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LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS APPROACH FOR DIRICHLET PSEUDO-OSCILLATION PROBLEM OF THE COUPLE-STRESS ELASTICITY

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The paper deals with the three-dimensional Dirichlet boundary value problem (BVP) of the couple-stress elasticity theory for anisotropic inhomogeneous solids and develops the generalized potential method based on the localized parametrix method. Using Green's integral representation formula and the properties of the localized layer and volume potentials, we reduce the Dirichlet BVP to the system of localized boundary-domain integral equations (LBDIE). The equivalence between the Dirichlet BVP and the corresponding LBDIE system is studied. We state that the obtained localized boundary-domain integral operator belongs to the Boutet de Monvel algebra and, using the Wiener–Hopf factorization method, we investigate the corresponding Fredholm properties and prove the invertibility of the localized operator in appropriate Sobolev function spaces.

1. Introduction

We consider the Dirichlet pseudo-oscillation boundary value problem (BVP) for a second order strongly elliptic system of partial differential equations in the divergence form with variable coefficients and develop the generalized integral potential method based on a *localized parametrix*.

A system of pseudo-oscillation equations is obtained by the Laplace transform of the dynamical system of equations (see [20]).

The BVP treated in the paper is well investigated in the literature by the variational method and, in the case of constant coefficients, by the classical potential method, when the corresponding fundamental solution is available in explicit form (see, e.g., [19, 21, 22]). However, as it is well known, for PDE systems with variable coefficients no fundamental solution is available in an analytical and/or cheaply calculated form, in general.

Our goal here is to develop a potential method for general second order strongly elliptic self-adjoint systems of partial differential equations with variable coefficients. We show that solutions of the problem can be represented by *localized parametrix-based potentials* and that the corresponding *localized boundary-domain integral operator* (LBDIO) is invertible, that is important for analysis of convergence and stability of LBDIE-based numerical methods for PDEs (see, e.g., [18, 23, 26, 28, 31, 32, 34, 35]).

Using Green's representation formula and the properties of the localized layer and volume potentials, we reduce the Dirichlet BVP to a system of Localized Boundary-Domain Integral Equations (LBDIEs). First, we establish the equivalence between the original boundary value problem and the corresponding LBDIE system, which appeared to be quite non-trivial task and plays a crucial role in our analysis. Afterwards, we establish that the localized boundary domain integral operator of the system belongs to the Boutet de Monvel operator algebra. Employing the Vishik–Eskin theory, based on the Wiener-Hopf factorization method, we investigate the corresponding Fredholm properties and prove the invertibility of the localized operator in appropriate Sobolev (Bessel potential) spaces.

In [6–12, 24], the LBDIE method has been developed for the case of scalar elliptic second order partial differential equations with variable coefficients, here we extend it to the PDE systems.

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2. Boundary Value Problem and Parametrix-based Operators

2.1. **Formulation of the boundary value problems and localized Green's third identity.** For isotropic inhomogeneous media, we consider the following system of pseudo-oscillation equations of couple-stress elasticity with respect to $U=(u,\phi)^\top=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3)^\top$ (see [20]):

$$\begin{aligned}\delta_{ij}\partial_l\big((\mu(x)+\varkappa(x))\partial_l\big)u_i+\partial_i\big((\lambda(x)+\mu(x))\partial_j\big)u_j-\rho_0(x)\tau^2u_i&\\ +\varkappa(x)\varepsilon_{ijk}\partial_j\phi_k=-\rho(x)f_i,\qquad i=1,2,3,&\end{aligned}\tag{2.1}$$

$$\begin{aligned}\delta_{ij}\partial_l\big(\gamma(x))\partial_l\big)\phi_i+\partial_i\big((\alpha(x)+\beta(x))\partial_j\big)\phi_j-I_0(x)\tau^2\phi_i+\varkappa(x)\varepsilon_{ijk}\partial_ju_k&\\ -2\varkappa(x)\phi_i=-\rho(x)X_i,\qquad i=1,2,3,&\end{aligned}\tag{2.2}$$

where δ_{ij} is the Kronecker symbol, $\tau=\sigma+i\omega$ is a complex parameter, $\sigma>\sigma_0>0$, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $\phi=(\phi_1,\phi_2,\phi_3)^\top$ is the vector of microrotation, (f_1,f_2,f_3) is the external body force per unit mass and X_i is the external body couple per unit mass. We employ the notation $\partial_x=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$.

The coefficients λ, μ, $\varkappa$, α, β, $\gamma\in C^\infty$ are the elastic coefficients, $I_0\in C^\infty$ is the coefficient of inertia, and ε_{ijk} is the Levi-Civita symbol (see [20]).

Due to the positiveness of internal energy, the coefficients of system (2.1)–(2.2) must satisfy the following conditions:

$$\begin{gathered}\varkappa>0,\qquad \varkappa+2\mu>0,\qquad \varkappa+2\mu+3\lambda>0,\\ \gamma>|\beta|,\qquad \beta+\gamma+3\alpha>0,\\ \rho_0>0,\qquad I_0>0,\end{gathered}\tag{2.3}$$

where $\rho_0\in C^\infty$ is the mass density.

Denote by

$$M(x,\partial_x,\tau)=\big[M_{ij}(x,\partial_x,\tau)\big]_{6\times 6}\tag{2.4}$$

the strongly elliptic second order matrix partial differential operator generated by the left-hand side expression in (2.1)–(2.2), where

$$\begin{aligned}\big[M_{ij}(x,\partial_x,\tau)\big]_{3\times3}&=\delta_{ij}\partial_l\big((\mu(x)+\varkappa(x))\partial_l\big)+\partial_i\big((\lambda(x)+\mu(x))\partial_j\big)-\rho_0(x)\tau^2\delta_{ij},\\ \big[M_{i,j+3}(x,\partial_x,\tau)\big]_{3\times3}&=\big[M_{i,j+3}(x,\partial_x,\tau)\big]_{3\times3}=\varkappa(x)\varepsilon_{ijk}\partial_j,\\ \big[M_{i+3,j+3}(x,\partial_x,\tau)\big]_{3\times3}&=\delta_{ij}\partial_l\big((\gamma(x))\partial_l\big)+\partial_i\big((\alpha(x)+\beta(x))\partial_j\big)-\big(2\varkappa(x)+I_0(x)\tau^2\big)\delta_{ij},\\ &\qquad i,j=1,2,3.\end{aligned}$$

Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed if not otherwise stated.

Further, let $\Omega=\Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega=S\in C^\infty$, $\overline{\Omega}=\Omega\cup S$. Throughout the paper, $n=(n_1,n_2,n_3)$ denotes the unit normal vector to S directed outward the domain Ω. Set $\Omega^-:=\mathbb{R}^3\setminus\overline{\Omega}$.

By $H^r(\Omega)=H^r_2(\Omega)$ and $H^r(S)=H^r_2(S)$, $r\in\mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary, while $\mathcal{D}(\mathbb{R}^3)$ and $\mathcal{D}(\Omega)$ stand for C^∞ functions with compact support in $\mathbb{R}^3$ and in Ω respectively, and $\mathcal{S}(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decreasing functions in $\mathbb{R}^3$. Recall that $H^0(\Omega)=L_2(\Omega)$ is a space of square integrable functions in Ω. For a vector $U=(u_1,\dots,u_6)^\top$, the inclusion $U=(u_1,\dots,u_6)^\top\in H^r$ implies that each component u_j belongs to the space H^r.

Let us denote by $\gamma^+U=\{U\}^+$ and $\gamma^-U=\{U\}^-$ the traces of U on S from the interior and exterior of Ω^+, respectively.

We also need the following subspace of $H^1(\Omega)$ (see, e.g., [13]):

$$H^{1,0}(\Omega;M):=\{U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3)^\top\in H^1(\Omega):MU\in H^0(\Omega)\}.$$

The Dirichlet boundary value problem reads as follows:
Find a vector-function $U=(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3)^\top\in H^{1,0}(\Omega,M)$ satisfying both the differential equation

$$MU=f\quad\text{in}\quad\Omega\tag{2.5}$$

and the Dirichlet boundary condition

$$\gamma^+ U = \varphi_0 \quad \text{on} \quad S, \tag{2.6}$$

where $\varphi_0 = (\varphi_{01}, \dots, \varphi_{06})^\top \in H^{1/2}(S)$ and $f = (f_1, \dots, f_6)^\top \in H^0(\Omega)$ are the given vector functions.

Equation (2.5) is understood in the distributional sense, while the Dirichlet boundary condition (2.6) is understood in the usual trace sense.

The stress differential operator of the couple-stress elasticity corresponding to the operator (2.4) is defined as follows (see [20]):

$$\mathcal{T}(x, \partial_x) = \big[\mathcal{T}_{ij}(x, \partial_x)\big]_{6\times 6},$$

where

$$\big[\mathcal{T}_{ij}(x, \partial_x)\big]_{3\times 3} = \lambda(x) n_i \partial_j + \mu(x) n_j \partial_i + \delta_{ij}(\mu(x) + \varkappa(x)) n_k \partial_k, \quad \big[\mathcal{T}_{i,j+3}(x, \partial_x)\big]_{3\times 3} = -\varkappa(x)\varepsilon_{ijk} n_k,$$
$$\big[\mathcal{T}_{i+3,j+3}(x, \partial_x)\big]_{3\times 3} = \alpha(x) n_i \partial_j + \beta(x) n_j \partial_i + \delta_{ij}\gamma(x) n_k \partial_k, \quad \big[\mathcal{T}_{i+3,j}(x, \partial_x)\big]_{3\times 3} = 0, \quad i, j = 1, 2, 3.$$

The corresponding stress operator of adjoint operator M^* of the operator M is (cf. [4])

$$\widetilde{T} = \widetilde{T}(x, \partial_x) = \big[\widetilde{T}_{ij}(x, \partial_x)\big],$$

where

$$\big[\widetilde{T}_{ij}(x, \partial_x)\big]_{3\times 3} = \lambda(x) n_i \partial_j + \mu(x) n_j \partial_i + \delta_{ij}(\mu(x) + \varkappa(x)) n_k \partial_k, \quad \big[\widetilde{T}_{i,j+3}(x, \partial_x)\big]_{3\times 3} = \varkappa(x)\varepsilon_{ijk} n_k,$$
$$\big[\widetilde{T}_{i+3,j+3}(x, \partial_x)\big]_{3\times 3} = \alpha(x) n_i \partial_j + \beta(x) n_j \partial_i + \delta_{ij}\gamma(x) n_k \partial_k, \quad \big[\widetilde{T}_{i+3,j}(x, \partial_x)\big]_{3\times 3} = 0, \quad i, j = 1, 2, 3.$$

For any complex-valued vector-functions $U = (u_1, \dots, u_6)^\top$, $U' = (u'_1, \dots, u'_6) \in H^2(\Omega)$, we have the following Green's formulas:

$$\int_\Omega [M(x, \partial_x, \tau) U \cdot U' + E(U, \overline{U'})]\, dx = \int_S \{TU\}^+ \cdot \{U'\}^+\, dS, \tag{2.7}$$

$$\int_\Omega [M(x, \partial_x, \tau) U \cdot U' - U \cdot M^*(x, \partial_x, \tau) U']\, dx = \int_S [\{TU\}^+ \cdot \{U'\}^+\, dS - \{U\}^+ \cdot \{\widetilde{T} U'\}^+] dS, \tag{2.8}$$

where $a \cdot b := \sum_{j=1}^{6} a_j \overline{b}_j$ is the bilinear product of two column-vectors a, $b \in \mathbb{C}^6$,

$$E(U, \overline{U'}) := (\mu + \varkappa)\partial_j u_j \partial_j \overline{u'}_i + \tau^2 \varrho_0 u_i \overline{u'}_i + \lambda \partial_j u_j \partial_i \overline{u'}_i + \mu \partial_i u_j \partial_j \overline{u'}_i + \varkappa \varepsilon_{ijk} \Phi_k \partial_j \overline{u'}_i$$
$$+ \gamma \partial_j \Phi_i \partial_j \overline{\Phi'}_i + (2\varkappa + \tau^2 I_0)\Phi_i \overline{\Phi}'_i + \alpha \partial_j \Phi_j \partial_i \overline{\Phi'}_i + \beta \partial_i \Phi_j \partial_j \overline{\Phi'}_i + \varkappa \varepsilon_{ijk} \partial_j u_i \overline{\Phi'}_k.$$

By the standard limit procedure, Green's formulas (2.7) and (2.8) can be extended to the vector functions $U \in H^{1,0}(\Omega)$ and $U' \in H^{1,0}(\Omega)$. With the help of Green's formula (2.7), we can correctly define the generalized trace vector $\{T(x, \partial_x) U\}^+ = \{TU\}^+ = T^+ U \in H^{-\frac{1}{2}}(S)$ for the vector-function $U \in H^{1,0}(\Omega)$ (cf., [22]). Moreover, by [13, Lemma 3.4] and [22, Lemma 4.3]), for any $U \in H^{1,0}(\Omega; M)$ and $U' \in H^1(\Omega)$, the first Green's identity in the form

$$\langle \{T(x, \partial_x) U\}^+, \{U'\}^+ \rangle_S := \int_\Omega [M(x, \partial_x, \tau) U \cdot U' + E(U, \overline{U'})] dx, \quad \forall U' \in H^{1,0}(\Omega; M), \tag{2.9}$$

holds, where $\langle \cdot\,, \cdot \rangle_S$ denotes the duality between the adjoint spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$, which extends the usual bilinear $L_2(S)$ inner product.

Remark 2.1. From condition (2.3), it follows that the quadratic form $E(U, U')$ is positive definite. Therefore Green's first identity (2.9) and Korn's inequality along with the Lax-Milgram lemma imply that the Dirichlet BVP (2.5)–(2.6) is uniquely solvable in the space $H^{1,0}(\Omega; M)$ (see, e.g., [19, 21, 22, 33]).

2.2. **Parametrix-based operators and integral identities.** As it has already been mentioned, our goal here is to develop the LBDIE method for the Dirichlet BVP (2.5)–(2.6).

Let $F_\Delta(x) := -1/4\pi|x|$ denote the scalar fundamental solution of the Laplace operator, where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$.

