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THE METHOD OF PROBABILISTIC SOLUTION FOR THE DIRICHLET GENERALIZED HARMONIC PROBLEM IN IRREGULAR PYRAMIDAL DOMAINS

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The Dirichlet generalized harmonic problem for irregular n-sided pyramidal domains is considered. The term "generalized" indicates that a boundary function has a finite number of first kind discontinuity curves. In the case under consideration, the pyramid edges appear in the role of the mentioned curves. The algorithm for a numerical solution of boundary problems consists of the following main steps: a) application of the method of probabilistic solution (MPS), which is in its turn based on a computer modeling of the Wiener process; b) finding the intersection point of the path of Wiener process simulation and the pyramid surface; c) development of a code for the numerical realization and checking the accuracy of calculated results; d) calculating the meaning of a sought for function at any chosen point.

For illustration two examples are considered. Numerical results are presented and discussed.

1. Introduction

In the present paper the MPS for numerical solution of the Dirichlet harmonic problem in irregular pyramidal domains with singularities in the boundary data is considered. It is known (see, e.g., [1,2,7,10,12]) that in practical stationary problems (for example, determination of electrical potential, temperature potential, gravitational potential, and so on) there are cases when it is necessary to consider the Dirichlet generalized harmonic problem.

As is well known (see, e.g., [7,8]), there are the methods for an approximate solution of classical boundary value problems that are: a) less applicable, or b) useless for solving generalized boundary problems. In the first case, convergence of the approximate process is very slow in the neighborhood of discontinuity curves. Consequently, the accuracy of numerical result is very low (see, e.g., [1, 2, 7, 10,12]). In the second case, the process is unstable. Namely, we have got a similar phenomenon while solving the Dirichlet generalized harmonic problem by the MFS.

Therefore researchers have tried to conduct preliminary "improvements" of the boundary value problem in question. For the Dirichlet generalized plane harmonic problems the following methods were elaborated: I) A method of reduction of the Dirichlet generalized harmonic problem to a classical problem (see, e.g., [9, 13]); II) A method of conformal mapping (see, e.g., [11]); III) A method of probabilistic solution (see, e.g., [3,5]). It is evident that in the case of 3D Dirichlet harmonic problems, from the above-mentioned methods we can apply only the third one.

For 3D Dirichlet generalized harmonic problems researchers face more significant difficulties. In particular, there does not exist a universal approach that can be applied to a wide class of domains.

The above-mentioned literature [1,2,7,10,12] deals with the simplified generalized problems. Mainly, the methods of separation of variables, particular solutions and heuristic methods are applied to their solution. Respectively, the accuracy of the solution is low. The heuristic methods do not guarantee to find the best solution. Moreover, in some cases, they may give an incorrect solution and, thus we have to check these solutions in order to establish how they satisfy all conditions of a problem (see, e.g., [7]). Therefore, the construction of effective computational schemes with a high accuracy

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for numerical solution of 3D Dirichlet generalized harmonic problems, applicable to a wide class of domains, are both of theoretical and practical importance.

It should be noted that in [10], the existence of discontinuity curves is ignored while solving the Dirichlet generalized harmonic simplest problems for a sphere. This fact and also the application of classical methods is the main reason of low accuracy. Therefore, for the numerical solution of three-dimensional Dirichlet generalized harmonic problems it is preferable to apply the methods in which there is no need to approximate a boundary function, or to consider the existence of discontinuity curves. It is on this basis that the Method of Prorabilistic Solutions is one of such methods.

2. Mathematical Formulation of the Main Problem

Let D be the interior of an irregular n-sided pyramid $P_n(h) \equiv P_n$ in the space R^3, where h is its height. We consider the cases, where h is a lateral edge of P_n (or its orthogonal projection lies in the base of P_n). In the first case, without loss of generality, we assume that h is located on Ox_3 of the right Cartesian coordinate system $Ox_1x_2x_3$ and the base of P_n lies in the plane Ox_1x_2. Also, suppose that the vertices of the base $A_1, A_2, \dots A_n$ of $P_n = MA_1A_2 \dots A_n$ are located in a counter-clockwise direction. Let us formulate the following problem for the pyramid $P_n \equiv \overline{D}$.

