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DYNAMICAL CONTACT PROBLEMS FOR A VISCOELASTIC HALF-SPACE WITH AN ELASTIC INCLUSION

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The dynamical contact problem for viscoelastic half-space which is reinforced by an elastic inclusion in the form of a strip, is considered. The solution of the problem is reduced to the integro-differential equation. Using the method of orthogonal polynomials, the integral equation is reduced to an infinite system of linear algebraic equations. The quasi-completely regularity of the obtained system is proved and the method of reduction for approximate solution is developed.

1. Statement of the problem

We investigate the dynamical contact problem for a viscoelastic half-space ($-\infty < x,\ z < \infty,\ y > 0$) which is reinforced by an elastic inclusion in the form of a strip ($0 \le y \le b,\ -\infty < z < \infty$) lying in the plane $x = 0$. The outer border of the inclusion is under the action of uniformly distributed shearing harmonic (acting along the oz axis) load of intensity $\tau_0 e^{-ikt}\delta(y)$, where $\delta(y)$ is the Dirac function, k is oscillation frequency, t is time. In the linear theory of viscoelasticity, for Kelvin–Voigt materials, only displacement component $\omega = \omega(x,y,t)$ and tangential stresses components $\tau_{yz} = G\frac{\partial \omega}{\partial y} + G_0\frac{\partial \dot\omega}{\partial y}$, $\tau_{xz} = G\frac{\partial \omega}{\partial x} + G_0\frac{\partial \dot\omega}{\partial x}$ are other than zero (the so-called anti-plane deformation), where G and G_0 are the elastic and viscoelastic shear modulus, respectively. The dot means a derivative with respect to the variable t, $\dot\omega \equiv \frac{\partial \omega}{\partial t}$.

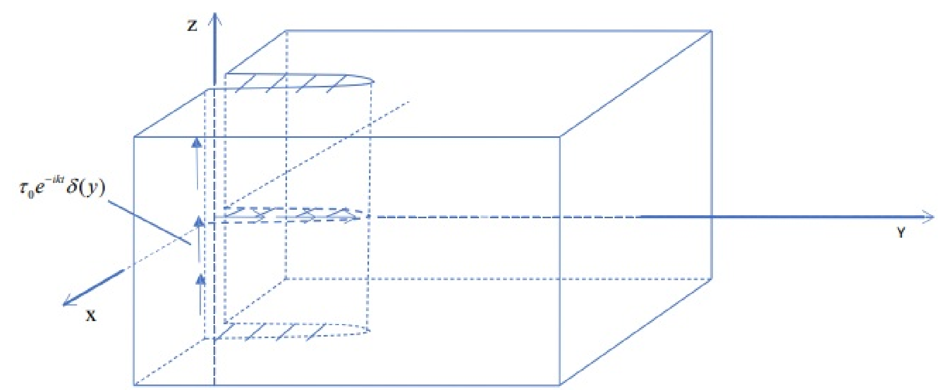

Figure 1

The problem is equivalent to the boundary value problem

$$G\Delta\omega + G_0\Delta\dot\omega = \rho\ddot\omega, \quad |x| < \infty, \quad y > 0, \quad \frac{\partial \omega(x,0,t)}{\partial y} + \frac{\partial \dot\omega(x,0,t)}{\partial y} = 0 \tag{1}$$

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(these equations are satisfied everywhere, except the domain occupied by the inclusion). ρ is the material density of the half-space [2,4–7,10,11].

Passing through the inclusion, the tangential stress has discontinuities, the displacement is continuous

$$\begin{gathered}\langle\tau_{xz}(0,y,t)\rangle=\mu(y,t),\quad 0<y<1;\quad \mu(y,t)=0,\quad y\geq 1\\ \omega(-0,y,t)=\omega(+0,y,t)=\omega^{(1)}(y,t)\end{gathered}\tag{2}$$

and the displacement of the points of an inclusion $\omega^{(1)}(y,t)$ satisfies the condition

$$\frac{\partial}{\partial y}h(y)\frac{\partial\omega^{(1)}(y,t)}{\partial y}-\frac{\rho_0 h(y)}{E_0}\ddot{\omega}^{(1)}(y,t)=-\frac{1}{E_0}\mu(y,t)-\frac{1}{E_0}\tau_0 e^{-ikt}\delta(y),\tag{3}$$

where $\mu(y,t)$ is an unknown contact stress at the point y at time moment t, acting onto the inclusion along the surface of its contact with a half-space, ρ_0 is a density and E_0 is the elasticity modulus of the inclusion material, $h(y)$ is its thickness. It is required to find fields of stresses and displacements.