Define a *localized matrix parametrix* for the matrix operator $I_6\Delta$ as

$$P(x) \equiv P_\chi(x) := P_\Delta(x)\, I_6 = \chi(x)\, F_\Delta(x)\, I_6 = -\frac{\chi(x)}{4\pi\,|x|}\, I_6, \tag{2.10}$$

where $P_\Delta(x) := \chi(x)\, F_\Delta(x)$ is a scalar function of the vector argument x, I_6 is the unit 6×6 matrix and χ is a localizing function (see Appendix A)

$$\chi \in X_+^k, \qquad k \geq 3, \quad \text{with} \quad \chi(0) = 1. \tag{2.11}$$

Throughout the paper, we assume that condition (2.11) is satisfied if not otherwise stated. Note that the function χ may have a compact support, useful for numerical implementations, but generally not necessary, and the class X_+^k includes the functions, not compactly supported, but sufficiently fast decreasing at infinity (see [7] and Appendix A below for details).

For sufficiently smooth vector-functions U and U', say $U, U' \in C^2(\overline{\Omega})$,the second Green's identity

$$\int\limits_\Omega \big[U' \cdot M(x,\partial_x,\tau)U - M^*(x,\partial_x,\tau)U' \cdot U\big]\, dx = \int\limits_S \big[\{U'\}^+ \cdot T^+U - \widetilde{T}^+U' \cdot \{U\}^+\big]\, dS \tag{2.12}$$

holds. Denote by $B(y,\varepsilon)$ a ball centered at the point y, of radius $\varepsilon > 0$, and let $S(y,\varepsilon) := \partial B(y,\varepsilon)$ be a sphere of radius ε. Let us take as $U'(x)$, successively, the columns of the matrix $P(x-y)$, where y is an arbitrarily fixed interior point in Ω, take also $\overline{U}$ instead of U and write the identity (2.12) for the region $\Omega_\varepsilon := \Omega\setminus B(y,\varepsilon)$ with $\varepsilon > 0$ such that $\overline{B(y,\varepsilon)} \subset \Omega$. Keeping in mind that $P^\top(x-y) = P(x-y)$, we arrive at the equality

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} \big[P(x-y)\; M(x,\partial_x,\tau)U(x) - \{M^*(x,\partial_x,\tau)P(x-y)\}\; U(x)\big]\, dx,\\
&\int\limits_S \big[P(x-y)\; T^+(x,\partial_x)U(x) - \{\widetilde{T}(x,\partial_x)P(x-y)\}\{U(x)\}^+\big]\, d_xS\\
&- \int\limits_{S(y,\varepsilon)} \big[P(x-y)\; T^+(x,\partial_x)U(x) - \{\widetilde{T}(x,\partial_x)P(x-y)\}\{U(x)\}^+\big]\, d_xS.
\end{aligned} \tag{2.13}$$

The normal vector on $S(y,\varepsilon)$ is directed inwards Ω_ε.

Let the operator $\mathcal{N}$ defined as

$$\mathcal{N}\,U(y) := \text{v.p.}\int\limits_\Omega [M^*(x,\partial_x,\tau)P(x-y)]U(x)\, dx := \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} [M^*(x,\partial_x,\tau)P(x-y)]U(x)\, dx \tag{2.14}$$

be the Cauchy principal-valued singular integral operator, which is well defined if the limit in the right-hand side exists. The similar operator with integration over the whole space $\mathbb{R}^3$ is denoted as

$$\mathbf{N}\,U(y) := \text{v.p.}\int\limits_{\mathbb{R}^3} [\,M^*(x,\partial_x,\tau)P(x-y)\,]\, U(x)\, dx\,. \tag{2.15}$$

Now, let us represent the differential operator $M(x,\partial_x,\tau)$ in the following form:

$$M(x,\partial_x,\tau) = M^{(0)}(x,\partial_x) + R(x,\partial_x,\tau), \tag{2.16}$$

where $M^{(0)}(x,\partial_x$ is the principal part of the operator $M(x,\partial_x,\tau)$ of the form

$$M^{(0)}(x,\partial_x)$$
$$:= \begin{bmatrix} \big[\delta_{ij}\big(\mu(x)+\varkappa(x)\big)\Delta + \big(\lambda(x)+\mu(x)\big)\partial_i\partial_j\big]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & \big[\delta_{ij}\gamma(x)\Delta + \big(\alpha(x)+\beta(x)\big)\partial_i\partial_j\big]_{3\times 3} \end{bmatrix}_{6\times 6}.$$

It is clear that the operator $M^{(0)}(x,\partial_x)$ is a positive definite, formally self-adjoint differential operator, and the operator $R(x,\partial_x,\tau)$ has the following form:

$$R(x,\partial_x,\tau)$$
$$:=\begin{bmatrix} \big[\delta_{ij}\partial_l\big(\mu(x)+\varkappa(x)\big)\partial_l+\partial_i\big(\lambda(x)+\mu(x)\big)\partial_j\big]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & \big[\delta_{ij}\partial_l\big(\gamma(x)\partial_l+\partial_i\big(\alpha(x)+\beta(x)\big)\partial_j\big]_{3\times 3} \end{bmatrix}_{6\times 6}$$
$$+\begin{bmatrix} -\rho_0(x)\tau^2 I_3 & \big[\varkappa(x)\varepsilon_{ijk}\partial_j\big]_{3\times 3} \\ \big[\varkappa(x)\varepsilon_{ijk}\partial_j\big]_{3\times 3} & -\big(I_0(x)\tau^2+2\varkappa(x)\big)I_3 \end{bmatrix}_{6\times 6}.$$

Consider the analogous representation of the formally adjoint differential operator M^*,

$$M^*(x,\partial_x,\tau)=M^{(0)}(x,\partial_x)+R^*(x,\partial_x,\tau),$$

where R^* is the operator, conjugate to R.

Note that

$$\frac{\partial^2}{\partial x_k\,\partial x_j}\,\frac{1}{|x-y|}=-\frac{4\,\pi\,\delta_{kj}}{3}\;\delta(x-y)+\text{v.p.}\,\frac{\partial^2}{\partial x_k\,\partial x_j}\frac{1}{|x-y|},\tag{2.17}$$

where $\delta(\,\cdot\,)$ is the Dirac distribution, the left-hand side in (2.17) is also understood in the distributional sense, while the second summand in the right-hand side is a Cauchy-integrable function. Therefore, in view of (2.10) and taking into account the fact that $\chi(0)=1$, we can write the following equality in the distributional sense:

$$\begin{aligned} M^*(x,\partial_x,\tau)P(x-y) &= M^{(0)}(x,\partial_x)P(x-y)+R^*(x,\partial_x,\tau)P(x-y)\\ &= \begin{bmatrix} \big[M^{(0)}_{ij}(x,\partial_x)F_\Delta(x-y)\big]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & [N^{(0)}_{ij}(x,\partial_x)F_\Delta(x-y)]_{3\times 3} \end{bmatrix}_{6\times 6}\\ &\quad +R^*(x,\partial_x,\tau)P(x-y)+R(x,y), \end{aligned}\tag{2.18}$$

where

$$M^{(0)}_{ij}(x,\partial_x):=\delta_{ij}\big(\mu(x)+\varkappa(x)\big)\Delta+\big(\lambda(x)+\mu(x)\big)\partial_i\partial_j,$$
$$N^{(0)}_{ij}(x,\partial_x):=\delta_{ij}\gamma(x)\Delta+\big(\alpha(x)+\beta(x)\big)\partial_i\partial_j,\quad i,j=1,2,3,$$
$$R(x,y):=\begin{bmatrix} \big[M^{(0)}_{ij}(x,\partial_x)\big(P_\Delta(x-y)-F_\Delta(x-y)\big)\big]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & [N^{(0)}_{ij}(x,\partial_x)\big(P_\Delta(x-y)-F_\Delta(x-y)\big)]_{3\times 3} \end{bmatrix}_{6\times 6}.$$

From equality (2.17), we have the following equalities:

$$\begin{aligned} M^{(0)}_{ij}(x,\partial_x)F_\Delta(x-y) &= \Big((\delta_{ij}\left(\mu(x)+\varkappa(x)\right)\Delta+\left(\lambda(x)+\mu(x)\right)\partial_i\partial_j\Big)F_\Delta(x-y)\\ = \delta_{ij}\left(\mu(x)+\varkappa(x)\right)\delta(x-y) &+ \left(\lambda(x)+\mu(x)\right)\frac{\delta_{ij}}{3}\delta(x-y)+\left(\lambda(x)+\mu(x)\right)\,\text{v.p.}\,\partial_i\partial_j F_\Delta(x-y)\\ &= \delta_{ij}\frac{1}{3}\left(\lambda(x)+4\mu(x)+3\varkappa(x)\right)\delta(x-y)+\left(\lambda(x)+\mu(x)\right)\ \text{v.p.}\,\partial_i\partial_j F_\Delta(x-y), \end{aligned}\tag{2.19}$$

and

$$\begin{aligned} N^{(0)}_{ij}(x,\partial_x)(F_\Delta(x-y) &= \Big(\delta_{ij}\left(\gamma(x)\right)\Delta+\left(\alpha(x)+\beta(x)\right)\partial_i\partial_j\Big)F_\Delta(x-y)\\ = \delta_{ij}\,\gamma(x)\,\delta(x-y) &+ \left(\alpha(x)+\beta(x)\right)\frac{\delta_{ij}}{3}\delta(x-y)+\left(\alpha(x)+\beta(x)\right)\,\text{v.p.}\,\partial_i\partial_j F_\Delta(x-y)\\ &= \delta_{ij}\frac{1}{3}\left(3\gamma(x)+\alpha(x)+\beta(x)\right)\delta(x-y)+\left(\alpha(x)+\beta(x)\right)\ \text{v.p.}\,\partial_i\partial_j F_\Delta(x-y). \end{aligned}\tag{2.20}$$

Applying the obtained equations (2.19) and (2.20) to equation (2.18), we get

$$M^*\left(x,\partial_x,\tau\right)\,P(x-y)=\boldsymbol{a}(x)\,\delta(x-y)+\text{v.p.}\,[M^*\left(x,\partial_x,\tau\right)P(x-y)],$$

where

$$\boldsymbol{a}(x) := \begin{pmatrix} \frac{1}{3}\big(\lambda(x)+4\mu(x)+3\varkappa(x)\big)I_3 & [0]_{3\times 3} \\ [0]_{3\times 3} & \frac{1}{3}\big(3\gamma(x)+\alpha(x)+\beta(x)\big)I_3 \end{pmatrix}_{6\times 6}, \tag{2.21}$$

and

$$\begin{gathered} \text{v.p.}\,[M^*(x,\partial_x,\tau)\,P(x-y)] \\ = \begin{bmatrix} \text{v.p.}\left[M_{ij}^{(0)}(x,\partial_x)F_\Delta(x-y)\right]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & \text{v.p.}\left[N_{ij}^{(0)}(x,\partial_x)F_\Delta(x-y)\right]_{3\times 3} \end{bmatrix}_{6\times 6} + R^{(1)}(x,y), \end{gathered}$$

which can be rewritten as follows:

$$\begin{gathered} \text{v.p.}\,[M^*(x,\partial_x,\tau)\,P(x-y)] \\ = \begin{bmatrix} \text{v.p.}\left[M_{ij}^{(0)}(y,\partial_x)F_\Delta(x-y)\right]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & \text{v.p.}\left[N_{ij}^{(0)}(y,\partial_x)F_\Delta(x-y)\right]_{3\times 3} \end{bmatrix}_{6\times 6} + R^{(2)}(x,y), \end{gathered} \tag{2.22}$$

where $F_\Delta(x-y) = -\frac{1}{4\pi|x-y|}$ is the fundamental solution of Laplace equations, and

$$\begin{gathered} R^{(1)}(x,y) := R^*(x,\partial_x,\tau)P(x-y) + R(x,y), \\ R^{(2)}(x,y) := R^{(1)}(x,y) + \Big[M^{(0)}(x,\partial_x) - M^{(0)}(y,\partial_x)\Big]F_\Delta(x-y). \end{gathered}$$

It is clear that $R(x,y)$, $R^{(1)}(x,y)$ and $R^{(2)}(x,y)$ have weak singularities $O(|x-y|^{-2})$, as $x \to y$.

Let us denote by $\overset{o}{E}$ zero the extension operator from Ω into Ω^-. From (2.13) and (2.14), it follows that

$$(\mathcal{N}U)(y) = (\mathbf{N}\overset{o}{E}\,U)(y) \text{ for } y\in\Omega, \quad U\in H^r(\Omega),\ r\geq 0.$$

The notation $\mathbf{N}$ can be expanded for smaller r as follows:

$$(\mathcal{N}U)(y) := (\mathbf{N}\widetilde{E}^r U)(y) \text{ for } y\in\Omega, \quad U\in H^r(\Omega),\ -1/2<r<1/2,$$

where $\widetilde{E}^r : H^r(\Omega) \to \widetilde{H}^r(\Omega)$ is the extension operator defined uniquely when $-1/2 < r < 1/2$ (see [23, Theorem 2.16]).

It follows from (2.21) (see [26], [2, Theorem 8.6.1]) that if $\chi \in X^k$ for integer $k \geq 2$, then the operators

$$\begin{gathered} r_\Omega\mathcal{N} = r_\Omega\mathbf{N}\overset{o}{E} \ : \ H^r(\Omega)\to H^r(\Omega), \quad 0\leq r, \\ r_\Omega\mathcal{N} = r_\Omega\mathbf{N}\widetilde{E}^r : H^r(\Omega)\to H^r(\Omega), \quad -1/2<r<1/2, \end{gathered}$$

are bounded, since the principal homogeneous symbol of $\mathbf{N}$ is rational (see (4.3) in Section 4) and the operators with the kernel functions $R(x,y)$ or $R^{(1)}(x,y)$, $R^{(2)}(x,y)$ map $H^r(\Omega)$ into $H^{r+1}(\Omega)$. Here and throughout the paper, r_Ω denotes the restriction operator to Ω.