Problem A. The function $g(y)$ given on the boundary S of the pyramid P_n is continuous everywhere, except the edges $l_1, l_2, \dots, l_{2n}$, of P_n, which represent the first kind discontinuity curves for the function $g(y)$. It is required to find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \cap C(\overline{D} \setminus \cup_{k=1}^{2n} l_k)$ satisfying the following conditions:

$$\Delta u(x) = 0, \qquad x \in D, \tag{2.1}$$

$$u(y) = g(y), \qquad y \in S, \quad y \,\overline{\in}\, l_k \subset S \ \ (k = \overline{1, 2n}), \tag{2.2}$$

$$|u(x)| < c, \qquad x \in \overline{D}, \tag{2.3}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, l_k $(k = \overline{1, 2n})$ are the edges of P_n, and c is a real constant.

It is shown (see [6, 14]) that Problem (2.1)–(2.3) has a unique solution depending continuously on the initial data. For the generalized solution $u(x)$, the generalized extremum principle

$$\min_{x \in S} u(x) < \underset{x \in D}{u(x)} < \max_{x \in S} u(x), \tag{2.4}$$

is valid, where it is supposed that $x \overline{\in} l_k$ $(k = \overline{1, 2n})$ for $x \in S$.

Note (see [14]) that the additional requirement (2.3) of the boundedness plays an important role in the extremum principle (2.4); it concerns only the neighborhoods of discontinuity curves of the function $g(y)$.

On the basis of (2.3), the values of $u(y)$ are, in general, not uniquely defined on the curves l_k. In particular, if Problem A concerns the determination of a thermal (or electric) field, then $u(y) = 0$ when $y \in l_k$, respectively. In this case, in the physical sense, the curves l_k are non-conductors (or dielectrics). Otherwise, l_k will not be a discontinuity curve.

It is evident that, actually, in the above-mentioned case, the boundary function $g(y)$ has the form

$$g(y) = \begin{cases} g_1(y), & y \in S_1, \\ g_2(y), & y \in S_2, \\ \dots\dots\dots\dots\dots & \\ g_n(y), & y \in S_n, \\ g_{n+1}(y), & y \in S_{n+1}, \\ 0, & y \in l_k \quad (k = \overline{1, 2n}), \end{cases} \tag{2.5}$$

where S_i $(i = \overline{1, n})$ and S_{n+1} are the lateral faces and the base of P_n without discontinuity curves (edges), respectively; the functions $g_i(y)$, $y \in S_i$ $(i = \overline{1, n+1})$ are continuous on the parts S_i of S. It is evident that $S = (\cup_{i=1}^{n+1} S_i) \cup (\cup_{k=1}^{2n} l_k)$.

Remark 1. a) If the interior of S is empty, we get the generalized problem with respect to closed shells; b) In Problem A, it is no need for all edges of the pyramid to be dielectric, moreover, we can consider the cases when faces, apothems, base diagonals, etc. are taken in the role of dielectrics.

3. The Method of Probabilistic Solution and Simulation of the Wiener Process

This section briefly describes the proposed algorithm for numerical solving the problems of type A. Its detailed description is suggested in [17]. The main theorem that allows us to apply the MPS is the following one (see, e.g., [6]).