2. Reduction to the Integral Equation

Considering steady oscillations of the half-space and inclusion, we assume that

$$\omega(x,y,t)=\omega_0(x,y)e^{-ikt},\quad \omega^{(1)}(y,t)=\omega_1(y)e^{-ikt},\quad \mu(y,t)=\mu_1(y)e^{-ikt}.$$

Thus from (1), (2), we obtain the following boundary value problem:

$$\begin{gathered}(G-ikG_0)\Delta\omega_0=-\rho k^2\omega_0,\quad |x|<\infty,\quad y>0,\quad \frac{\partial\omega_0(x,0)}{\partial y}=0,\\ (G-ikG_0)\langle\frac{\partial\omega_0(0,y)}{\partial x}\rangle=\mu_1(y),\quad 0<y<1,\quad \mu_1(y)\equiv 0,\quad y\geq 1.\end{gathered}\tag{4}$$

Based on the condition (3), the amplitude of the displacement of boundary points on the inclusion satisfies the condition

$$\frac{d}{dy}h(y)\frac{d\omega_1(y)}{dy}+\frac{\rho_0 h(y)}{E_0}k^2\omega_1(y)=-\frac{1}{E_0}\mu_1(y)-\frac{1}{E_0}\tau_0\delta(y),\quad 0<y<1.\tag{5}$$

Multiplying equations (4) by $e^{i\alpha x}$ and integrating by parts separately on the intervals $(-\infty,0)$ and $(0,\infty)$, for the Fourier transform, we obtain the one-dimensional boundary value problem [12,13]

$$\omega_\alpha''(y)-(\alpha^2-k_0^2)\omega_\alpha(y)=f(y),\quad 0<y<\infty,\quad \omega_\alpha'(0)=0,\tag{6}$$

where

$$k_0^2=\frac{\rho k^2}{\widetilde{G}},\quad f(y)=-\frac{\mu_1(y)}{\widetilde{G}},\quad \widetilde{G}=G-ikG_0.$$

The decreasing at infinity fundamental function of equation (6) is defined by the methods of integral transformations and contour integration. Since Green's function $G_\alpha(y,\eta)$ of the boundary value problem (6) must satisfy the equation $G_\alpha(0,\eta)=0$, it can be constructed in the form of a simple combination of the above-mentioned fundamental functions, that is,

$$G_\alpha(y,\eta)=\Phi(y,\eta)+\Phi(y,-\eta).$$

Thus

$$\omega_\alpha(y)=\int_0^1[\Phi(y,\eta)+\Phi(y,-\eta)]f(\eta)d\eta=\int_{-1}^1\Phi(y,\eta)f(\eta)d\eta.$$

We have taken here into account the fact that the right-hand side of equation (6) is equal to zero for $y>1$, and its continuation is realized evenly by negative values of the argument.

Consequently, a solution of the boundary value problem (6) can be represented in the form

$$\widetilde{G}\omega_\alpha(y)=\int_{-1}^1\frac{e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}}{2\sqrt{\alpha^2-k_0^2}}\mu_1(\eta)d\eta.$$

Using the inverse transformation, we find that

$$\widetilde{G}\omega_0(x,y)=\int\limits_{-1}^{1}\mu_1(\eta)d\eta\int\limits_0^\infty\frac{e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}\cos\alpha x d\alpha}{2\sqrt{\alpha^2-k_0^2}}\,. \tag{7}$$

For the conditions of diverging wave to be fulfilled, it is assumed that $\gamma(\alpha)=\sqrt{\alpha^2-k_0^2}\to|\alpha|$, as $|\alpha|\to\infty$, and when k_0 is a real number, $\sqrt{\alpha^2-k_0^2}=-i\sqrt{k_0^2-\alpha^2}$, that is, the real axis of the complex plane $z=\alpha+i\sigma$ goes around the branch points $-k_0$ from above and k_0 from below.