Further, by direct calculations one can easily verify that

$$\lim_{\varepsilon\to 0}\int\limits_{S(y,\varepsilon)} P(x-y)\,T(x,\partial_x)U(x)\,d_xS = 0, \tag{2.23}$$

$$\begin{gathered} \lim_{\varepsilon\to 0}\int\limits_{S(y,\varepsilon)} \{T(x,\partial_x)\,P(x-y)\}U(x)d_xS \\ = \begin{bmatrix} \Big[\frac{\lambda(y)+\mu(y)}{4\pi}\int\limits_{\Sigma_1}\eta_j\eta_i d\Sigma_1 + \frac{\mu(y)+\varkappa(y)}{4\pi}\int\limits_{\Sigma_1}\eta_k\eta_k d\Sigma_1\delta_{ij}\Big]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & \Big[\frac{\alpha(y)+\beta(y)}{4\pi}\int\limits_{\Sigma_1}\eta_j\eta_i d\Sigma_1 + \frac{\gamma(y)}{4\pi}\int\limits_{\Sigma_1}\eta_k\eta_k d\Sigma_1\delta_{ij}\Big]_{3\times 3} \end{bmatrix}_{6\times 6} U(y) \end{gathered}$$

$$= \begin{bmatrix} \Big[\frac{\lambda(y)+\mu(y)}{4\pi}\frac{4\pi\delta_{ij}}{3}+\frac{\mu(y)+\varkappa(y)}{4\pi}4\pi\delta_{ij}\Big]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & \Big[\frac{\alpha(y)+\beta(y)}{4\pi}\frac{4\pi\delta_{ij}}{3}+\frac{\gamma(y)}{4\pi}4\pi\delta_{ij}\Big]_{3\times 3} \end{bmatrix}_{6\times 6} U(y) = \boldsymbol{a}(y)U(y), \quad (2.24)$$

where Σ_1 is a unit sphere, $\eta = (\eta_1, \eta_2, \eta_3) \in \Sigma_1$, $\boldsymbol{a}$ is defined by (2.21) and $i, j = 1, 2, 3$.

Passing to the limit in (2.13) as $\varepsilon \to 0$ and using relations (2.14), (2.23) and (2.24), we obtain

$$\boldsymbol{a}(y)\,U(y) + \mathcal{N}\,U(y) - V(T^+U)(y) + W(\gamma^+U)(y) = \mathcal{P}\big(MU\big)(y), \qquad y \in \Omega, \quad (2.25)$$

where $\mathcal{N}$ is a *localized singular integral operator* given by (2.14), while V, W and $\mathcal{P}$ are, respectively, the *localized vector single layer, double layer and Newtonian volume potentials*,

$$Vg(y) := -\int\limits_S P(x-y)\,g(x)\,d_xS, \quad (2.26)$$

$$Wg(y) := -\int\limits_S \big[\,T(x,\partial_x)\,P(x-y)\big]\;g(x)\,d_xS, \quad (2.27)$$

$$\mathcal{P}h(y) := \int\limits_\Omega P(x-y)\,h(x)\,dx. \quad (2.28)$$

Here, the densities g and h are the six-dimensional vector-functions. Introducing the localised scalar Newtonian volume potential

$$\mathcal{P}_\Delta h_0(y) := \int\limits_\Omega P_\Delta(x-y)\,h_0(x)\,dx \quad (2.29)$$

with h_0 being a scalar density function, we evidently obtain

$$[\mathcal{P}h(y)]_p = \mathcal{P}_\Delta h_p(y), \qquad p = \overline{1,6},$$

for any vector function $h = (h_1, \dots, h_6)^\top$.

We will need the localised vector Newtonian volume potential similar to (2.28), but with integration over the whole space $\mathbb{R}^3$,

$$\mathbf{P}h(y) := \int\limits_{\mathbb{R}^3} P(x-y)\,h(x)\,dx. \quad (2.30)$$

The mapping properties of potentials (2.26)–(2.30) have been investigated in [7, 12] and given in Appendix B.

We refer to relation (2.25) as *Green's third identity.* Due to the density of $\mathcal{D}(\overline{\Omega})$ in $H^{1,0}(\Omega; M)$ (see [25, Theorem 3.12]) and the mapping properties of the potentials, Green's third identity (2.25) is valid also for $u \in H^{1,0}(\Omega; M)$. In this case, the co-normal derivative T^+U is understood in the sense of definition (2.9). In particular, (2.25) holds true for the solutions of the above-formulated Dirichlet BVP (2.5)–(2.6).

On the other hand, applying the first Green's identity (2.9) on Ω_ε to $U \in H^1(\Omega)$ and to $P(x-y)$ as $U'(x)$, and taking the limit as $\varepsilon \to 0$, one can easily derive another, more general, form of the third Green's identity,

$$\boldsymbol{a}(y)\,U(y) + \mathcal{N}\,U(y) + W(\gamma^+U)(y) = \mathcal{Q}\,U(y), \qquad \forall\, y \in \Omega, \quad (2.31)$$

where for the i-th component of the vector $\mathcal{Q}\,u(y)$, we have

$$\begin{aligned}[\mathcal{Q}U(y)]_i := -\int\limits_\Omega \big[&(\mu(x)+\varkappa(x))\,\partial_j P_\Delta(x-y)\partial_j u_i(x) + \tau^2\rho_0(x)P_\Delta(x-y)u_i(x) \\ &+ \lambda(x)\partial_i P_\Delta(x-y)\partial_j u_j(x) + \mu(x)\partial_j P_\Delta(x-y)\partial_i u_j(x) + \varkappa(x)\varepsilon_{ijk}\partial_j P_\Delta(x-y)\phi_k(x)\big]dx, \quad i = 1,2,3,\end{aligned}$$

$$\begin{aligned}[\mathcal{Q}U(y)]_{i+3} := -\int\limits_\Omega \big]&\gamma(x)\partial_j P_\Delta(x-y)\partial_j \phi_i\,(x) + (2\varkappa(x)+\tau^2 I_0(x))P_\Delta(x-y)\phi_i\,(x) \\ &+ \alpha(x)\partial_i P_\Delta(x-y)\partial_j \phi_j\,(x) + \beta(x)\partial_j P_\Delta(x-y)\partial_i \phi_j\,(x) + \varkappa(x)\varepsilon_{kji}P_\Delta(x-y)\partial_j u_k(x)\big]dx, \quad i = 1,2,3,\end{aligned}$$

whence we obtain

$$[\mathcal{Q}U(y)]_i = \partial_j P_\Delta\left((\mu+\varkappa)\partial_j u_i\right)(y) - \tau^2 P_\Delta\left(\rho\, u_i\right)(y) + \partial_i P_\Delta\left(\lambda\,\partial_j u_j\right)(y)$$
$$+\,\partial_j P_\Delta\left(\mu\,\partial_i u_j\right)(y) + \varepsilon_{ijk}\partial_j P_\Delta\left(\varkappa\,\phi_k\right)(y), \qquad i=1,2,3, \quad \forall y\in\Omega$$
$$[\mathcal{Q}U(y)]_{i+3} = \partial_j P_\Delta\left(\gamma\,\partial_j\phi_i\right)(y) - 2P_\Delta\left(\varkappa\,\phi_i\right)(y) - \tau^2 P_\Delta\left(I_0\,\phi_i\right)(y)$$
$$+\,\partial_i P_\Delta\left(\alpha\,\partial_j\phi_j\right)(y) + \partial_j P_\Delta\left(\beta\,\partial_i\phi_j\right)(y) - \varepsilon_{kji}P_\Delta\left(\varkappa\,\partial_j u_k\right)(y), \qquad i=1,2,3, \quad \forall y\in\Omega.$$

Using the properties of localized potentials described in Appendix B (see Theorems B.1 and B.4) and taking the trace of equation (2.25) on S, for $U\in H^{1,\,0}(\Omega^+;M)$, we arrive at the relation

$$\mathcal{N}^+U - \mathcal{V}(T^+U) + (\boldsymbol{a}-\boldsymbol{b})\,\gamma^+U + \mathcal{W}(\gamma^+U) = \mathcal{P}^+\big(MU\big) \quad \text{on } S, \tag{2.32}$$

where the localized boundary integral operators $\mathcal{V}$ and $\mathcal{W}$ generated by the localized single and double layer potentials are defined in (B.1) and (B.2), the matrix $\boldsymbol{b}$ is defined in Theorem B.4, while

$$\mathcal{N}^+ := \gamma^+\mathcal{N}, \qquad \mathcal{P}^+ := \gamma^+\mathcal{P}.$$

Now, we prove the following technical

Lemma 2.2. *Let $\chi\in X^3$, $f\in H^0(\Omega)$, $F\in H^{1,0}(\Omega,\Delta)$, $\psi\in H^{-\frac12}(S)$ and $\varphi\in H^{\frac12}(S)$. Moreover, let $U\in H^1(\Omega)$ and the following equation*

$$\boldsymbol{a}(y)U(y) + \mathcal{N}U(y) - V\psi(y) + W\varphi(y) = F(y) + \mathcal{P}f(y), \quad y\in\Omega \tag{2.33}$$

hold. Then $U\in H^{1,0}(\Omega,M)$.

Proof. Note that by Theorem B.1, $\mathcal{P}f\in H^2(\Omega)$ for arbitrary $f\in H^0(\Omega)$, while by Theorem B.2, the inclusions $V\psi, W\varphi\in H^{1,0}(\Omega,\Delta)$ hold for arbitrary $\psi\in H^{-\frac12}(S)$ and $\varphi\in H^{\frac12}(S)$. In view of relations (2.31)–(2.33), equation (2.33) can be rewritten component-wise as

$$\partial_j P_\Delta\left((\mu+\varkappa)\partial_j u_i\right)(y) - \tau^2 P_\Delta\left(\rho\, u_i\right)(y) + \partial_i P_\Delta\left(\lambda\,\partial_j u_j\right)(y) + \partial_j P_\Delta\left(\mu\,\partial_i u_j\right)(y)$$
$$+\,\varepsilon_{ijk}\partial_j P_\Delta\left(\varkappa\,\phi_k\right)(y) \;=\; F_i(y) + \mathcal{P}_\Delta f_i\,(y) + [V\psi(y)]_i - \big[W\big(\varphi-\{U\}^+\big)(y)\big]_i, \quad y\in\Omega \quad i=\overline{1,3},$$
$$\partial_j P_\Delta\left(\gamma\,\partial_j\phi_i\right)(y) - 2P_\Delta\left(\varkappa\,\phi_i\right)(y) - \tau^2 P_\Delta\left(I_0\,\phi_i\right)(y)$$
$$+\,\partial_i P_\Delta\left(\alpha\,\partial_j\phi_j\right)(y) + \partial_j P_\Delta\left(\beta\,\partial_i\phi_j\right)(y) - \varepsilon_{kji}P_\Delta\left(\varkappa\,\partial_j u_k\right)(y)$$
$$= F_{i+3}(y) + \mathcal{P}_\Delta f_{i+3}\,(y) + [V\psi(y)]_{i+3} - \big[W\big(\varphi-\{U\}^+\big)(y)\big]_{i+3}, \qquad y\in\Omega \quad i=\overline{1,3}.$$

Due to Theorems B.1 and B.2, it follows that the right-hand side functions in the above equalities belong to the space

$$H^{1,0}(\Omega,\Delta) := \{v\in H^1(\Omega)\,:\,\Delta v\in H^0(\Omega)\},$$

since $\gamma^+U\in H^{\frac12}(S)$. We have

$$\Delta_x\, P_\Delta(x-y) = \delta(x-y) + R_\Delta(x-y), \tag{2.34}$$

where

$$R_\Delta(x-y) := -\frac{1}{4\,\pi}\Big\{\frac{\Delta\,\chi(x-y)}{|x-y|} + 2\,\frac{\partial\,\chi(x-y)}{\partial x_l}\,\frac{\partial}{\partial x_l}\frac{1}{|x-y|}\Big\}. \tag{2.35}$$

Clearly, $R_\Delta(x-y)=\mathcal{O}(|x-y|^{-2})$ as $x\to y$ and by (2.34) and (2.35), one can establish that for an arbitrary scalar test function $\phi\in\mathcal{D}(\Omega)$, the relation (see, e.g., [27])

$$\Delta\,\mathcal{P}_\Delta\phi(y) = \phi(y) + \mathcal{R}_\Delta\phi(y), \quad y\in\Omega \tag{2.36}$$

holds, where

$$\mathcal{R}_\Delta\phi(y) := \int\limits_\Omega R_\Delta(x-y)\,\phi(x)\,dx. \tag{2.37}$$