Theorem 1. *If a finite domain $D \in R^3$ is bounded by a piecewise smooth surface S and $g(y)$ is a continuous (or discontinuous) bounded function on S, then the solution of the Dirichlet classical (or generalized) boundary value problem for the Laplace equation at the fixed point $x \in D$ has the following form:*

$$u(x) = E_x g(x(\tau)). \tag{3.1}$$

In (3.1): $E_x g(x(\tau))$ is the mathematical expectation of values of the boundary function $g(y)$ at the random intersection points of Wiener process trajectory and the boundary S; τ is the random moment of the first exit of Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random intersection points $y^j = (y_1^j, y_2^j, y_3^j) \in S$ $(j = 1, 2, \dots, N)$ is sufficiently large, then according to the law of large numbers, from (3.1), we have

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{j=1}^{N} g(y^j) \tag{3.2}$$

or $u(x) = \lim u_N(x)$, as $N \to \infty$, in a probability sense. Thus, if we have the Wiener process, the approximate value of the probabilistic solution of Problem A at the point $x \in D$ is calculated by using formula (3.2).

In order to simulate the Wiener process, we construct the following recursion (see, e.g., [14, 17]):

$$\begin{aligned} x_1(t_k) &= x_1(t_{k-1}) + \gamma_1(t_k)/nq, \\ x_2(t_k) &= x_2(t_{k-1}) + \gamma_2(t_k)/nq, \\ x_3(t_k) &= x_3(t_{k-1}) + \gamma_3(t_k)/nq, \\ (k = 1, 2, \dots), &\quad x(t_0) = x, \end{aligned} \tag{3.3}$$

according to which the coordinates of the point $x(t_k) = (x_1(t_k), x_2(t_k), x_3(t_k))$ are being determined. In (3.3): $\gamma_1(t_k), \gamma_2(t_k), \gamma_3(t_k)$ are three normally distributed independent random numbers for the k-th step, with means, equal to zero and variances, equal to 1 (the generation of the above numbers takes place apart); nq is a quantification number (nq) such that $1/nq = \sqrt{t_k - t_{k-1}}$ and when $nq \to \infty$, then the discrete process approaches to the continuous Wiener process. In the implementation, the random process is simulated at each step of the walk and continues until it crosses the boundary.

In the case under consideration, calculations and generation of random numbers are performed in MATLAB.

4. An Auxiliary Problem

It should be noted that in 3D case, there are, in general (except for a special case), no exact test solutions for generalized problems of type A, therefore, for the verification of the scheme needed for the numerical solution of Problem A, the reliability of obtained results can be demonstrated in the following way.

If we take $g_i(y) = 1/|y - x^0|$ in the boundary conditions (2.5), where $y \in S_i$ $(i = \overline{1, n+1})$, $x^0 = (x_1^0, x_2^0, x_3^0) \,\overline{\in}\, \overline{D}$, and $|y - x^0|$ denotes the distance between the points y and x^0, then we can see that the curves l_k $(k = \overline{1, 2n})$ are the removable discontinuity curves for the boundary function $g(y)$. In the mentioned case, instead of Problem A we obtain the next Dirichlet classical harmonic problem.

Problem B. Find a Function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \cap C(\overline{D})$ under the following conditions:

$$\Delta u(x) = 0, \quad x \in D,$$
$$u(y) = 1/|y - x^0|, \quad y \in S, \;\; x^0 \overline{\in} \overline{D}.$$

We solve this problem by using the MPS with the algorithm constructed for Problem A. It is known that Problem B is well posed, i.e., the solution exists, is unique and depends continuously on the data. An exact solution of Problem B has the form

$$u(x^0, x) = \frac{1}{|x - x^0|}, \quad x \in \overline{D}, \;\; x^0 \overline{\in} \overline{D}. \tag{4.1}$$

Note that the process of solving the Dirichlet classical harmonic problems numerically by MPS is quite interesting and important (see, e.g., [4, 18]). In the present paper, Problem B plays an auxiliary role and is used for checking the reliability of the scheme, and the corresponding program is needed for a numerical solution of Problem A. First, we solve Problem B and then compare the obtained results with the exact solution and solve Problem A under the boundary conditions (2.5).