Since the integrand of the interior integral in formula (7) may have at infinity the behavior α^{-1}, its Fourier transformation (in a sense of the theory of generalized functions) is represented as a sum of its principal and regular part [10]:

$$R(x,|y-\eta|)=\frac{1}{2}\ln\frac{1}{x^2+(y-\eta)^2}+R_0(x,|y-\eta|), \tag{8}$$

where

$$R_0(x,|y-\eta|)=\int\limits_0^\infty\bigg(\frac{e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}\cos\alpha x}{\sqrt{\alpha^2-k_0^2}}-\frac{e^{-\alpha|y-\eta|}\cos\alpha x-e^{-\alpha|\eta|}}{\alpha}\bigg)d\alpha.$$

Thus the function can be represented as follows:

$$\widetilde{G}\omega_0(x,y)=\frac{1}{4\pi}\int\limits_{-1}^{1}\ln\frac{1}{x^2+(y-\eta)^2}\mu_1(\eta)d\eta+\int\limits_{-1}^{1}R_0(x,|y-\eta|)\mu_1(\eta)d\eta.$$

Taking into account the contact condition of the inclusion and the half-space $\omega_0(0,y)=\omega_1(y)$, in view of formulas (8) and (5), we obtain the following integro-differential equation:

$$\Big(\frac{d}{dy}h(y)\frac{d}{dy}+\frac{\rho_0k^2h(y)}{E_0}\Big)\Big(\frac{1}{2\pi\widetilde{G}}\int\limits_{-1}^{1}\ln\frac{1}{|y-\eta|}\mu_1(\eta)d\eta+\frac{1}{\widetilde{G}}\int\limits_{-1}^{1}R_0(0,|y-\eta|)\mu_1(\eta)d\eta\Big)$$
$$=-\frac{1}{E_0}\mu_1(y)-\frac{1}{E_0}\tau_0\delta(y) \tag{9}$$

under the condition that

$$\int\limits_{-1}^{1}\mu_1(\eta)d\eta=2\tau_0. \tag{10}$$

The subject of our investigation is the integro-differential equation (9) with condition (10).

3. Reduction of Problem (9), (10) to an Infinite System of Linear Algebraic Equations

A solution of problem (9), (10) will be sought in the form

$$\mu_1(y)=\frac{a_0}{\sqrt{1-y^2}}+\frac{1}{\sqrt{1-y^2}}\sum_{m=1}^\infty a_mT_m(y), \tag{11}$$

where $T_m(y)$ is the first kind Chebyshev's orthogonal polynomial, $\{a_n\}_{n\geq1}$ are unknown sequences.

By virtue of the equilibrium conditions of inclusion (10), we obtain $a_0=\frac{2\tau_0}{\pi}$.

a) If $h(y)=h=\text{const}$, using Rodrigue's formula for Jacobi's polynomials and the following spectral relation

$$\frac{1}{\pi}\int\limits_{-1}^{1}\ln\frac{1}{|x-y|}\frac{T_m(y)dy}{\sqrt{1-y^2}}=\mu_mT_m(x),\qquad \mu_m=\begin{cases}\ln2, & m=0,\\ \dfrac{1}{m}, & m\neq0,\end{cases}$$

from the integro-differential equation (9), we have [14]:

$$\frac{\sqrt{\pi}}{8}\sum_{m=2}^{\infty} a_m\mu_m \frac{(m+1)!m}{\Gamma(m+2^{-1})}\, P_{m-2}^{(3/2,3/2)}(y)+\frac{\rho_0 k^2}{2E_0}\sum_{m=0}^{\infty} a_m\mu_m T_m(y)+\sum_{m=0}^{\infty} a_m\int_{-1}^{1} K(|y-\eta|)\frac{T_m(\eta)}{\sqrt{1-\eta^2}}d\eta$$

$$=-\frac{\widetilde{G}}{E_0 h}\frac{1}{\sqrt{1-y^2}}\sum_{m=0}^{\infty} a_m T_m(y)-\frac{\widetilde{G}}{E_0 h}\tau_0\delta(y),$$

where $K(|y-\eta|)=\frac{\partial^2 R_0(0,|y-\eta|)}{\partial y^2}+\frac{\rho_0 k^2}{E_0}R_0(0,|y-\eta|)$.