Evidently, (2.36) remains true also for $\phi\in H^0(\Omega)$, since $\mathcal{D}(\Omega)$ is dense in $H^0(\Omega)$. It is easy to see that (see [7])

$$\mathcal{R}_\Delta\,:\,H^0(\Omega)\to H^1(\Omega)\,. \tag{2.38}$$

Consequently,

$$\partial_j[\Delta_{\,y}P_\Delta\left((\mu+\varkappa)\partial_j u_i\right)(y)] - \tau^2\Delta_y\,P_\Delta\left(\rho\,u_i\right)(y) + \partial_i[\Delta_y P_\Delta\left(\lambda\,\partial_j u_j\right)(y)] + \partial_j[\Delta_y P_\Delta\left(\mu\,\partial_i u_j\right)(y)]$$

$$
\begin{aligned}
&+\varepsilon_{ijk}\Delta_y\partial_j P_\Delta\,(\varkappa\phi_k)\,(y) = \partial_j\,((\mu+\varkappa)\partial_j u_i)\,(y) + \partial_j[\,R_\Delta\,((\mu+\varkappa)\partial_j u_i)\,(y)] - \tau^2\,(\rho u_i)\,(y)\\
&-\tau^2 R_\Delta\,(\rho u_i)\,(y) + \partial_i\,(\lambda\,\partial_j u_j)\,(y) + \partial_i R_\Delta\,(\lambda\,\partial_j u_j)\,(y) + \partial_j\,(\mu\,\partial_i u_j)\,(y) + \partial_j R_\Delta\,(\mu\,\partial_i u_j)\,(y)\\
&+\varepsilon_{ijk}\partial_j\,(\varkappa\,\phi_k)\,(y) + \varepsilon_{ijk}\partial_j R_\Delta\,(\varkappa\,\phi_k)\,(y) = [MU]_i + \ \partial_j[R_\Delta\,((\mu+\varkappa)\partial_j u_i)\,(y)]\\
&-\tau^2\,R_\Delta\,(\rho\,u_i)\,(y)\ + \partial_j R_\Delta\,(\lambda\,\partial_i u_i)\,(y) + \partial_i R_\Delta\,(\mu\,\partial_j u_i)\,(y) + \varepsilon_{ijk}\partial_j\,(\varkappa\,\phi_k)\,(y)\\
&+\varepsilon_{ijk}\partial_j R_\Delta\,(\varkappa\,\phi_k)\,(y) = [MU(y)]_i\ + \big[R^{(1)}_\Delta U\,(y)\big]_i, \qquad \forall y\in\Omega, \quad i=1,2,3,
\end{aligned}
\tag{2.39}
$$

and

$$
\begin{aligned}
&\partial_j\Delta_y P_\Delta\,(\gamma\,\partial_j\phi_i)\,(y) - 2\Delta_y P_\Delta\,(\varkappa\,\phi_i)\,(y) - \tau^2\Delta_y P_\Delta\,(I_0\,\phi_i)\,(y)\\
&\partial_i\Delta_y P_\Delta\,(\alpha\,\partial_j\phi_j)\,(y) + \partial_j\Delta_y P_\Delta\,(\beta\,\partial_i\phi_j)\,(y) - \varepsilon_{kji}\Delta_y P_\Delta\,(\varkappa\,\partial_j u_k)\,(y)\,] = \partial_j\,(\gamma\,\partial_j\phi_i)\,(y)\\
&+\partial_j R_\Delta\,(\gamma\,\partial_j\phi_i)\,(y) - 2\,(\varkappa\,\phi_i)\,(y) - \tau^2\,(I_0\,\phi_i)\,(y) - 2R_\Delta\,(\varkappa\,\phi_i)\,(y) - \tau^2 R_\Delta\,(I_0\,\phi_i)\,(y)\\
&+\partial_i\,(\alpha\,\partial_j\phi_j)\,(y) + \partial_i R_\Delta\,(\alpha\,\partial_j\phi_j)\,(y) + \partial_j\,(\beta\,\partial_i\phi_j)\,(y) + \partial_j R_\Delta\,(\beta\,\partial_i\phi_j)\,(y) - \varepsilon_{ijk}\,(\varkappa\,\partial_j u_k)\,(y)\\
&-\varepsilon_{ijk} R_\Delta\,(\varkappa\,\partial_j u_k)\,(y) = [MU(y)]_{i+3} + \partial_j R_\Delta\,(\gamma\,\partial_j\phi_i)\,(y)\ - 2R_\Delta\,(\varkappa\,\phi_i)\,(y) - \tau^2 R_\Delta\,(I_0\,\phi_i)\,(y)\\
&+\partial_j\,(\alpha\,\partial_i\phi_i)\,(y) + \partial_j R_\Delta\,(\alpha\,\partial_i\phi_i)\,(y)\ + \partial_i\,(\beta\,\partial_j\phi_i)\,(y)\ + \partial_i R_\Delta\,(\beta\,\partial_j\phi_i)\,(y)\ - \varepsilon_{kji}\,(\varkappa\,\partial_j u_k)\,(y)\\
&-\varepsilon_{kji}\Delta_y R_\Delta\,(\varkappa\,\partial_j u_k)\,(y)\ = [M(y)]_{i+3}\ + \big[R^{(1)}_\Delta U\,(y)\big]_{i+3}, \qquad \forall y\in\Omega \quad i=1,2,3.
\end{aligned}
\tag{2.40}
$$

Evidently, $\mathcal{R}^{(1)}_\Delta U \in H^0(\Omega)$, whence the embedding $MU \in H^0(\Omega)$ follows from (2.39), (2.40) due to (2.37). □

Actually, the continuity of the operator in (2.38) and identities (2.40), (2.39) in the proof of Lemma 2.2 imply by (2.31) the following assertion.

Corollary 2.3. *If $\chi \in X^3$, then the operator*

$$\boldsymbol{a} + \mathcal{N}\ :\ H^{1,0}(\Omega, M) \to H^{1,0}(\Omega, \Delta)$$

is bounded.

Remark 2.4. Note that the localized parametrix can be determined by the scalar fundamental solution of the Helmholtz operator $\Delta - \tau^2$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, i.e.,

$$P(x) = -\frac{\chi(x)e^{-\tau|x|}}{4\pi|x|}\,.$$

3. LBDIE, Formulation of the Dirichlet Problem, and the Equivalence Theorem

Let $U \in H^{1,0}(\Omega, M)$ be a solution to the Dirichlet BVP (2.5)–(2.6) with $\varphi_{{}_0} \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$. As we have derived above, relations (2.25) and (2.32) hold and now can be rewritten in the form

$$(\boldsymbol{a} + \mathcal{N})U - V\psi = \mathcal{P}f - W\varphi_{{}_0} \quad \text{in}\ \Omega, \tag{3.1}$$

$$\mathcal{N}^+U - \mathcal{V}\psi = \mathcal{P}^+f - (\boldsymbol{a} - \boldsymbol{b})\,\varphi_{{}_0} - \mathcal{W}\varphi_{{}_0} \quad \text{on}\ S, \tag{3.2}$$

where $\psi := T^+U \in H^{-\frac{1}{2}}(S)$ and $\boldsymbol{b}$ is defined in Theorem B.4. One can consider these relations as the LBDIE system with respect to the unknown vector-functions U and ψ. Now, we prove the following equivalence theorem.

Theorem 3.1. *Let $\chi \in X^3_+, \varphi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega)$.*

(i) *If a vector-function $U \in H^{1,0}(\Omega, M)$ solves the Dirichlet BVP* (2.5)–(2.6)*, then the solution is unique and the pair $(U, \psi) \in H^{1,0}(\Omega, M) \times H^{-\frac{1}{2}}(S)$ with*

$$\psi = T^+U\,, \tag{3.3}$$

solves the LBDIE system (3.1)–(3.2)*.*

(ii) *Vice versa, if the pair $(U, \psi) \in H^{1,0}(\Omega, M) \times H^{-\frac{1}{2}}(S)$ solves the LBDIE system* (3.1)–(3.2)*, then the solution is unique, the vector-function U solves the Dirichlet BVP* (2.5)–(2.6)*, and relation* (3.3) *holds.*

Proof. (i) The first part of the theorem is trivial and folows directly from relations (2.25), (2.32), (3.3) and Remark 2.1.

(ii) Now, let the pair $(U,\psi)\in H^{1,0}(\Omega,M)\times H^{-\frac{1}{2}}(S)$ solve the LBDIE system (3.1)–(3.2). Taking the trace of (3.1) on S and comparing it with (3.2), we get

$$\gamma^+U=\varphi_0 \quad \text{on } \ S. \tag{3.4}$$

Further, since $U\in H^{1,0}(\Omega,M)$, in view of (3.4), the Green's third identity (2.25) can be rewritten as

$$(\boldsymbol{a}+\mathcal{N})U-V(T^+U)=\mathcal{P}\big(MU\big)-W\varphi_{_0} \quad \text{in } \ \Omega. \tag{3.5}$$

From (3.1) and (3.5), it follows that

$$V(T^+U-\psi)+\mathcal{P}\big(MU-f\big)=0 \ \text{ in } \ \Omega,$$

whence by Lemma 6.3 in [7], we have

$$MU=f \ \text{ in } \ \Omega \quad \text{and} \quad T^+U=\psi \ \text{ on } \ S.$$

Thus U solves the Dirichlet BVP (2.5)–(2.6) and equation (3.3) holds.

The uniqueness of a solution to the LBDIE system (3.1)–(3.2) in the space $H^{1,0}(\Omega,M)\times H^{-\frac{1}{2}}(S)$ follows directly from the above-proven equivalence result and the uniqueness theorem for the Dirichlet problem (2.5)–(2.6) (see Remark 2.1). □

4. Symbols and Invertibility of a Domain Operator in a Half-space

In what follows, in our analysis, we need the explicit expression of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{N})(y,\xi)$ of the singular integral operator $\mathcal{N}$ which due to (2.14), (2.15) and (2.16) reads as

$$\mathfrak{S}(\mathcal{N})(y,\xi)=\mathfrak{S}(\mathbf{N})(y,\xi)=\begin{pmatrix}\mathfrak{S}(\mathbf{N}_1)(y,\xi) & 0\\ 0 & \mathfrak{S}(\mathbf{N}_2)(y,\xi)\end{pmatrix}_{6\times6},$$

where

$$\begin{aligned}
\mathfrak{S}(\mathbf{N}_1)(y,\xi)&=-\frac{1}{4\pi}\Big[\mathcal{F}_{z\to\xi}\Big(\text{v.p.}\delta_{ij}(\mu(y)+\varkappa(y))\Delta\frac{1}{|z|}+\text{v.p.}(\lambda(y)+\mu(y))\frac{\partial^2}{\partial z_i\partial z_j}\frac{1}{|z|}\Big)\Big]_{3\times3}\\
&=-\frac{1}{4\pi}\Big[\text{v.p.}(\lambda(y)+\mu(y))\mathcal{F}_{z\to\xi}\frac{\partial^2}{\partial z_i\partial z_j}\frac{1}{|z|}\Big]_{3\times3}=-\frac{1}{4\pi}\Big[(\lambda(y)+\mu(y))\mathcal{F}_{z\to\xi}\Big(\frac{4\pi\delta_{ij}}{3}\delta(z)+\frac{\partial^2}{\partial z_i\partial z_j}\frac{1}{|z|}\Big)\Big]_{3\times3}\\
&=\Big[-\frac{1}{3}(\lambda(y)+\mu(y))\delta_{ij}-\frac{1}{4\pi}(\lambda(y)+\mu(y))(-i\xi_i)(-i\xi_j)\mathcal{F}_{z\to\xi}\frac{1}{|z|}\Big]_{3\times3}\\
&=\Big[-\frac{1}{3}(\lambda(y)+\mu(y))\delta_{ij}+\frac{(\lambda(y)+\mu(y))\xi_i\,\xi_j}{|\xi|^2}\Big]_{3\times3}\\
&=\Big[-(\mu(y)+\varkappa(y))\delta_{ij}-\frac{1}{3}(\lambda(y)+\mu(y))\delta_{ij}+\frac{\delta_{ij}(\mu(y)+\varkappa(y))|\xi|^2+(\lambda(y)+\mu(y))\xi_i\,\xi_j}{|\xi|^2}\Big]_{3\times3}\\
&=-\boldsymbol{a}_1(y)+\frac{M_1(y,\xi)}{|\xi|^2},
\end{aligned}$$

$$\begin{aligned}
M_1(y,\xi)&=\big[\mu(y)+\varkappa(y))|\xi|^2\delta_{ij}+(\lambda(y)+\mu(y))\xi_i\,\xi_j\big]_{3\times3},\\
\boldsymbol{a}_1(y)&=\frac{1}{3}\,(\lambda(y)+4\mu(y)+3\varkappa(y))\,I_3,
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{S}(\mathbf{N}_2)(y,\xi)&=-\frac{1}{4\pi}\Big[\mathcal{F}_{z\to\xi}\Big(\text{v.p.}\delta_{ij}\gamma(y)\Delta\frac{1}{|z|}+(\alpha(y)+\beta(y))\frac{\partial^2}{\partial z_i\partial z_j}\frac{1}{|z|}\Big)\Big]_{3\times3}\\
&=-\frac{1}{4\pi}\Big[(\alpha(y)+\beta(y))\mathcal{F}_{z\to\xi}\text{v.p.}\frac{\partial^2}{\partial z_i\partial z_j}\frac{1}{|z|}\Big]_{3\times3}=-\frac{1}{4\pi}\Big[(\alpha(y)+\beta(y))\mathcal{F}_{z\to\xi}\Big(\frac{4\pi\delta_{ij}}{3}\delta(z)+\frac{\partial^2}{\partial z_i\partial z_j}\frac{1}{|z|}\Big)\Big]_{3\times3}\\
&=\Big[-\frac{1}{3}(\alpha(y)+\beta(y))\delta_{ij}-\frac{1}{4\pi}(\alpha(y)+\beta(y))(-i\xi_i)(-i\xi_j)\mathcal{F}_{z\to\xi}\frac{1}{|z|}\Big]_{3\times3}
\end{aligned}$$

$$= -\frac{1}{3}(\alpha(y)+\beta(y))\delta_{ij} + \frac{(\alpha(y)+\beta(y))\xi_i\,\xi_j}{|\xi|^2}$$
$$= \Big[-\gamma(y)\delta_{ij} - \frac{1}{3}(\alpha(y)+\beta(y))\delta_{ij} + \frac{\delta_{ij}\gamma(y)|\xi|^2 + (\alpha(y)+\beta(y))\xi_i\,\xi_j}{|\xi|^2}\Big]_{3\times 3} = -\boldsymbol{a}_2(y) + \frac{M_2(y,\xi)}{|\xi|^2},$$

$$M_2(y,\xi) = \big[\delta_{ij}\gamma(y)|\xi|^2 + (\alpha(y)+\beta(y))\xi_i\,\xi_j\big]_{3\times 3},$$
$$\boldsymbol{a}_2(y) = \frac{1}{3}\,(\alpha(y)+\beta(y)+3\gamma(y))\,I_3.$$

From the above expressions, we obtain

$$\mathfrak{S}(\mathcal{N})(y,\xi) = \mathfrak{S}(\mathbf{N})(y,\xi) = \begin{pmatrix} \mathfrak{S}(\mathbf{N}_1)(y,\xi) & 0 \\ 0 & \mathfrak{S}(\mathbf{N}_2)(y,\xi)\end{pmatrix}_{6\times 6}$$
$$= -\mathbf{a}(y) + \frac{1}{|\xi|^2}\begin{pmatrix} M_1(y,\xi) & [0]_{3\times 3} \\ [0]_{3\times 3} & M_2(y,\xi)\end{pmatrix}_{6\times 6} = -\mathbf{a}(y) + \frac{M(y,\xi)}{|\xi|^2}, \quad y\in\Omega, \quad \xi\in\mathbb{R}^3, \tag{4.1}$$

where

$$M(y,\xi) := \begin{pmatrix} M_1(y,\xi) & [0]_{3\times 3} \\ [0]_{3\times 3} & M_2(y,\xi)\end{pmatrix}_{6\times 6}, \qquad \boldsymbol{a}(y) := \begin{pmatrix} \boldsymbol{a}_1(y) & [0]_{3\times 3} \\ [0]_{3\times 3} & \boldsymbol{a}_2(y)\end{pmatrix}_{6\times 6},$$

and the Fourier transform operator $\mathcal{F}$ is defined as

$$\mathcal{F}g(\xi) = \mathcal{F}_{z\to\xi}[g(z)] = \int\limits_{\mathbb{R}^3} g(z)\,e^{i\,z\cdot\xi}\,dz.$$

Here, we have applied $\mathcal{F}_{z\to\xi}[(4\pi|z|)^{-1}] = |\xi|^{-2}$ (see, e.g., [17]).