In this paper, the MPS is applied to two examples. In the tables, N denotes a number of trajectories in the simulated Wiener process for the given points $x^i = (x_1^i, x_2^i, x_3^i) \in D$, and nq is the number of quantifications. For the sake of simplicity, in the examples under consideration, the values of nq and N are the same. The tables below present for the problems of type B the numerical absolute errors Δ^i at the points $x^i \in D$ of $u_N(x)$, in the MPS approximation, for $nq = 200$ and various values of N, and the numbers are given in scientific format. In particular,

$$\Delta^i = \max |u_N(x^i) - u(x^0, x^i)|, \quad (i = 1, 2, \dots, 5),$$

where $u_N(x^i)$ is the approximate solution of Problem B at the point x^i, which is defined by using formula (3.2), and the exact solution $u(x^0, x^i)$ of the test problem is given by (4.1). In the tables, for problems of type A, the probabilistic solution $u_N(x)$ is calculated at the points x^i, defined by (3.2).

Remark 2. Problems of type A and B for ellipsoidal, spherical, cylindrical, conic, prismatic, regular pyramidal, axisymmetric finite domains with a cylindrical hole, external Dirichlet generalized problem for a sphere are considered in [14–17, 19, 20].

5. Numerical Examples

In order to determine the intersection points $y^j = (y_1^j, y_2^j, y_3^j)$ $(j = \overline{1, N})$ of the simulated process path and the surface S of P_n, first of all, for each current point $x(t_k)$ we check whether it belongs to P_n or not.

Knowing two parameters n, h and coordinates of the vertices $M, A_1, A_2, \dots, A_n$ of P_n, we can find: 1) angles α_n of inclination of the lateral faces with respect to the plane of the base of P_n; 2) equations of lateral faces; 3) equations of edges of P_n.

Example 1. In the first example, the domain D is the interior of the irregular 3-sided pyramid $P_3(h)$, where h is its height.

We consider the case, where h, lying on Ox_3,is the lateral edge of P_3. Besides, the angle between lateral faces containing h is to equal $\pi/2$, and $A_1 = (a, 0, 0)$, $A_2 = (0, b, 0)$, $A_3 \equiv O = (0, 0, 0)$, $M = (0, 0, h)$.

In the rectangular coordinate system $Ox_1x_2x_3$, the equation of the plane passing through the points A_1, A_2, M, has the following form:

$$bhx_1 + ahx_2 + abx_3 - abh = 0. \tag{5.1}$$

It is evident that the equations of the planes A_1MA_3, A_2MA_3 and $A_1A_3A_2$ are $x_2 = 0$, $x_1 = 0$, $x_3 = 0$, respectively.

The equations of the lines A_1A_2, A_1M, A_2M are

$$(-b/a)x_1 - x_2 + b = 0, \; (-h/a)x_1 - x_3 + h = 0, \quad (-h/b)x_2 - x_3 + h = 0, \tag{5.2}$$

respectively.

TABLE 5.1A. Results for Problem A (in Example 1).

x^i	$(0.8, 0.6, 0.5)$	$(0.8, 0.6, 1)$	$(0.6, 0.4, 1.2)$	$(0.4, 0.2, 1.5)$	$(0.2, 0.1, 1.6)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.95330	1.53880	1.44480	1.37130	1.31410
$1E+4$	1.94440	1.53160	1.45600	1.36585	1.31100
$5E+4$	1.94815	1.53196	1.44918	1.36937	1.31430
$1E+5$	1.94504	1.53303	1.45021	1.37063	1.31180
$5E+5$	1.94667	1.53151	1.45245	1.37135	1.31160
$1E+6$	1.94656	1.53133	1.45113	1.37101	1.31272

TABLE 5.1B. Results for Problem B (in Example 1).

x^i	$(0.8, 0.6, 0.5)$	$(0.8, 0.6, 1)$	$(0.6, 0.4, 1.2)$	$(0.4, 0.2, 1.5)$	$(0.2, 0.1, 1.6)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.20E-3$	$0.47E-4$	$0.18E-3$	$0.30E-4$	$0.12E-4$
$1E+4$	$0.97E-4$	$0.24E-3$	$0.34E-4$	$0.88E-4$	$0.66E-5$
$5E+4$	$0.10E-3$	$0.83E-4$	$0.70E-4$	$0.65E-5$	$0.55E-4$
$1E+5$	$0.32E-4$	$0.97E-4$	$0.27E-4$	$0.77E-4$	$0.47E-4$
$5E+5$	$0.57E-5$	$0.44E-4$	$0.50E-4$	$0.53E-4$	$0.37E-4$
$1E+6$	$0.18E-5$	$0.44E-4$	$0.54E-4$	$0.35E-4$	$0.33E-4$

