3.7) in the form

$$\begin{aligned}
u(x) &= u_0(x) + \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(\tilde{u}^{(1)}_{mk}(r) X_{mk}(\theta,\varphi) + \sqrt{k(k+1)}\big[\tilde{v}^{(1)}_{mk}(r) Y_{mk}(\theta,\varphi) + \tilde{w}^{(1)}_{mk}(r) Z_{mk}(\theta,\varphi)\big]\Big),\\
\omega(x) &= \omega_0(x) + \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(\tilde{u}^{(2)}_{mk}(r) X_{mk}(\theta,\varphi) + \sqrt{k(k+1)}\big[\tilde{v}^{(2)}_{mk}(r) Y_{mk}(\theta,\varphi) + \tilde{w}^{(2)}_{mk}(r) Z_{mk}(\theta,\varphi)\big]\Big),\\
w(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big(u^{(3)}_{mk}(r) X_{mk}(\theta,\varphi) + \sqrt{k(k+1)}\big[v^{(3)}_{mk}(r) Y_{mk}(\theta,\varphi) + w^{(3)}_{mk}(r) Z_{mk}(\theta,\varphi)\big]\Big), \qquad (3.19)\\
v(x) &= v_0(x) + \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \tilde{u}^{(4)}_{mk}(r) Y^{(m)}_k(\theta,\varphi),\\
\vartheta(x) &= \vartheta_0(x) + \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \tilde{u}^{(5)}_{(mk)}(r) Y^{(m)}_k(\theta,\varphi),
\end{aligned}$$

where

$$\begin{aligned}
u_0(x) &= Ax + \frac{1}{R} \sum_{m=-1}^{1} \Big(A^{(1)}_{m1}\big[X_{m1}(\theta,\varphi) + \sqrt{2} Y_{m1}(\theta,\varphi)\big] + \sqrt{2} r A^{(3)}_{m1} Z_{m1}(\theta,\varphi)\Big), \quad A = \frac{1}{\sqrt{\pi}} a_4 A^{(0)}_{00},\\
\omega_0(x) &= \frac{1}{R} \sum_{m=-1}^{1} A^{(3)}_{m1}\big[X_{m1}(\theta,\varphi) + \sqrt{2} Y_{m1}(\theta,\varphi)\big],\\
v_0(x) &= B, \quad B = -\frac{3}{\sqrt{\pi}} \frac{a_1}{\lambda_4^2} A^{(0)}_{00}, \qquad (3.20)\\
\vartheta_0(x) &= C, \quad C = \frac{3}{\sqrt{\pi}} \lambda_0 A^{(0)}_{00};
\end{aligned}$$

the functions $\tilde{u}^{(j)}_{mk}(r)$ $j = 1, 2, 4, 5$, $\tilde{v}^{(j)}_{mk}(r)$, $\tilde{w}^{(1)}_{mk}(r)$ $j = 1, 2$ can be easily written down.

From (3.6), we get the following formulas:

$$\begin{aligned}
&\sum_{m=-1}^{1} \big[X_{m1}(\theta,\varphi) + \sqrt{2} Y_{m1}(\theta,\varphi)\big] A^{(j)}_{m1} = \operatorname{grad} \sum_{m=-1}^{1} r Y^{(m)}_1(\theta,\varphi) A^{(j)}_{m1}, \quad j = 1, 3,\\
&\sum_{m=-1}^{1} \sqrt{2} r Z_{m1}(\theta,\varphi) A^{(3)}_{m1} = -\Big[x \times \operatorname{grad} \sum_{m=-1}^{1} r Y^{(m)}_1(\theta,\varphi) A^{(3)}_{m1}\Big], \qquad (3.21)
\end{aligned}$$

Using the relations for the Legendre polynomials [16]

$$\begin{aligned}
\mathbf{P}^{(m)}_k(\cos\theta) &= (-1)^m \sin^m\theta \frac{d^m \mathbf{P}_k(\cos\theta)}{d(\cos\theta)^m},\\
\mathbf{P}^{(-m)}_k(\cos\theta) &= (-1)^m \frac{(k-m)!}{(k+m)!} \mathbf{P}^{(m)}_k(\cos\theta),\\
\mathbf{P}^{(0)}_1(\cos\theta) &= \mathbf{P}_1(\cos\theta) = \cos\theta,
\end{aligned}$$

we obtain

$$rY_1^{(0)}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}x_3,\ \ rY_1^{(1)}(\theta,\varphi)=-\frac{1}{2}\sqrt{\frac{3}{2\pi}}\big(x_1+ix_2\big),\ \ rY_1^{(-1)}(\theta,\varphi)=\frac{1}{2}\sqrt{\frac{3}{2\pi}}\big(x_1-ix_2\big),$$

$$\operatorname{grad}\sum_{m=-1}^{1} rY_1^{(m)}(\theta,\varphi)A_{m1}^{(j)}=\frac{1}{2}\sqrt{\frac{3}{2\pi}}\Big(A_{-11}^{(j)}e_1-A_{11}^{(j)}e_2+\sqrt{2}A_{01}^{(j)}e_3\Big),\ j=1,3,$$

where the vectors $e_1,\ e_2,\ e_3$ have the form (3.15).

Introduce the notation

$$\begin{aligned}a&\equiv\frac{1}{2R}\sqrt{\frac{3}{2\pi}}\Big(A_{-11}^{(3)}e_1-A_{11}^{(3)}e_2+\sqrt{2}A_{01}^{(3)}e_3\Big),\\ b&\equiv\frac{1}{2R}\sqrt{\frac{3}{2\pi}}\Big(A_{-11}^{(1)}e_1-A_{11}^{(1)}e_2+\sqrt{2}A_{01}^{(1)}e_3\Big).\end{aligned}\tag{3.22}$$

Evidently, a and b are arbitrary three-dimensional constant vectors.