As we see, the entries of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{N})(y,\xi)$ of the operator $\mathcal{N}$ are the even rational homogeneous functions in ξ of order 0. It can easily be verified that both the characteristic function of the singular kernel in (2.22) and the symbol (4.1) satisfy the Tricomi condition, i.e., their integral averages over the unit sphere vanish (cf., [27]).

Relation (4.1) implies that the principal homogeneous symbols of the singular integral operators $\mathbf{N}$ and $\boldsymbol{a}+\mathbf{N}$ read as

$$\begin{aligned} &\mathfrak{S}(\mathbf{N})(y,\xi) = |\xi|^{-2}M(y,\xi) - \boldsymbol{a} \qquad \forall y\in\overline{\Omega}, \quad \forall\,\xi\in\mathbb{R}^3\setminus\{0\}, \\ &\mathfrak{S}(\boldsymbol{a}+\mathbf{N})(y,\xi) = |\xi|^{-2}M(y,\xi) \qquad \forall\,y\in\overline{\Omega}, \quad \forall\,\xi\in\mathbb{R}^3\setminus\{0\}. \end{aligned} \tag{4.2}$$

Due to (2.3), the symbol matrix (4.2) is positive definite,

$$[\mathfrak{S}(\boldsymbol{a}+\mathbf{N})(y,\xi)\,\zeta]\cdot\bar{\zeta} = |\xi|^{-2}\,\bar{\zeta}\cdot M(y,\xi)\,\zeta \ge c_1\,|\zeta|^2 \quad \forall\,y\in\overline{\Omega}, \quad \forall\xi\in\mathbb{R}^3\setminus\{0\}, \quad \forall\,\zeta\in\mathbb{C}^6,$$

where c_1 is the same positive constant.

Denote

$$\mathbf{B} := \boldsymbol{a} + \mathbf{N}.$$

By (4.2), the principal homogeneous symbol matrix of the operator $\mathbf{B}$ reads as

$$\mathfrak{S}(\mathbf{B})(y,\xi) = |\xi|^{-2}M(y,\xi) \qquad \text{for} \qquad y\in\overline{\Omega}, \quad \xi\in\mathbb{R}^3\setminus\{0\}, \tag{4.3}$$

is an even rational homogeneous matrix-function of order 0 in ξ and due to (2.3), it is positive definite,

$$[\mathfrak{S}(\mathbf{B})(y,\xi)\zeta]\cdot\bar{\zeta} \ge\; c_1\,|\zeta|^2 \qquad \text{for all} \qquad y\in\overline{\Omega}, \quad \xi\in\mathbb{R}^3\setminus\{0\} \text{ and } \zeta\in\mathbb{C}^6.$$

Consequently, $\mathbf{B}$ is a strongly elliptic pseudodifferential operator of zero order (i.e., the Cauchy-type singular integral operator) and the partial indices of factorization of symbol (4.3) are equal to zero (cf., [3, 5, 30]).

In our further analysis, we need some auxiliary assertions. To formulate them, let $\widetilde{y}\in S=\partial\Omega$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B})(\widetilde{y},\xi)\equiv\mathfrak{S}(\mathbf{B})(\xi)$ of the operator $\mathbf{B}$ written in the chosen local co-ordinate system. Further, let $\widehat{\mathbf{B}}$ denote the pseudodifferential operator with the symbol

$$\widehat{\mathfrak{S}}(\mathbf{B})(\xi', \xi_3) := \mathfrak{S}(\mathbf{B})\big((1+|\xi'|)\omega, \xi_3\big), \quad \text{where } \omega = \frac{\xi'}{|\xi'|}, \ \ \xi = (\xi', \xi_3), \ \ \xi' = (\xi_1, \xi_2).$$

Then the frozen principal homogeneous symbol matrix $\mathfrak{S}(\mathbf{B})(\xi)$ is likewise the principal homogeneous symbol matrix of the operator $\widehat{\mathbf{B}}$. It can be factorized with respect to the variable ξ_3 as

$$\mathfrak{S}(\mathbf{B})(\xi) = \mathfrak{S}^{(-)}(\mathbf{B})(\xi)\ \mathfrak{S}^{(+)}(\mathbf{B})(\xi),$$

where

$$\mathfrak{S}^{(\pm)}(\mathbf{B})(\xi) = \frac{1}{\Theta^{(\pm)}(\xi', \xi_3)}\, M^{(\pm)}(\xi', \xi_3).$$

Here, $\Theta^{(\pm)}(\xi', \xi_3) := \xi_3 \pm i|\xi'|$ are the "plus" and "minus" factors of the symbol $\Theta(\xi) := |\xi|^2$, and $M^{(\pm)}(\xi', \xi_3)$ are the "plus" and "minus" polynomial matrix factors of the first order in ξ_3 of the positive definite polynomial symbol matrix $M(\xi', \xi_3) \equiv M(y, \xi', \xi_3)$ corresponding to the frozen differential operator $M(y, \partial_x)$ at the point $y \in S$ (see [14–16]), i.e.,

$$M(\xi', \xi_3) = M^{(-)}(\xi', \xi_3)\ M^{(+)}(\xi', \xi_3) \tag{4.4}$$

with $\det M^{(+)}(\xi', \tau) \neq 0$ for $\operatorname{Im}\tau > 0$ and $\det M^{(-)}(\xi', \tau) \neq 0$ for $\operatorname{Im}\tau < 0$. Moreover, the entries of the matrices $M^{(\pm)}(\xi', \xi_3)$ are the homogeneous functions in $\xi = (\xi', \xi_3)$ of order 1.

Denote by $a^{(\pm)}(\xi')$ the coefficients at ξ_3^6 in the determinants $\det M^{(\pm)}(\xi', \xi_3)$. Evidently,

$$a^{(-)}(\xi')\, a^{(+)}(\xi') = \det\, M(0,0,1) > 0 \quad \text{for} \quad \xi' \neq 0. \tag{4.5}$$

It is easy to see that the factor-matrices $M^{(\pm)}(\xi', \xi_3)$ have the following structure

$$\Big(\big[M^{(\pm)}(\xi', \xi_3)\big]^{-1}\Big)_{ij} = \frac{1}{\det M^{(\pm)}(\xi', \xi_3)}\, p_{ij}^{(\pm)}(\xi', \xi_3), \quad i,j = \overline{1,6},$$

where $p_{ij}^{(\pm)}(\xi', \xi_3)$ are the co-factors of the matrix $M^{(\pm)}(\xi', \xi_3)$ which can be written in the form

$$p_{ij}^{(\pm)}(\xi', \xi_3) = \sum_{k=0}^{5} c_{ij}^{(\pm),k}(\xi')\, \xi_3^{5-k}. \tag{4.6}$$

Here, $c_{ij}^{(\pm),k}$, $k = \overline{0,5}$, $i,j = \overline{1,6}$ are the homogeneous functions of order k with respect to ξ'.

From the above-said it follows that the entries of the factor-symbol matrices $\mathfrak{b}_{kj}^{(\pm)}(\omega, r, \xi_3) := \mathfrak{S}_{kj}^{(\pm)}(\mathbf{B})(\xi', \xi_3)$, $k,j = 1,2,3$, with $\omega = \xi'/|\xi'|$ and $r = |\xi'|$, satisfy the following relations:

$$\frac{\partial^l \mathfrak{b}_{kj}^{(\pm)}(\omega, 0, -1)}{\partial r^l} = (-1)^l\, \frac{\partial^l \mathfrak{b}_{kj}^{(\pm)}(\omega, 0, +1)}{\partial r^l}\,, \qquad l = 0,1,2,\dots .$$

These relations imply that the entries of the matrices $\mathfrak{S}^{(\pm)}(\mathbf{B})(\xi', \xi_3)$ belong to the class of symbols D_0 introduced in [17, Ch. III, §10]

$$\mathfrak{S}^{(\pm)}(\mathbf{B})(\xi', \xi_3) \in D_0.$$

Denote by $\Pi^{\pm}$ the Cauchy type integral operators

$$\Pi^{\pm} h(\xi) := \pm\, \frac{i}{2\pi} \lim_{t \to 0+} \int\limits_{-\infty}^{+\infty} \frac{h(\xi', \eta_3)\, d\eta_3}{\xi_3 \pm i\,t - \eta_3},$$

which are well defined at any $\xi \in \mathbb{R}^3$ for a bounded smooth function $h(\xi', \cdot)$ satisfying the relation $h(\xi', \eta_3) = \mathcal{O}(1 + |\eta_3|)^{-\kappa}$ with some $\kappa > 0$.

Let $\mathring{E}_+$ be the extension by zero operator from $\mathbb{R}^3_+$ onto the whole space $\mathbb{R}^3$ and let $r_+ := r_{\mathbb{R}^3_+} : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3_+)$ be the restriction operator to the half-space $\mathbb{R}^3_+$. First, we prove the following assertion.

Lemma 4.1. *Let $s \geq 0$ and $\chi \in X_+^k$ with integer $k \geq 2$. The operator*

$$r_+ \widehat{\mathbf{B}} \mathring{E}_+ \ : \ H^s(\mathbb{R}^3_+) \to H^s(\mathbb{R}^3_+)$$

is invertible.

Moreover, for $f \in H^s(\mathbb{R}^3_+)$, the unique solution of the equation

$$r_+ \widehat{\mathbf{B}} \mathring{E}_+ U = f$$

for $U \in H^s(\mathbb{R}^3_+)$ can be represented in the form $U = r_+ U_+$, where

$$U_+ = \mathring{E} U = \mathcal{F}^{-1}\Big\{ [\widehat{\mathfrak{S}}^{(+)}(\mathbf{B})]^{-1} \Pi^+ \Big([\widehat{\mathfrak{S}}^{(-)}(\mathbf{B})]^{-1} \mathcal{F}(f_*) \Big) \Big\},$$

and $f_* \in H^s(\mathbb{R}^3)$ is an extension of $f \in H^s(\mathbb{R}^3_+)$ (i.e. $r_+ f_* = f$) such that $\|f_*\|_{H^s(\mathbb{R}^3)} = \|f\|_{H^s(\mathbb{R}^3_+)}$.

Lemma 4.2. *Let the factor matrix $M^{(+)}(\xi', \tau)$ be as in (4.4), and $a^{(+)}$ and let $c_{ij}^{(+),0}$ be as in (4.5) and (4.6), respectively. Then the following equality*

$$\frac{1}{2\pi i} \int\limits_{\Gamma^-} \big[M^{(+)}(\xi', \tau) \big]^{-1} d\tau = \frac{1}{a^{(+)}(\xi')} C^{(+),0}(\xi'),$$

holds, where $C^{(+),0}(\xi') = \big[c_{ij}^{(+),0}(\xi') \big]_{ij=1}^6$ and $\det \big[C^{(+),0}(\xi') \big] \neq 0$ for $\xi' \neq 0$. Here, Γ^- is a contour in the lower complex half-plane enclosing all the roots of the polynomial $\det M^{(+)}(\xi', \tau)$ with respect to τ.

Proof. Note that $\det M^{(+)}(\xi', \tau)$ is a third-order polynomial in τ, while $p_{ij}^{(+)}(\xi', \tau)$ is a second-order polynomial in τ defined in (4.6).

Let Γ_R be a circle centred at the origin and having sufficiently large radius R. By the Cauchy theorem, we then derive

$$\begin{aligned} \frac{1}{2\pi i} \int\limits_{\Gamma^-} \big\{ \big[M^{(+)}(\xi', \tau) \big]^{-1} \big\}_{ij} \, d\tau &= \frac{1}{2\pi i} \int\limits_{\Gamma^-} \frac{p_{ij}^{(+)}(\xi', \tau)}{\det M^{(+)}(\xi', \tau)} \, d\tau = \frac{1}{2\pi i} \int\limits_{\Gamma_R} \frac{p_{ij}^{(+)}(\xi', \tau)}{\det M^{(+)}(\xi', \tau)} \, d\tau \\ &= \frac{1}{2\pi i} \frac{c_{ij}^{(+),0}(\xi')}{a^{(+)}(\xi')} \int\limits_{\Gamma_R} \frac{1}{\tau} \, d\tau + \int\limits_{\Gamma_R} Q_{ij}(\xi', \tau) \, d\tau = \frac{c_{ij}^{(+),0}(\xi')}{a^{(+)}(\xi')} + \int\limits_{\Gamma_R} Q_{ij}(\xi', \tau) \, d\tau, \end{aligned} \tag{4.7}$$

where $Q_{ij}(\xi', \tau) = O(|\tau|^{-2})$ as $|\tau| \to \infty$.

It is clear that

$$\lim_{R \to \infty} \int\limits_{\Gamma_R} Q_{ij}(\xi', \tau) \, d\tau = 0.$$

Therefore by passing to the limit in (4.7), as $R \to \infty$, we obtain

$$\frac{1}{2\pi i} \int\limits_{\Gamma^-} \big\{ \big[M^{(+)}(\xi', \tau) \big]^{-1} \big\}_{ij} \, d\tau = \frac{c_{ij}^{(+),0}(\xi')}{a^{(+)}(\xi')}.$$

Now, we show that $\det[C^{(+),0}] \neq 0$. We introduce the notations

$$P^{(+)}(\xi', \xi_3) = [p_{ij}^{(+)}(\xi', \xi_3)]_{ij=1}^6 = \sum_{k=0}^{5} C^{(+),k}(\xi') \xi_3^{5-k},$$

where

$$C^{(+),k}(\xi') = [c_{ij}^{(+),k}(\xi')]_{ij=1}^6, \quad k = \overline{0,5}.$$

Since $\det[M^{(+)}(\xi', \xi_3)]^{-1} \neq 0$ for $\xi = (\xi', \xi_3) \neq 0$, therefore $\det P^{(+)}(\xi', \xi_3) \neq 0$ for $\xi = (\xi', \xi_3) \neq 0$.