It is clear that the equations of the lines A_1A_3, A_2A_3, A_3M are: $x_2 = 0$ and $x_3 = 0$; $x_1 = 0$ and $x_3 = 0$; $x_1 = 0$ and $x_2 = 0$, respectively. The angle α_1 of inclination of the plane A_1A_2M with respect to the plane of the base of P_3 is $\alpha_1 = \arctan(h/\Delta_1)$, where Δ_1 is a distance between the point O and the line A_1A_2. By the equation of line A_1A_2 (see (5.2)), $\Delta_1 = |b|/\sqrt{kk^2+1}$, where $kk = -b/a$. It is evident that the analogous angles for the planes A_2MA_3 and A_1MA_3 are $\alpha_2 = \alpha_3 = \pi/2$.

We have now the necessary information about the pyramid P_3 in order to establish whether each current point $x(t_k)$ defined from (3.3) belongs to P_3 or not. For each step of the simulated Wiener process we calculate the angles β_m $(m = \overline{1,3})$ of inclination of the planes passing through the points $x(t_k)$, A_m, A_{m+1} ($A_4 \equiv A_1$, with respect to the plane of the base of P_3. It is easy to see that

$$\beta_1 = \arctan(x_3(t_k)/\Delta^*), \quad \beta_2 = \arctan(x_3(t_k)/x_1(t_k)), \quad \beta_3 = \arctan(x_3(t_k)/x_2(t_k)),$$

where Δ^* is a distance between the point $(x_1(t_k), x_2(t_k))$ and the line A_1A_2. It is known that

$$\Delta^* = |kkx_1(t_k) - x_2(t_k) + b|/\sqrt{(kk)^2+1}.$$

After having calculated the angles β_m $(m = 1,2,3)$, we compare them with the angle $\alpha_m(m = 1,2,3)$. In particular: (1^*) if $\beta_m < \alpha_m$ and $0 < x_3(t_k) < h$ and $x_1(t_k) > 0$ and $x_2(t_k) > 0$ for $m = 1,2,3$, then the process continues until it crosses the boundary S; (2^*) if $\beta_1 = \alpha_1$ and $0 < x_3(t_k) < h$ and $x_1(t_k) > 0$ and $x_2(t_k) > 0$, then $x(t_k) \in \overline{S_1}$ or $y^j = (y_1^j, y_2^j, y_3^j) = x(t_k)$; ($3^*$) if $\beta_1 > \alpha_1$ and $0 < x_3(t_k) < h$, this means that the trajectory of the simulated Wiener process intersects the lateral face $S_1 \equiv A_1A_2M$ of P_3 or $x(t_{k-1}) \in D$ for the moment $t = t_{k-1}$, and $x(t_k) \overline{\in} P_3$ for the moment $t = t_k$. In this case, for an approximate determination of the point y^j, a parametric equation of a line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is obtained initially; it has the following form:

$$\begin{cases} x_1 = x_1(t_{k-1}) + (x_1(t_k) - x_1(t_{k-1}))\theta, \\ x_2 = x_2(t_{k-1}) + (x_2(t_k) - x_2(t_{k-1}))\theta, \\ x_3 = x_3(t_{k-1}) + (x_3(t_k) - x_3(t_{k-1}))\theta, \end{cases} \tag{5.3}$$

where (x_1, x_2, x_3) is the current point of L and θ is a parameter $(-\infty < \theta < \infty)$.