Multiplying both parts of the above equality by $(1-y^2)^{3/2}P_{n-2}^{(3/2,3/2)}(y)$, integrating in the interval $(-1,1)$ and based on the orthogonality of Jacobi's polynomials, we obtain the infinite system of linear algebraic equations

$$\gamma_n\alpha_n+\sum_{m=1}^{\infty} R_{nm}a_m=\tau_0 f_n,\quad n=2,3,\dots, \tag{12}$$

where

$$R_{nm}=\frac{\rho_0 k^2}{2\sqrt{\pi}E_0}R_{mn}^{(1)}+\frac{1}{\sqrt{\pi}}R_{mn}^{(2)}+\frac{\widetilde{G}}{\sqrt{\pi}E_0 h}R_{mn}^{(3)},\qquad R_{nm}^{(1)}=\frac{1}{m}\int_{-1}^{1}(1-y^2)^{3/2}P_{n-2}^{(3/2,3/2)}(y)T_m(y)dy,$$

$$R_{nm}^{(2)}=\int_{-1}^{1}(1-y^2)^{3/2}P_{n-2}^{(3/2,3/2)}(y)\biggl(\int_{-1}^{1} K|y-\eta|\frac{T_m(\eta)d\eta}{\sqrt{1-\eta^2}}\biggr)dy,$$

$$R_{nm}^{(3)}=\int_{-1}^{1}(1-y^2)P_{n-2}^{(3/2,3/2)}(y)T_m(y)dy,$$

$$f_n=-\frac{\rho_0 k^2\ln 2}{\pi\sqrt{\pi}E_0}\int_{-1}^{1}(1-y^2)^{3/2}P_{n-2}^{(3/2,3/2)}(y)dy-\frac{2}{\pi\sqrt{\pi}}\int_{-1}^{1}(1-y^2)^{3/2}l(y)P_{n-2}^{(3/2,3/2)}(y)dy$$

$$-\frac{2\widetilde{G}}{\pi\sqrt{\pi}E_0 h}\int_{-1}^{1}(1-y^2)P_{n-2}^{(3/2,3/2)}(y)-\frac{\widetilde{G}}{\pi E_0 h}\int_{-1}^{1}(1-y^2)^{3/2}P_{n-2}^{(3/2,3/2)}(y)\delta(y)dy,$$

$$l(y)=\int_{-1}^{1}\frac{K(|y-\eta|)d\eta}{\sqrt{1-\eta^2}},\qquad \gamma_n=\frac{\Gamma(n+1/2)}{n\Gamma(n-1)}\,.$$

Using Stirling's formula for the Gamma function $\Gamma(z)$ [1], we have

$$\gamma_n=O(n^{1/2}),\quad n\to\infty. \tag{13}$$

Using now Rodrigue's formula and Darboux asymptotic formula for the Jacobi's polynomials [14], after some calculations, we get

$$R_{nm}^{(1)}=\frac{1}{\sqrt{\pi(n-2)}m}\begin{cases}0, & m\neq n,\quad m\neq n\pm 2,\\ \pi, & m=n\\ -\pi/2, & m=n\pm 2,\end{cases}+O(n^{-3/2})\frac{1}{m}\begin{cases}0, & m\neq 2,4\\ -\pi/4, & m=2,\quad n\to\infty\\ -\pi/16, & m=4\end{cases},$$

$$R_{mn}^{(2)}=\frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma(m+1/2)(n-2)m(m-1)}\int_{-1}^{1}\frac{d}{dy}(1-y^2)^{5/2}P_{n-3}^{(5/2,5/2)}(y)$$