Let us introduce new coordinates $r = |\xi'|$, $\omega = \xi'/|\xi'|$ and denote

$$\mathcal{P}^{(+)}(\omega, r, \xi_3) := P^{(+)}(\xi', \xi_3) = P^{(+)}(\omega \, r, \xi_3).$$

Then we have

$$\det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \det P^{(+)}(\xi', \xi_3) = \det \Big(\sum_{k=0}^{5} C^{(+),k}(\omega) \xi_3^{5-k}\, r^k \Big) \quad \text{for all } \ \xi_3 \neq 0,$$

whence

$$\lim_{r\to 0} \det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \xi_3^{30} \, \det C^{(+),0}(\omega).$$

Consequently, $\det C^{(+),0}(\omega) \neq 0$ and Lemma 4.2 is proved. □

Further, let us introduce an auxiliary operator Π' defined as

$$\Pi'(g)(\xi') := \lim_{x_3\to 0+} r_{\mathbb{R}^3_+} \mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi', \xi_3)] = \frac{1}{2\pi} \lim_{x_3\to 0+} \int\limits_{-\infty}^{+\infty} g(\xi', \xi_3) e^{-ix_3\xi_3}\, d\xi_3 = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} g(\xi', \xi_3)\, d\xi_3$$

for $g(\xi', \cdot) \in L_1(\mathbb{R})$.

The operator Π' can be extended to the class of functions $g(\xi', \xi_3)$, being rational in ξ_3 with the denominator, not vanishing for real non-zero $\xi = (\xi', \xi_3) \in \mathbb{R}^3 \setminus \{0\}$, homogeneous of order $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ in ξ and infinitely differentiable with respect to ξ for $\xi' \neq 0$. Then it can be shown that (see Appendix C in [12])

$$\Pi'(g)(\xi') = \lim_{x_3\to 0+} r_{\mathbb{R}_+} \mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi', \xi_3)] = -\frac{1}{2\pi} \int\limits_{\Gamma^-} g(\xi', \zeta)\, d\zeta,$$

where $r_{\mathbb{R}_+}$ denotes the restriction operator onto $\mathbb{R}_+ = (0, +\infty)$ with respect to x_3, Γ^- is a contour in the lower complex half-plane in ζ, orientated anticlockwise and enclosing all the poles of the rational function $g(\xi', \cdot)$. It is clear that if $g(\xi', \zeta)$ is holomorphic in ζ in the lower complex half-plane ($\operatorname{Im} \zeta < 0$), then $\Pi'(g)(\xi') = 0$.

5. Invertibility of the Dirichlet LBDIO

From Theorem 3.1, it follows that the LBDIE system (3.1)–(3.2) with a special right-hand side is uniquely solvable in the space $H^{1,0}(\Omega, M) \times H^{-1/2}(S)$. Let us investigate the localized boundary-domain integral operator, generated by the left-hand side expressions in (3.1)–(3.2), in appropriate functional spaces.

The LBDIE system (3.1)–(3.2) with arbitrary right-hand side vector-functions from the space $H^1(\Omega) \times H^{1/2}(S)$ can be written as

$$\mathbf{B}\mathring{E}U \ - V\psi = F_1 \quad \text{in } \ \Omega, \tag{5.1}$$

$$\mathbf{N}^+\mathring{E}U - \mathcal{V}\psi = F_2 \quad \text{on } \ S, \tag{5.2}$$

where $\mathbf{B} = \boldsymbol{a} + \mathbf{N}$, $F_1 \in H^1(\Omega)$ and $F_2 \in H^{1/2}(S)$. We denote by $\mathfrak{D}$ the localized boundary-domain integral operator generated by the left-hand side expressions in LBDIE system (5.1)–(5.2),

$$\mathfrak{D} := \begin{bmatrix} r_\Omega \mathbf{B}\mathring{E} & -r_\Omega V \\ \mathbf{N}^+\mathring{E} & -\mathcal{V} \end{bmatrix}.$$

We would like to prove the following assertion.

Theorem 5.1. *Let the localising function $\chi \in X^\infty_+$ and $r > -\frac{1}{2}$. Then the operator*

$$\mathfrak{D} \ : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S) \tag{5.3}$$

is invertible.

We reduce the theorem proof to several lemmas.

Lemma 5.2. *Let $\chi \in X^\infty$. The operator $r_\Omega \mathbf{B}\mathring{E} \, : \, H^s(\Omega) \to H^s(\Omega)$ for $s \geq 0$ is Fredholm with zero index.*

Proof. Since (4.3) is a rational function in ξ, we can apply the theory of pseudo-differential operators with the symbol satisfying the transmission conditions (see [2, 3, 17, 29, 30]). Now, using the local principle (see Lemma 23.9 in [17]) and Lemma 4.1, we deduce that the operator

$$\mathcal{B} := r_{\Omega}\mathbf{B}\,\mathring{E}\ :\ H^s(\Omega) \to H^s(\Omega)$$

is Fredholm for all $s \geq 0$.

To show that $\operatorname{Ind}\mathcal{B} = 0$, we use the fact that the operators $\mathcal{B}$ and

$$\mathcal{B}_t = r_{\Omega}(\boldsymbol{a} + t\,\mathbf{N})\mathring{E}$$

are homotopic, and $t \in [0,1]$. Note that $\mathcal{B} = \mathcal{B}_1$. The principal homogeneous symbol of the operator $\mathcal{B}_t$ has the form

$$\mathfrak{S}(\mathcal{B}_t)(y,\xi) = \boldsymbol{a}(y) + t\ \mathfrak{S}(\mathbf{N})(y,\xi) = (1-t)\boldsymbol{a}(y) + t\mathfrak{S}(\mathbf{B})(y,\xi).$$

It is easy to see that the symbol $\mathfrak{S}(\mathcal{B}_t)(y,\xi)$ is positive definite,

$$[\mathfrak{S}(\mathcal{B}_t)(y,\xi)\zeta]\cdot\bar{\zeta} = (1-t)[\boldsymbol{a}(y)\,\zeta]\cdot\bar{\zeta} + t[\mathfrak{S}(\mathbf{B})(y,\xi)\zeta]\cdot\bar{\zeta} \geq c|\zeta|^2$$

for all $y \in \overline{\Omega}$, $\xi \neq 0$, $\zeta \in \mathbb{C}^6$ and $t \in [0,1]$, where c is some positive number.

Since $\mathfrak{S}(\mathcal{B}_t)(y,\xi)$ is rational, even and homogeneous of order zero in ξ, we conclude, as above, that the operator

$$\mathcal{B}_t\ :\ H^s(\Omega) \to H^s(\Omega)$$

is Fredholm for all $s \geq 0$ and for all $t \in [0,1]$. Therefore $\operatorname{Ind}\mathcal{B}_t$ is the same for all $t \in [0,1]$. On the other hand, due to the equality $\mathcal{B}_0 = r_{\Omega}\,I$, we get

$$\operatorname{Ind}\mathcal{B} = \operatorname{Ind}\mathcal{B}_1 = \operatorname{Ind}\mathcal{B}_t = \operatorname{Ind}\mathcal{B}_0 = 0. \qquad \square$$

Lemma 5.3. *Let $\chi \in X^{\infty}$. The operator $\mathfrak{D}$ given by* (5.3) *is Fredholm.*

Proof. To investigate Fredholm properties of the operator $\mathfrak{D}$, we apply the local principle (cf., e.g., [1, 17], § 19 and § 22). Due to this principle, we have to show first that the operator $\mathfrak{D}$ is locally Fredholm at an arbitrary "frozen" interior point $\widetilde{y} \in \Omega$ and, secondly, that the so-called generalized *Šapiro–Lopatinskiĭ condition* for the operator $\mathfrak{D}$ holds at an arbitrary "frozen" boundary point $\widetilde{y} \in S$. To obtain the explicit form of this condition, we proceed as follows. Let $\widetilde{U}$ be a neighbourhood of a fixed point $\widetilde{y} \in \overline{\Omega}$ and let $\widetilde{\psi}_0, \widetilde{\varphi}_0 \in \mathcal{D}(\widetilde{U})$ such that

$$\operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0 \neq \emptyset, \qquad \widetilde{y} \in \operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0.$$

Consider the operator $\widetilde{\psi}_0\mathfrak{D}\,\widetilde{\varphi}_0$ separately in two possible cases: case (1) $\widetilde{y} \in \Omega$, and case (2) $\widetilde{y} \in S$.

Case (1). If $\widetilde{y} \in \Omega$ then we can choose a neighbourhood $\widetilde{U}$ such that $\overline{\widetilde{U}} \subset \Omega$. Therefore the operator $\widetilde{\psi}_0\mathfrak{D}\,\widetilde{\varphi}_0$ has the same Fredholm properties as the operator $\widetilde{\psi}_0\mathbf{B}\,\widetilde{\varphi}_0$ (see the similar arguments in the proof of Theorem 22.1 in [17]). Then by Lemma 5.2, we conclude that $\widetilde{\psi}_0\mathfrak{D}\,\widetilde{\varphi}_0$ is a locally Fredholm operator at the interior points of Ω.

Case (2). If $\widetilde{y} \in S$, then at this point we have to "froze" the operator $\widetilde{\psi}_0\,\mathfrak{D}\,\widetilde{\varphi}_0$, implying that we can choose a neighbourhood $\widetilde{U}$ sufficiently small such that in the local co-ordinate system with the origin at the point $\widetilde{y}$ and the third axis coinciding with the normal vector at the point $\widetilde{y} \in S$, the following decomposition

$$\widetilde{\psi}_0\mathfrak{D}\,\widetilde{\varphi}_0 = \widetilde{\psi}_0\left(\widehat{\mathfrak{D}} + \widetilde{\mathbf{K}} + \widetilde{\mathbf{T}}\right)\widetilde{\varphi}_0 \tag{5.4}$$

holds, where

$$\widetilde{\mathbf{K}}\ : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2)$$

is a bounded operator with a small norm, while

$$\widetilde{\mathbf{T}}\ : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+2}(\mathbb{R}^3_+) \times H^{r+3/2}(\mathbb{R}^2)$$

is a bounded operator. The operator

$$\widehat{\mathfrak{D}} := \begin{bmatrix} r_+\widehat{\mathbf{B}}\mathring{E} & -r_+\widehat{V} \\ \widehat{\mathbf{N}}^+\mathring{E} & -\widehat{\mathcal{V}} \end{bmatrix}$$

with $r_+ = r_{\mathbb{R}^3_+}$, is defined in the upper half-space $\mathbb{R}^3_+$ and possesses the following mapping property:

$$\widehat{\widehat{\mathfrak{D}}} : H^{r+1}(\mathbb{R}^3_+) \times H^{r-1/2}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r+1/2}(\mathbb{R}^2). \tag{5.5}$$

The operators appearing in the expression of $\widehat{\mathfrak{D}}$ are defined as follows: for the operator $\mathcal{M}$, the operator $\widehat{\mathcal{M}}$ denotes the operator in $\mathbb{R}^3$ constructed by the symbol

$$\widehat{\mathfrak{S}}(\mathcal{M})(\xi) = \mathfrak{S}(\mathcal{M})\big((1+|\xi'|)\omega, \xi_3\big),$$

where $\omega = \frac{\xi'}{|\xi'|}$, $\xi = (\xi', \xi_3)$, $\xi' = (\xi_1, \xi_2)$.

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of operator (5.5). Indeed, let us write the system corresponding to the operator $\widehat{\mathfrak{D}}$ as

$$r_+ \widehat{\mathbf{B}} \mathring{E} \widetilde{U} - r_+ \widehat{V} \widetilde{\psi} = \widetilde{F}_1 \quad \text{in } \mathbb{R}^3_+, \tag{5.6}$$

$$\widehat{\mathbf{N}}^+ \mathring{E} \widetilde{U} - \widehat{\mathcal{V}} \widetilde{\psi} = \widetilde{F}_2 \quad \text{on } \mathbb{R}^2, \tag{5.7}$$

where $\widetilde{F}_1 \in H^1(\mathbb{R}^3_+)$, $\widetilde{F}_2 \in H^{1/2}(\mathbb{R}^2)$.
Note that the operator $r_+ \widehat{\mathbf{B}} \mathring{E}$ is a singular integral operator with an even rational elliptic principal homogeneous symbol. Then due to Lemma 4.1, the operator

$$r_+ \widehat{\mathbf{B}} \mathring{E} : H^{r+1}(\mathbb{R}^3_+) \to H^{r+1}(\mathbb{R}^3_+)$$

is invertible. We can determine $\widetilde{U}$ from equation (5.6) and write

$$\mathring{E}\widetilde{U} = \mathring{E}\big[r_+ \widehat{\mathbf{B}} \mathring{E}\big]^{-1} \widetilde{f} = \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\mathbf{B})\big]^{-1}\, \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\mathbf{B})\big]^{-1} \mathcal{F}(\widetilde{f}_*)\Big)\Big\}, \tag{5.8}$$

where $\widetilde{f}_* = \widetilde{F}_{1*} + \widehat{V}\widetilde{\psi}$ is an extension of $\widetilde{f} = \widetilde{F}_1 + r_+ \widehat{V} \widetilde{\psi}$ from $\mathbb{R}^3_+$ to $\mathbb{R}^3$ preserving the function space. The symbols $\widehat{\mathfrak{S}}^{(\pm)}(M)$ denote the so-called "plus" and "minus" factors in the factorization of the symbol $\widehat{\mathfrak{S}}(M)$ with respect to the variable ξ_3. Note that the function $\mathring{E}\widetilde{u}$ in (5.8) does not depend on the chosen extension $\widetilde{f}_*$ of $\widetilde{f}$.