If we substitute the expressions of x_1, x_2, x_3 defined from (5.3) into (5.1), then we obtain the equation with respect to θ, which has the form

$$(b1+b2+b3)\theta = abh - a1 - a2 - a3. \tag{5.4}$$

In (5.4),

$$b1 = bh(x_1(t_k) - x_1(t_{k-1})); \quad b2 = ah(x_2(t_k) - x_2(t_{k-1})); \quad b3 = ab(x_3(t_k) - x_3(t_{k-1}));$$
$$a1 = bhx_1(t_{k-1}); \quad a2 = ahx_2(t_{k-1}); \quad a3 = abx_3(t_{k-1}).$$

In the considered case, due to the existence of the intersection point, we have $(b1+b2+b3) \neq 0$ and $y^j = (x_1(\theta), x_2(\theta), x_3(\theta))$, where θ is defined by (5.4).

It is easy to see that on the basis of (5.3) and the equations of faces $S_2 = A_2A_3M$, $S_3 = A_1A_3M$, $S_4 = A_1A_2A_3$, for the parameter θ, we have

$$\begin{aligned} S_2 &: x_1 = 0, \theta = -x_1(t_{k-1})/(x_1(t_k) - x_1(t_{k-1})), \\ S_3 &: x_2 = 0, \theta = -x_2(t_{k-1})/(x_2(t_k) - x_2(t_{k-1})), \\ S_4 &: x_3 = 0, \theta = -x_3(t_{k-1})/(x_3(t_k) - x_3(t_{k-1})), \end{aligned} \tag{5.5}$$

respectively.

It is evident that in the cases under consideration: if (the intersection point) $y^j \in S_2$, then $y^j = (0, x_2(\theta), x_3(\theta)$; if $y^j \in S_3$, then $y^j = (x_1(\theta), 0, x_3(\theta))$, if $y^j \in S_4$, then $y^j = (x_1(\theta), x_2(\theta), 0)$, where θ is defined by (5.5), accordingly.

Remark 3. In addition, during a numerical implementation, with the help of the corresponding equations (see (5.2)), it is checked whether the intersection point y^j is situated on the edge or not.

Problems A and B are solved when $h = 2$, $a = 4$, $b = 3$, $x^0 = (2, 1, -4)$, and in Problem A, the boundary function $g(y) \equiv g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 1.5, & y \in S_1, \\ 2, & y \in S_2, \\ 1, & y \in S_3, \\ 3, & y \in S_4, \\ 0, & y \in l_k \quad (k = \overline{1,6}). \end{cases} \tag{5.6}$$

In (5.6): S_i $(i = \overline{1,3})$ and S_4 are the lateral faces and the base of P_3 without discontinuity curves (edges), respectively; l_k $(k = \overline{1,6})$ are the edges of P_3. In the physical sense, l_k are non-conductors (or dielectrics).

In the examples, considered by us for determination of the intersection points $y^j = (y_1^j, y_2^j, y_3^j)$ $(j = \overline{1,N})$ of the trajectory of the Wiener process and the surface S, we have used the scheme, described above. As it is noted in Section 3, for verification, first of all, we solve the auxiliary Problem B with the calculating program of Problem A.

In Table 5.1B, the numerical absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points $x^i \in D$ $(i = \overline{1,5})$ are presented.

The results, presented in Table 5.1B, show that the algorithm elaborated for Problem A is effective.

As $nq \to \infty$ (see Section 3), the discrete process approaches to the continuous Wiener process and the accuracy of probabilistic solution is increasing. We conducted numerical experiment, i.e., calculated the probabilistic solution of Problem B at the point (0.8,0.6,0.5) for $N = 1E+5$, $nq = 400$, and obtained $\Delta^1 = 0.23E-4$ (see, Table 5.1B). This result agrees with the above noted. If more accuracy is needed, then the calculations for sufficiently large values of nq and N must be carried out.

In Table 5.1A, the values of approximate solution $u_N(x)$ of Problem A at the same points x^i $(i = \overline{1,5})$ are given. The results have sufficient accuracy for many practical problems and are in agreement with the real physical picture.