$$\times\biggl(\int_{-1}^{1} K(|y-\eta|)\frac{d^2}{d\eta^2}(1-\eta^2)^{3/2}P_{m-2}^{(3/2,3/2)}(\eta)d\eta\biggr)dy$$

$$= \frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma(m+1/2)(n-2)m(m-1)} \int_{-1}^{1} (1-y^2)^{5/2} P_{n-3}^{(5/2,5/2)}(y)$$

$$\times \biggl(\int_{-1}^{1} (1-\eta^2)^{3/2} P_{m-2}^{(3/2,3/2)}(\eta) \frac{\partial^3 K(|y-\eta|)}{\partial y \partial \eta^2} d\eta \biggr) dy,$$

$$R_{mn}^{(3)} \sim \frac{2((-1)^{m+n}+1)}{\sqrt{\pi}\sqrt{(n-2)}} \Bigl[\frac{1}{(m+n)^2-1} + \frac{1}{(m-n)^2-1)} \Bigr] + O(n^{-3/2}) \frac{(-1)^m+1}{m^2-1}, \quad n \to \infty,$$

$$f_n = O\Bigl(\frac{1}{\sqrt{n}}\Bigr), \quad n \to \infty. \tag{14}$$

Now, investigating the regularity of the infinite system (12) and taking into account estimations (13), (14), for the system (12), we obtain the following conditions:

$$\sum_{m=1,n=2} \biggl(\frac{R_{mn}}{\gamma_n} \biggr)^2 < \infty, \qquad \sum_{n=2} \biggl(\frac{f_n}{\gamma_n} \biggr)^2 < \infty. \tag{15}$$

b) If $h(x) = h_0\sqrt{1-x^2}$, $|x| < 1$, a solution of problem (9), (10) will be sought in the form (11) and from (9), we have

$$\frac{h_0}{2} \sum_{m=1}^{\infty} \frac{mT_m(y)}{\sqrt{1-y^2}} a_m + \frac{\rho_0 k^2 h_0}{2E_0} \sum_{m=0}^{\infty} a_m \mu_m \sqrt{1-y^2} T_m(y)$$

$$+ \sum_{m=0}^{\infty} a_m \int_{-1}^{1} \widetilde{K}(|y-\eta|) \frac{T_m(\eta)}{\sqrt{1-\eta^2}} d\eta = -\frac{\widetilde{G}}{E_0} \frac{1}{\sqrt{1-y^2}} \sum_{m=0}^{\infty} a_m T_m(y) - \frac{\widetilde{G}}{E_0} \tau_0 \delta(y),$$

where

$$\widetilde{K}(|y-\eta|) = \frac{\partial}{\partial y} \sqrt{1-y^2}\, \frac{\partial R_0(0,|y-\eta|)}{\partial y} + \frac{\rho_0 k^2}{E_0} \sqrt{1-y^2} R_0(0,|y-\eta|).$$

Multiplying both parts of the above equality by $T_n(y)$, integrating in the interval $(-1,1)$ and using the conditions of orthogonality of Chebyshev's polynomials of the first kind, we obtain the infinite system of linear algebraic equations

$$\delta_n a_n + \sum_{m=1}^{\infty} L_{mn} a_m = \tau_0 g_m, \qquad n = 1,2,3,\dots, \tag{16}$$

where

$$L_{mn} = \frac{\rho_0 k^2 h_0}{2E_0} L_{mn}^{(1)} + L_{mn}^{(2)}, \qquad L_{mn}^{(1)} = \frac{1}{m} \int_{-1}^{1} \sqrt{1-y^2} T_m(y) T_n(y) dy,$$

$$L_{mn}^{(2)} = \int_{-1}^{1} T_n(y) \biggl(\int_{-1}^{1} \widetilde{K}(|y-\eta|) \frac{T_m(\eta) d\eta}{\sqrt{1-\eta^2}} \biggr) dy,$$