Substituting (5.8) into (5.7), we obtain the following pseudo-differential equation with respect to the unknown function $\widetilde{\psi}$:

$$\widehat{\widehat{\mathbf{N}}}^+ \mathcal{F}^{-1}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\widetilde{\mathbf{B}})\big]^{-1}\, \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\widetilde{\mathbf{B}})\big]^{-1} \mathcal{F}(\widehat{\widehat{V}}\widetilde{\psi})\Big)\Big\} - \widehat{\widehat{\mathcal{V}}}\widetilde{\psi} = \widetilde{F}| \quad \text{on } \mathbb{R}^2, \tag{5.9}$$

where

$$\widetilde{F} = \widetilde{F}_2 - \widehat{\mathbf{N}}^+ \mathring{E}\, \big[r_+ \widehat{\mathbf{B}} \mathring{E}\big]^{-1} \widetilde{F}_1.$$

It is easy to see that

$$\mathbf{N}^+\, v\,(\widetilde{y}\,') = \Big[\mathcal{F}^{-1}_{\xi\to\widetilde{y}}\big[\mathfrak{S}(\mathbf{N})(\xi)\ \mathcal{F}(v)(\xi)\big]\Big]_{\widetilde{y}_3=0+} = \mathcal{F}^{-1}_{\xi\,'\to\widetilde{y}\,'}\Big[\Pi'\big[\mathfrak{S}(\mathbf{N})\ \mathcal{F}(v)\big](\xi')\Big].$$

In view of the relation (see, e.g., [13, Eq. (4.1)], [12, Eqs. (B.5), (B.6)])

$$V\widetilde{\psi}(y) = -\langle \gamma P(\cdot - y), \widetilde{\psi}\rangle_S = -\langle P(\cdot - y), \gamma^*\widetilde{\psi}\rangle_{\mathbb{R}^3} = -\mathbf{P}(\gamma^*\widetilde{\psi})(y),$$

the operator γ^* is dual to the trace operator γ. When the surface S coincides with $\mathbb{R}^2 = \partial\mathbb{R}^3_+$, we have $\gamma^*\widetilde{\psi} = \widetilde{\psi}(\widetilde{y}') \otimes \delta_3$ with δ_3, being the one-dimensional Dirac distribution in the $\widetilde{y}_3$ direction. Thus we arrive at the equality

$$\begin{aligned}
&\widehat{\mathbf{N}}^+ \mathcal{F}^{-1}_{\xi\to\widetilde{x}}\Big\{\big[\widehat{\mathfrak{S}}^{(+)}(\mathbf{B})(\xi)\big]^{-1}\, \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\mathbf{B})\big]^{-1} \mathcal{F}(\widehat{V}\widetilde{\psi})\Big)(\xi)\Big\}\,(\widetilde{y}\,')\\
&= -\mathcal{F}^{-1}_{\xi'\to\widetilde{y}\,'}\Big\{\Pi'\Big[\widehat{\mathfrak{S}}(\mathbf{N})\big[\widehat{\mathfrak{S}}^{(+)}(\mathbf{B})\big]^{-1}\, \Pi^+\Big(\big[\widehat{\mathfrak{S}}^{(-)}(\mathbf{B})\big]^{-1} \widehat{\mathfrak{S}}(\mathbf{P})\Big)\Big](\xi')\mathcal{F}_{\widetilde{x}\,'\to\xi'}\widetilde{\psi}\Big\}.
\end{aligned}$$

Using these relations, equation (5.9) can be rewritten in the form

$$\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\,\big[\,\widehat{e}\,(\xi')\,\mathcal{F}(\widetilde{\psi})(\xi')\big] = \widetilde{F}(\widetilde{y}\,') \quad \text{on } \mathbb{R}^2, \tag{5.10}$$

where

$$\widehat{e}(\xi') = e\Big((1+|\xi'|)\,\omega\Big), \qquad \omega = \frac{\xi'}{|\xi'|},$$

with e, being a homogeneous function of order -1 given by the equality

$$e(\xi') = -\Pi'\Big\{\mathfrak{S}(\mathbf{N})\big[\mathfrak{S}^{(+)}(\mathbf{B})\big]^{-1}\Pi^{+}\Big(\big[\mathfrak{S}^{(-)}(\mathbf{B})\big]^{-1}\mathfrak{S}(\mathbf{P})\Big)\Big\}(\xi') - \mathfrak{S}(\mathcal{V})(\xi'), \quad \forall\,\xi' \neq 0. \tag{5.11}$$

If the function $\det e(\xi')$ is other than zero for all $\xi' \neq 0$, then $\det \widehat{e}(\xi') \neq 0$ for all $\xi' \in \mathbb{R}^2$, and the corresponding pseudo-differential operator

$$\widehat{\mathbf{E}} : H^s(\mathbb{R}) \to H^{s+1}(\mathbb{R}) \quad \text{for all } \; s \in \mathbb{R}$$

generated by the left-hand side expression in (5.10) is invertible. In particular, it follows that the system of equation (5.6)–(5.7) is uniquely solvable with respect to $(\widetilde{U},\, \widetilde{\psi})$ in the space $H^1(\mathbb{R}^3_+) \times H^{-1/2}(\mathbb{R}^2)$ for arbitrary right-hand sides $(\widetilde{F}_1,\, \widetilde{F}_2) \in H^1(\mathbb{R}^3_+) \times H^{1/2}(\mathbb{R}^2)$. Consequently, the operator $\widehat{\mathfrak{D}}$ in (5.5) is invertible, which implies that operator (5.4) possesses a left and a right regularizer. In its turn, this implies that operator (5.3) possesses a left and right regularizer, as well. Thus operator (5.3) is Fredholm if

$$\det e(\xi') \neq 0 \qquad \forall\,\xi' \neq 0.$$

This condition is called the *Šapiro–Lopatinskiĭ condition* (cf., [17], Theorems 12.2 and 23.1, and also formulas (12.27), (12.25)). Let us show that in our case the Šapiro–Lopatinskiĭ condition holds. To this end let us note that the principal homogeneous symbols $\mathfrak{S}(\mathbf{N})$, $\mathfrak{S}(\mathbf{B})$, $\mathfrak{S}(\mathbf{P})$ and $\mathfrak{S}(\mathcal{V})$ of the operators $\mathbf{N}$, $\mathbf{B}$, $\mathbf{P}$ and $\mathcal{V}$ in the chosen local co-ordinate system appearing in formula (5.11) read as

$$\mathfrak{S}(\mathbf{N})(\xi) = |\xi|^{-2}M(\xi) - \boldsymbol{a}, \quad \mathfrak{S}(\mathbf{B})(\xi) = |\xi|^{-2}M(\xi), \quad \mathfrak{S}(\mathbf{P})(\xi) = -|\xi|^{-2}\,I, \quad \mathfrak{S}(\mathcal{V})(\xi') = \frac{1}{2|\xi'|}\,I,$$
$$\xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2).$$

Rewrite (5.11) in the form

$$\begin{aligned} e(\xi') &= -\Pi'\big\{\big(\mathfrak{S}(\mathbf{B}) - \boldsymbol{a}\big)[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi') - \mathfrak{S}(\mathcal{V})(\xi') \\ &= e_1(\xi') + e_2(\xi') - \mathfrak{S}(\mathcal{V})(\xi'), \end{aligned} \tag{5.12}$$

where

$$e_1(\xi') = -\Pi'\big\{\mathfrak{S}(\mathbf{B})[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi'), \tag{5.13}$$

$$e_2(\xi') = \boldsymbol{a}\,\Pi'\big\{[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi'), \tag{5.14}$$

$$\mathfrak{S}(\mathcal{V})(\xi') = \frac{1}{2|\xi'|}\,I. \tag{5.15}$$

Direct calculations result in

$$\begin{aligned} \Pi^{+}\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)(\xi') &= \frac{i}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}\frac{\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)(\xi',\eta_3)\,d\eta_3}{\xi_3 + i\,t - \eta_3} \\ = -\frac{i}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}&\frac{[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi',\eta_3)\,d\eta_3}{(\xi_3 + i\,t - \eta_3)\,(|\xi'|^2 + \eta_3^2)} = \frac{i}{2\pi}\lim_{t\to 0+}\int_{\Gamma^-}\frac{[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi',\tau)\,d\tau}{(\xi_3 + i\,t - \tau)\,(|\xi'|^2 + \tau^2)} \\ = \frac{i}{2\pi}\lim_{t\to 0+}&\frac{2\pi i\,[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|)}{(\xi_3 + i\,t + i|\xi'|)\,2\,(-i|\xi'|)} = -\frac{i\,[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|)}{2\,|\xi'|\,\Theta^{(+)}(\xi', \xi_3)}. \end{aligned} \tag{5.16}$$

Now, from (5.13), by virtue of (5.16), we derive

$$\begin{aligned} e_1(\xi') &= -\Pi'\Big\{\mathfrak{S}^{(-)}(\mathbf{B})\mathfrak{S}^{(+)}(\mathbf{B})[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)\Big\}(\xi') \\ &= -\Pi'\big\{\mathfrak{S}^{(-)}(\mathbf{B})\Pi^{+}\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi') = \Pi'\Big\{\frac{\mathfrak{S}^{(-)}(\mathbf{B})}{\Theta^{(+)}}\Big\}(\xi')\,\Big(\frac{i\,[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|)}{2\,|\xi'|}\Big) \\ &= -\frac{1}{2\,\pi}\int_{\Gamma^-}\frac{\mathfrak{S}^{(-)}(\mathbf{B})(\xi',\tau)}{\tau + i|\,\xi'|}\,d\tau\,\Big(\frac{i\,[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|)}{2\,|\xi'|}\Big) \end{aligned}$$

$$= -i\, \mathfrak{S}^{(-)}(\mathbf{B})(\xi', -i\,|\xi'|)\, \frac{i\,[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|)}{2\,|\xi'|} = \frac{1}{2\,|\xi'|}\, I. \tag{5.17}$$

Quite similarly, from (5.14), with the help of (5.16), we get

$$\begin{aligned} e_2(\xi') &= \boldsymbol{a}\, \Pi'\big\{[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^+\big([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi') = \\ &\quad - \boldsymbol{a}\, \Pi'\Big\{\frac{[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}}{\Theta^{(+)}}\Big\}(\xi')\, \Big(\frac{i\,[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|)}{2\,|\xi'|}\Big) \\ &= -\frac{i\,\boldsymbol{a}}{2\,|\xi'|}\Big(- \frac{1}{2\,\pi}\int\limits_{\Gamma^-} \frac{[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}(\xi', \tau)}{\tau + i|\,\xi'|}\, d\tau\Big)[\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}(\xi', -i|\xi'|) \\ &= \frac{i\,\boldsymbol{a}}{4\,\pi\,|\xi'|}\int\limits_{\Gamma^-}[M^{(+)}(\xi', \tau)]^{-1}d\tau\ (-2\,i\,|\xi'|)\ [M^{(-)}(\xi', -i\,|\xi'|)]^{-1} \\ &= i\boldsymbol{a}\,\Big\{\frac{1}{2\,\pi\, i}\int\limits_{\Gamma^-}[M^{(+)}(\xi', \tau)]^{-1}d\tau\Big\}[M^{(-)}(\xi', -i|\xi'|)]^{-1}. \end{aligned}$$

Therefore due to (5.12), (5.15), (5.17) and Lemma 4.2, we have

$$e_2(\xi') = \frac{i}{a^{(+)}(\xi')}\boldsymbol{a}\, C^{(+),0}(\xi')\ [M^{(-)}(\xi', -i\,|\xi'|)]^{-1},$$

where $\det \boldsymbol{a} \neq 0$, $\det C^{(+),0}(\xi') \neq 0$ and $\det M^{(-)}(\xi', -i\,|\xi'|) \neq 0$ for all $\xi' \neq 0$. Then it is clear that

$$\det e(\xi') = -\frac{i}{\big(a^{(+)}(\xi')\big)^3}\det \boldsymbol{a}\ \det C^{(+),0}(\xi')\det[M^{(-)}(\xi', -i\,|\xi'|)]^{-1} \neq 0$$

for all $\xi' \neq 0$.

Thus we have found that for the operator $\mathfrak{D}$, the Šapiro–Lopatinskiĭ condition holds. Therefore the operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm for $r > -\frac{1}{2}$. □

Lemma 5.4. *Let $\chi \in X^\infty$. The operator $\mathfrak{D}$ given by* (5.3) *is Fredholm with a zero index.*

Proof. For $t \in [0, 1]$, let us consider the operator

$$\mathfrak{D}_t := \begin{bmatrix} r_\Omega \mathbf{B}_t \mathring{E} & -r_\Omega V \\ t\,\mathbf{N}^+ \mathring{E} & -\mathcal{V} \end{bmatrix}$$

with $\mathbf{B}_t = \boldsymbol{a} + t\,\mathbf{N}$ and establish that it is homotopic to the operator $\mathfrak{D} = \mathfrak{D}_1$. We have to check that for the operator $\mathfrak{D}_t$, the Šapiro–Lopatinskiĭ condition is satisfied for all $t \in [0, 1]$. Indeed, in this case, the Šapiro–Lopatinskiĭ condition reads as

$$\det e_t(\xi') \neq 0 \quad \text{for all } \ \xi' \neq 0,$$

where (cf. (5.11))

$$\begin{aligned} e_t(\xi') &= -\Pi'\big\{\big(\mathfrak{S}(\mathbf{B}_t) - \boldsymbol{a}\big)[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+\big([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi') - \mathfrak{S}(\mathcal{V})(\xi') \\ &= e_t^{(1)}(\xi') + e_t^{(2)}(\xi') - \mathfrak{S}(\mathcal{V})(\xi') \end{aligned} \tag{5.18}$$

with

$$e_t^{(1)}(\xi') = -\Pi'\big\{\mathfrak{S}(\mathbf{B}_t)[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+\big([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi') = \frac{1}{2\,|\xi'|}\, I, \tag{5.19}$$

$$e_t^{(2)}(\xi') = \boldsymbol{a}\,\Pi'\big\{[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+\big([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\big)\big\}(\xi'),$$

$$\mathfrak{S}(\widetilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|}\, I. \tag{5.20}$$