It can be shown that formula (3.20) takes the form

$$\begin{aligned}u_0(x)=&Ax+[a\times x]+b,\\ \omega_0(x)=&a,\\ v_0(x)=&B,\\ \vartheta_0(x)=&C,\end{aligned}\tag{3.23}$$

where a and b are arbitrary three-dimensional constant vectors introduced above and A,B and C are arbitrary constants.

Now, consider the vector $\widetilde{U}=(Ax+[a\times x]+b,a,0,B,C)^\top$. It satisfies the system of equations (2.1) in Ω^+ and boundary condition $P\big(\partial,n\big)\widetilde{U}(x)=0$ on $\partial\Omega^+$ if the following conditions are satisfied

$$\begin{aligned}3\alpha_0'A+\mu_0B&=\beta_0C,\\ 3\mu_0A+\eta B&=\beta_1C,\end{aligned}$$

whence we obtain

$$A=\frac{1}{3}\,\frac{\beta_0\eta-\mu_0\beta_1}{\alpha_0'\eta-\mu_0^2}=\frac{1}{3}p'C,\qquad B=\frac{\alpha_0'\beta_1-\mu_0\beta_0}{\alpha_0'\eta-\mu_0^2}=q'C;\tag{3.24}$$

here, the constants p', q', α_0' are given by formulas (2.8).

Taking into account (3.24), we establish that the vector

$$\widetilde{U}=\Big(u_0(x),\omega_0(x),0,v_0(x),\vartheta_0(x)\Big)^\top=\Big(1/3p'Cx+[a\times x]+b,a,0,q'C,C\Big)^\top,\tag{3.25}$$

which implies that a solution of the Neumann problem is defined modulo a rigid displacement vector of type (3.25).

If we substitute the solutions of systems (3.17) and (3.18) into (3.7), then the vector $U=(u,\omega,w,v,\theta)^\top$, given by (3.7), will be a formal solution of Problem $(II)^+$. To justify our approach, we have to show the convergence of the series (3.7) and (3.10).

The following asymptotic representations are valid for $k\to\infty$ [16]:

$$g_k(k_jr)\approx\Big(\frac{r}{R}\Big)^k,\qquad g_k'(k_jr)\approx\frac{k}{r}\Big(\frac{r}{R}\Big)^k,\quad r<R.\tag{3.26}$$

We need the following technical results [1].

Theorem 3.2. *The following inequalities are valid for $k\geq 0$:*

$$|X_{mk}(\theta,\varphi)|\leq\sqrt{\frac{2k+1}{4\pi}},\quad k\geq 0,\qquad |Y_{mk}(\theta,\varphi)|\leq\sqrt{\frac{2k(k+1)}{2k+1}},\quad k\geq 1,$$

$$|Z_{mk}(\theta,\varphi)|\leq\sqrt{\frac{2k(k+1)}{2k+1}},\quad k\geq 1,\tag{3.27}$$

Theorem 3.3. *If $f^{(j)} \in C^l(\partial\Omega)$, $j = 1,2,3$, then the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$ and $\gamma_{mk}^{(j)}$, satisfy the asymptotic relation*

$$\alpha_{mk}^{(j)} = \mathcal{O}(k^{-l}), \quad \beta_{mk}^{(j)} = \mathcal{O}(k^{-l-1}), \quad \gamma_{mk}^{(j)} = \mathcal{O}(k^{-l-1}).$$

Theorem 3.4. *If $f_j \in C^l(\partial\Omega)$, $j = 4,5$, then the coefficients $\alpha_{mk}^{(j)}$, $j = 4,5$, satisfy the asymptotic relation*

$$\alpha_{mk}^{(j)} = \mathcal{O}(k^{-l}).$$

In view of (3.26) and (3.2), the series (3.7) and (3.10) are absolutely and uniformly convergent for $x \in \partial\Omega$ $(r = R)$, provided the following majorant series

$$\alpha_0 \sum_{k=k_0}^{\infty} k^{3/2} \Big[\sum_{j=1}^{3} \Big(|\alpha_{mk}^{(j)}| + k|\beta_{mk}^{(j)}| + k|\gamma_{mk}^{(j)}| \Big) + k|\alpha_{mk}^{(4)}| + |\alpha_{mk}^{(5)}| \Big] \tag{3.28}$$

is convergent. Here, $\alpha_{mk}^{(j)}$ $\beta_{mk}^{(j)}$ $\gamma_{mk}^{(j)}$, $j = 1,2,3$, $\alpha_{mk}^{(4)}$, $\alpha_{mk}^{(5)}$ are the Fourier–Laplace coefficients. In turn, the series (3.28) are convergent if the coefficients have the following asymptotic behaviour

$$\alpha_{mk}^{(j)} = \mathcal{O}(k^{-3}), \quad \beta_{mk}^{(j)} = \mathcal{O}(k^{-4}), \quad \gamma_{mk}^{(j)} = \mathcal{O}(k^{-4}), \ \ j = 1,2,3, \quad \alpha_{mk}^{(j)} = \mathcal{O}(k^{-4}), \ \ j = 4,5. \tag{3.29}$$

According to Theorems 3.3 and 3.4, the estimates (3.29) hold if the boundary vector-functions satisfy the following smoothness conditions:

$$f^{(j)} \in \mathcal{C}^3(\partial\Omega), \quad j = 1,2,3, \qquad f_j \in \mathcal{C}^3(\partial\Omega), \quad j = 4,5. \tag{3.30}$$

Therefore if the boundary vector-functions satisfy conditions (3.30), then the vector $U = (u, \omega, w, v, \theta)^\top$ represented by equalities (3.7) is a solution of Problem $(II)^+$.

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