Example 2. Here, in the capacity of the domain D, the interior of an irregular 4-sided pyramid $P_4(h)$ is taken, where h is its height.

TABLE 5.2A. Results for Problem A (in Example 2).

x^i	$(2, 1.5, 0.5)$	$(2, 1.5, 1)$	$(2, 1.5, 1.8)$	$(1, 1.5, 0.5)$	$(3, 1.5, 0.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	2.43350	2.02430	1.82350	2.11640	2.12180
$1E+4$	2.43150	2.04070	1.83740	2.11030	2.11750
$5E+4$	2.42634	2.03708	1.83515	2.12035	2.11918
$1E+5$	2.43157	2.04039	1.83449	2.11672	2.11452
$5E+5$	2.43160	2.04011	1.83373	2.11512	2.11525
$1E+6$	2.43125	2.04055	1.83390	2.11447	2.11542

TABLE 5.2B. Results for Problem B (in Example 2).

x^i	$(2, 1.5, 0.5)$	$(2, 1.5, 1)$	$(2, 1.5, 1.8)$	$(1, 1.5, 0.5)$	$(3, 1.5, 0.5)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.56E-3$	$0.76E-3$	$0.24E-4$	$0.15E-3$	$0.31E-3$
$1E+4$	$0.17E-3$	$0.25E-3$	$0.17E-3$	$0.12E-3$	$0.15E-3$
$5E+4$	$0.25E-4$	$0.31E-3$	$0.69E-4$	$0.30E-4$	$0.12E-3$
$1E+5$	$0.77E-5$	$0.75E-4$	$0.58E-4$	$0.10E-4$	$0.65E-5$
$5E+5$	$0.58E-5$	$0.15E-4$	$0.74E-4$	$0.37E-4$	$0.63E-4$
$1E+6$	$0.40E-4$	$0.54E-4$	$0.64E-4$	$0.10E-4$	$0.60E-4$

In this case, we consider Problem A for the irregular pyramid $MA_1A_2A_3A_4$, whose base $A_1A_2A_3A_4$ is a rectangle with the vertices $A_1 = (a, 0, 0)$, $A_2 = (a, b, 0)$, $A_3 = (0, b, 0)$, $A_4 = (0, 0, 0)$, $M = (a/2, b/2, h)$. It is evident that the base of P_4 lies in the first quarter of the plane Ox_1x_2 and a project of the vertex M is the centre of the rectangle $A_1A_2A_3A_4$.

First of all, we have to define the angles α_n $(n = 1, 2, 3, 4)$ of inclination of the lateral faces: $S_1 = MA_1A_2$; $S_2 = MA_2A_3$; $S_3 = MA_3A_4$; $S_4 = MA_4A_1$, with respect to the plane of the base of P_4. It is clear that

$$\alpha_1 = \arctan(2h/a), \quad \alpha_2 = \arctan(2h/b), \quad \alpha_3 = \alpha_1, \quad \alpha_4 = \alpha_2.$$

Analogously to Example 1, for each step of the simulated Wiener process we have to calculate the following angles: β_m $(m = 1, 2, 3, 4)$, which are the angles of inclination of the planes, passing through the points $x(t_k)$, A_m, A_{m+1} $(A_5 \equiv A_1)$, with respect to the plane of P_4 base. In our case, $\beta_1 = \arctan(x_3(t_k)/(a - x_1(t_k)))$, $\beta_2 = \arctan(x_3(t_k)/(b - x_2(t_k)))$, $\beta_3 = \arctan(x_3(t_k)/x_1(t_k))$, $\beta_4 = \arctan(x_3(t_k)/x_2(t_k))$.