By direct calculations, we get

$$
\begin{aligned}
e_t^{(2)}(\xi') &= \boldsymbol{a}\,\Pi'\Big\{[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^{+}\Big([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\Big)\Big\}(\xi') \\
&= -\boldsymbol{a}\,\Pi'\Big\{\frac{[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}}{\Theta^{(+)}}\Big\}(\xi')\ \Big(\frac{i\,[\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}(\xi',-i|\xi'|)}{2\,|\xi'|}\Big) \\
&= -\frac{i\,\boldsymbol{a}}{2\,|\xi'|}\Big(-\frac{1}{2\,\pi}\int\limits_{\Gamma^-}\frac{[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}(\xi',\tau)}{\tau+i|\,\xi'|}\,d\tau\Big)[\mathfrak{S}^{-}(\mathbf{B}_t)]^{-1}(\xi',-i|\xi'|) \\
&= \frac{i\,\boldsymbol{a}}{4\,\pi\,|\xi'|}\int\limits_{\Gamma^-}[M_t^{(+)}(\xi',\tau)]^{-1}d\tau\ (-2\,i\,|\xi'|)\ [M_t^{(-)}(\xi',-i\,|\xi'|)]^{-1} \\
&= i\,\boldsymbol{a}\Big\{\frac{1}{2\,\pi\,i}\int\limits_{\Gamma^-}[M_t^{(+)}(\xi',\tau)]^{-1}d\tau\Big\}\ [M_t^{(-)}(\xi',-i\,|\xi'|)]^{-1},
\end{aligned}
\tag{5.21}
$$

where $M_t(\xi)=(1-t)\,|\xi|^2\,\boldsymbol{a}+t\,M(\xi)$, $M_t(\xi',\xi_3)=M_t^{(-)}(\xi',\xi_3)M_t^{(+)}(\xi',\xi_3)$ and $M_t^{(\pm)}(\xi',\xi_3)$ are the "plus" and "minus" polynomial matrix factors in ξ_3 of the polynomial symbol matrix $M_t(\xi',\xi_3)$. Due to (5.18), (5.19), (5.20), (5.21) and Lemma 4.2, we have

$$
e_t^{(2)}(\xi')=\frac{i}{a_t^{(+)}(\xi')}\boldsymbol{a}\,C_t^{(+),0}(\xi')\ [M_t^{(-)}(\xi',-i\,|\xi'|)]^{-1},
$$

where $C_t^{(+),0}(\xi')=\big[c_{ij,t}^{(+)}(\xi')\big]_{ij=1}^{6}$ and $c_{ij,t}^{(+),0}$, $i,j=\overline{1,6}$, are the main coefficients of the co-factors $p_{ij,t}^{(+)}(\xi',\tau)$ of the polynomial matrix $M_t^{(+)}(\xi',\tau)$ and $a^{(+)}$ is the coefficient at τ^3 in the determinant $\det M_t^{(+)}(\xi',\tau)$. In addition,

$$
\det\boldsymbol{a}\neq 0,\quad \det C_t^{(+),0}(\xi')\neq 0,\quad \det M_t^{(-)}(\xi',-i\,|\xi'|)\neq 0
$$

for all $\xi'\neq 0$ and $t\in[0,1]$.

Then it is clear that

$$
\det e_t(\xi')=-\frac{i}{\big(a_t^{+}(\xi')\big)^3}\det\boldsymbol{a}\,\det C_t^{(+),0}(\xi')\det[M_t^{(-)}(\xi',-i\,|\xi'|)]^{-1}\neq 0
$$

for all $\xi'\neq 0$ and for all $t\in[0,1]$, which implies that for the operator $\mathfrak{D}_t$, the Šapiro–Lopatinskiĭ condition is satisfied.

Therefore the operator

$$
\mathfrak{D}_t\ :H^{r+1}(\Omega)\times H^{r-1/2}(S)\to H^{r+1}(\Omega)\times H^{r+1/2}(S)
$$

is Fredholm for all $r>-\frac{1}{2}$ and for all $t\in[0,1]$. Consequently,

$$
\operatorname{Ind}\mathfrak{D}=\operatorname{Ind}\mathfrak{D}_1=\operatorname{Ind}\mathfrak{D}_t=\operatorname{Ind}\mathfrak{D}_0=0. \qquad\square
$$

Theorem 5.1 proof. Since the operator $\mathfrak{D}$ is by Lemma 5.4 Fredholm with zero index, its injectivity implies the invertibility. Thus it remains to prove that the null space of the operator $\mathfrak{D}$ is trivial for $r>-\frac{1}{2}$. Assume that $V=(U,\psi)^\top\in H^{r+1}(\Omega)\times H^{r-1/2}(S)$ is a solution to the homogeneous equation

$$
\mathfrak{D}\,V=0. \tag{5.22}
$$

The operator

$$
\mathfrak{D}\ :\ H^{r+1}(\Omega)\times H^{r-1/2}(S)\to H^{r+1}(\Omega)\times H^{r+1/2}(S)
$$

is Fredholm with index zero for all $r>-\frac{1}{2}$. It is well known that there exists a left regularizer $\mathfrak{L}$ of the operator $\mathfrak{D}$,

$$
\mathfrak{L}\ :H^{r+1}(\Omega)\times H^{r+1/2}(S)\to H^{r+1}(\Omega)\times H^{r-1/2}(S),
$$

such that

$$
\mathfrak{L}\,\mathfrak{D}=I+\mathfrak{T},
$$

where $\mathfrak{T}$ is the operator of order -1 (cf., proofs of Theorems 22.1 and 23.1 in [17]), i.e.,

$$\mathfrak{T}\ : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{5.23}$$

Therefore from (5.22), we have

$$\mathfrak{L}\,\mathfrak{D}\, V = V + \mathfrak{T} V = 0 \tag{5.24}$$

and from (5.23), we can see that

$$\mathfrak{T}\, V \in H^{r+2}(\Omega) \times H^{r+1/2}(S).$$

Consequently, in view of (5.24),

$$V = (U, \psi)^\top \in H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{5.25}$$

If $r \geq 0$, this implies $U \in H^{1,\,0}(\Omega, M)$. If $-\frac{1}{2} < r < 0$, we iterate the above reasoning for V satisfying (5.25) to obtain

$$V = (U, \psi)^\top \in H^{r+3}(\Omega) \times H^{r+3/2}(S)$$

which again implies $U \in H^{1,\,0}(\Omega, M)$. Next, we can apply the equivalence Theorem 3.1 to conclude that a solution $V = (U, \psi)^\top$ to the homogeneous equation (5.22) is trivial, i.e.,

$$U = 0 \quad \text{in} \;\; \Omega, \qquad \psi = 0 \;\; \text{on} \;\; S.$$

Thus $\operatorname{Ker} \mathfrak{D} = \{0\}$ in the class $H^{r+1}(\Omega) \times H^{r-1/2}(S)$ and therefore the operator

$$\mathfrak{D}\ : H^{r+1}(\Omega) \times H^{r-1/2}(S) \to H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is invertible for all $r > -\frac{1}{2}$.

For localizing function χ of finite smoothness, we have the following result.

Corollary 5.5. *Let a localising function $\chi \in X^3_+$. Then the operator*

$$\mathfrak{D}\ :\ H^1(\Omega) \times H^{-1/2}(S) \to H^1(\Omega) \times H^{1/2}(S)$$

is invertible.

Proof. Can be presented by word for word arguments employed in the proofs of Lemmas 5.2–5.4 and Theorem 5.1, with $r = 0$, and by using the mapping properties of the localized potentials for a localizing function of finite smoothness (see Appendix B). □

Relying on Lemma 2.2, Theorem 3.1 and Corollaries 2.3 and 5.5, we have the following assertion.

Corollary 5.6. *Let a localising function $\chi \in X^3_+$. Then the operator*

$$\mathfrak{D}\ :\ H^{1,0}(\Omega, M) \times H^{-1/2}(S) \to H^{1,0}(\Omega, \Delta) \times H^{1/2}(S)$$

is invertible.

Appendix A. Classes of Localising Functions

Here, we present the classes of localizing functions used in the main text (see [7] for details).

Definition A.1. We say $\chi \in X^k$ for integer $k \geq 0$ if $\chi(x) = \breve{\chi}(|x|)$, $\breve{\chi} \in W^k_1(0, \infty)$ and $\varrho\breve{\chi}(\varrho) \in L_1(0, \infty)$. We say $\chi \in X^k_+$ for integer $k \geq 1$ if $\chi \in X^k$, $\chi(0) = 1$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\hat{\chi}_s(\omega)}{\omega} > 0 & \text{for } \; \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho\breve{\chi}\,(\varrho)\, d\varrho & \text{for } \; \omega = 0, \end{cases} \tag{A.1}$$

and $\hat{\chi}_s(\omega)$ denotes the sine-transform of the function $\breve{\chi}$,

$$\hat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}\,(\varrho)\, \sin(\varrho\,\omega)\, d\varrho. \tag{A.2}$$

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$ and $X^{k_1}_+ \subset X^{k_2}_+$ for $k_1 > k_2$. The class X^k_+ is defined in terms of the sine-transform. The following lemma from [7] provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class.

Lemma A.2. *Let $k \geq 1$. If $\chi \in X^k$, $\breve{\chi}(0) = 1$, $\breve{\chi}(\varrho) \geq 0$ for all $\varrho \in (0, \infty)$, and $\breve{\chi}$ is a non-increasing function on $[0, +\infty)$, then $\chi \in X^k_+$.*

The following (and other) examples for χ are presented in [7],

$$\chi_{1k}(x) = \begin{cases} \Big[1 - \dfrac{|x|}{\varepsilon}\Big]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.3}$$

$$\chi_2(x) = \begin{cases} \exp\Big[\dfrac{|x|^2}{|x|^2 - \varepsilon^2}\Big] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A.4}$$

One can observe that $\chi_{1k} \in X^k_+$ for $k \geq 1$, while $\chi_2 \in X^\infty_+$, due to Lemma A.2.

Appendix B. Properties of Localized Potentials

Here, we collect some assertions describing the mapping properties of the localized potentials. The proofs coincide with or are similar to the ones in [7] and [12, Appendix B] (see also [19], Chapter 8 and references therein).

Let us introduce the boundary operators generated by the localized layer potentials associated with the localized parametrix $P(x - y) \equiv P_\chi(x - y)$:

$$\mathcal{V} g(y) := -\int_S P(x-y)\, g(x)\, dS_x, \qquad y \in S, \tag{B.1}$$

$$\mathcal{W} g(y) := -\int_S \big[T(x, \partial_x)\, P(x-y)\big]\; g(x)\, dS_x, \qquad y \in S, \tag{B.2}$$

$$\mathcal{W}' g(y) := -\int_S \big[T(y, \partial_y)\, P(x-y)\big]\, g(x)\, dS_x, \qquad y \in S, \tag{B.3}$$

$$\mathcal{L}^\pm g(y) := T^\pm(y, \partial_y)\, W g(y), \qquad y \in S. \tag{B.4}$$

Theorem B.1. *The operators*

$$\mathcal{P} : \widetilde{H}^s(\Omega) \to H^{s+2,s}(\Omega; \Delta), \qquad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1, \tag{B.5}$$

$$: H^s(\Omega) \to H^{s+2,s}(\Omega; \Delta), \qquad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1, \tag{B.6}$$

$$: H^s(\Omega) \to H^{\frac{5}{2}-\varepsilon, \frac{1}{2}-\varepsilon}(\Omega; \Delta), \quad \frac{1}{2} \leq s < \frac{3}{2}, \quad \forall\, \varepsilon \in (0,1), \;\; \chi \in X^2, \tag{B.7}$$

are continuous and Δ is the Laplace operator.

Theorem B.2. *The operators*

$$V : H^{s-\frac{3}{2}}(S) \to H^s(\mathbb{R}^3), \qquad s < \frac{3}{2}, \quad \textit{if } \chi \in X^1, \tag{B.8}$$

$$: H^{s-\frac{3}{2}}(S) \to H^{s,s-1}(\Omega^\pm; \Delta), \qquad \frac{1}{2} < s < \frac{3}{2}, \quad \textit{if } \chi \in X^2, \tag{B.9}$$

$$W : H^{s-\frac{1}{2}}(S) \to H^s(\Omega^\pm), \qquad s < \frac{3}{2}\,, \quad \textit{if } \chi \in X^2, \tag{B.10}$$

$$: H^{s-\frac{1}{2}}(S) \to H^{s,s-1}(\Omega^\pm; \Delta), \qquad \frac{1}{2} < s < \frac{3}{2}, \qquad \textit{if } \chi \in X^3. \tag{B.11}$$

are continuous.

Theorem B.3. *If $\chi \in X^k$ has a compact support and $-\frac{1}{2} \le s \le \frac{1}{2}$, then the following localized operators*

$$V : H^s(S) \to H^{s+\frac{3}{2}}(\Omega^\pm) \qquad \text{for } k = 2, \tag{B.12}$$

$$W : H^{s+1}(S) \to H^{s+\frac{3}{2}}(\Omega^\pm) \qquad \text{for } k = 3 \tag{B.13}$$

are continuous.

Theorem B.4. *Let $\psi \in H^{-\frac{1}{2}}(S)$ and $\varphi \in H^{\frac{1}{2}}(S)$. Then the following jump relations hold on S*

$$\gamma^\pm V\psi = \mathcal{V}\psi, \qquad \chi \in X^1, \tag{B.14}$$

$$\gamma^\pm W\varphi = \mp\, \boldsymbol{b}\,\varphi + \mathcal{W}\varphi, \qquad \chi \in X^2, \tag{B.15}$$

$$T^\pm V\psi = \pm\, \boldsymbol{b}\,\psi + \mathcal{W}'\psi, \qquad \chi \in X^2, \tag{B.16}$$

where

$$\mathbf{b}(y) := \frac{1}{2}\begin{bmatrix} [\delta_{ij}\,(\mu(y) + \varkappa(y)) + (\lambda(y) + \mu(y))\, n_i(y) n_j(y)]_{3\times 3} & [0]_{3\times 3} \\ [0]_{3\times 3} & [\delta_{ij}\,(\gamma(y)) + (\alpha(y) + \beta(y))\, n_i(y) n_j(y)]_{3\times 3} \end{bmatrix}_{6\times 6},$$

$y \in S$ and $\boldsymbol{b}(y)$ is positive definite due to (2.3).

Theorem B.5. *Let $-\frac{1}{2} \le s \le \frac{1}{2}$. The operators*

$$\mathcal{V} : H^s(S) \to H^{s+1}(S), \qquad \chi \in X^2, \tag{B.17}$$

$$\mathcal{W} : H^{s+1}(S) \to H^{s+1}(S), \qquad \chi \in X^3, \tag{B.18}$$

$$\mathcal{W}' : H^s(S) \to H^s(S), \qquad \chi \in X^3, \tag{B.19}$$

$$\mathcal{L}^\pm : H^{s+1}(S) \to H^s(S), \qquad \chi \in X^3, \tag{B.20}$$

are continuous.

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