It is not difficult to show that on the basis of (5.3) and the equations of S_1, S_2, S_3, S_4, $S_5 \equiv A_1A_2A_3A_4$, for the parameter θ, we have

$$\begin{aligned}
S_1 : 2hx_1 + ax_3 - 2ah = 0, \quad & \theta = (2ah - 2hx_1(t_{k-1}) - ax_3(t_{k-1}))/(2hC_1 + aC_3);\\
S_2 : 2hx_2 + x_3 - 2bh = 0, \quad & \theta = (2bh - 2hx_2(t_{k-1}) - bx_3(t_{k-1}))/(2hC_2 + bC_3);\\
S_3 : 2hx_1 - ax_3 = 0, \quad & \theta = (ax_3(t_{k-1}) - 2hx_1(t_{k-1}))/(2hC_1 - aC_3);\\
S_4 : 2hx_2 - bx_3 = 0, \quad & \theta = (bx_3(t_{k-1}) - 2hx_2(t_{k-1}))/(2hC_2 - bC_3);\\
S_5 : x_3 = 0, \quad & \theta = -x_3(t_{k-1})/C_3,
\end{aligned} \tag{5.7}$$

where $C_1 = x_1(t_k) - x_1(t_{k-1})$, $C_2 = x_2(t_k) - x_2(t_{k-1})$, $C_3 = x_3(t_k) - x_3(t_{k-1})$.

It is evident that on the basis of (5.7), if the intersection point $y^j \in S_i$ $(i = \overline{1, 5})$, then $y^j = (x_1(\theta), x_2(\theta), x_3(\theta))$, where θ is defined by (5.7), according to i.

The problems A and B are solved for $h = 2$, $a = 4$, $b = 3$, $x^0 = (2, 1.5, -4)$ and the boundary function $g(y)$ has the form

$$g(y) = \begin{cases} 1.5, & y \in S_1, \\ 2, & y \in S_2, \\ 1.5, & y \in S_3, \\ 2, & y \in S_4, \\ 3, & y \in S_5, \\ 0, & y \in l_k \quad (k = \overline{1,8}), \end{cases} \tag{5.8}$$

In (5.8): S_i $(i = \overline{1,4})$ and S_5 are the lateral faces and the base of P_4 without discontinuity curves, respectively; l_k $(k = \overline{1,8})$ are the edges of P_4; the edges l_k are non-conductors.

In Table 5.2B, the numerical absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points $x^i \in D$ $(i = \overline{1,5})$ are presented.

The values of numerical solution $u_N(x)$ of Problem A at the points $x^i \in D$ $(i = \overline{1,3})$ are presented in Table 5.2B. Since the boundary function (5.8) is symmetric with respect to the plane $x_1 = 2$, therefore, in the role of x^i $(i = 4, 5)$, the points which are symmetric with respect to the plane $x_1 = 2$ are taken for a control. The obtained results have sufficient accuracy for many practical problems and are in agreement with the real physical picture(see Table 5.2A).

In this work, we have solved the problems of type A, while boundary functions $g_i(y)$ $(i = \overline{1, n+1})$ are the constants. We conclude that the obtained results agree with real physical picture. It is evident that solving Problem A under condition (2.5) is as easy as Problem B.

The analysis of numerical experiments show that the results obtained by the proposed algorithm are reliable and effective for numerical solution of problems of type A and B. In particular, the algorithm is sufficiently simple for numerical implementation.

The numerical solution of the considered examples by the MPS demonstrate that unlike regular pyramids (see, e. g., [16]), an individual approach is required for numerical solution of problems of type A and B by the same method.

6. Concluding Remarks

1. This paper demonstrates that the suggested algorithm is ideally suited for numerical solution of problems A and B in difficult domains such as irregular pyramids.

2. According to this algorithm, there is no need to approximate the boundary function.

3. The computational outlays of this algorithm is low and the accuracy is sufficient for practical purposes.

4. The next steps of our research are related to:

∗ The numerical solution of the Dirichlet classical and generalized harmonic problems for the infinite space R^3 with a finite number of spherical cavities.

∗ The MPS for the same type problems in finite domains which are bounded by several closed surfaces.

∗ The MPS for the same type problem in infinite 2D domains with a finite number of circular holes.

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