$$g_n = -\frac{\widetilde{G}}{E_0} \int_{-1}^{1} T_n(y)\delta(y) dy - \frac{\rho_0 k^2 h_0}{\pi E_0} \ln 2 \int_{-1}^{1} \sqrt{1-y^2} T_n(y) - \frac{2}{\pi} \int_{-1}^{1} T_n(y) \biggl(\int_{-1}^{1} \frac{\widetilde{K}(|y-\eta|) d\eta}{\sqrt{1-\eta^2}} \biggr) dy,$$

$$\delta_n = \frac{\pi h_0}{4} n + \frac{\pi \widetilde{G}}{2E_0}\,.$$

Using the properties of the first kind Chebyshev's orthogonal polynomials and Gamma function, we have

$$L_{mn}^{(1)} = \frac{1}{m}\begin{cases}\pi/8 & m=n=1\\ \pi/4, & m=n\neq 1\\ -\pi/8, & m=n\pm 2\\ 0, & m\neq n,\quad m\neq n\pm 2\end{cases},$$

$$L_{mn}^{(2)} = \frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma(m+1/2)m(m-1)}\int_{-1}^{1} T_n(y)\times\left(\int_{-1}^{1}\widetilde{K}(|y-\eta|)\frac{d^2}{d\eta^2}(1-\eta^2)^{3/2}P_{m-2}^{(3/2,3/2)}(\eta)d\eta\right)dy$$

$$= \frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma(m+1/2)m(m-1)}\int_{-1}^{1} T_n(y)\times\left(\int_{-1}^{1}(1-\eta^2)^{3/2}P_{m-2}^{(3/2,3/2)}(\eta)\frac{\delta^2\widetilde{K}(|y-\eta|)}{\delta\eta^2}d\eta\right)dy,$$

$$g_n = -\frac{\widetilde{G}}{E_0}\cos\frac{\pi n}{2} - \frac{2}{\pi}\int_{-1}^{1} T_n(y)\left(\int_{-1}^{1}\frac{\widetilde{K}(|y-\eta|)d\eta}{\sqrt{1-\eta^2}}\right)dy, \qquad n\neq 2$$

$$g_2 = \frac{\widetilde{G}}{E_0} + \frac{\rho_0 k^2 h_0}{4E_0}\ln 2 - \frac{2}{\pi}\int_{-1}^{1} T_2(y)\left(\int_{-1}^{1}\frac{\widetilde{K}(|y-\eta|)d\eta}{\sqrt{1-\eta^2}}\right)dy,$$

$$\delta_n = O(n), \quad n\to\infty.$$

If we rewrite the system (16) in following form

$$a_n + \sum_{m=1}^{\infty}\frac{L_{nm}}{\delta_n}a_m = \tau_0\frac{g_n}{\delta_n}, \quad n=1,2,3,\dots \tag{17}$$

based on the previous representations for system (17), we obtain the conditions

$$\sum_{n=1,m=1}^{\infty}\left(\frac{L_{nm}}{\delta_n}\right)^2 < \infty, \qquad \sum_{n=1}^{\infty}\left(\frac{g_n}{\delta_n}\right)^2 < \infty. \tag{18}$$

Conditions (15) and (18) prove that the infinite systems (12) and (17) are quasi-completely regular in the space l_2, that is, their solutions satisfy the condition $\sum\limits_{n=1}^{\infty} a_n^2 < \infty$.

On the basis of the Hilbert alternative [8,9], if the determinants of the corresponding finite system of linear algebraic equations are nonzero, then systems (12), (17) will have a unique solution in the class l_2, and problem (9), (10) has the unique solution in the form (11).

The results of [8, p. 534], are applicable to an infinite system (17). Relying on this fact, the system

$$a_n^N + \sum_{m=1}^{N}\widetilde{L}_{nm}a_m^N = \widetilde{g}_n, \qquad n=1,2,\dots,N, \qquad \widetilde{g}_n = \tau_0\frac{g_n}{\delta_n}, \qquad \widetilde{L}_{nm} = \frac{L_{nm}}{\delta_n} \tag{19}$$

is solvable for sufficiently large N and the convergence of approximate solutions $\{a_n^N\}_{n=1,\dots,N}$ to $\{a_n\}_{n\geq 1}$ is valid in the sense of the norm of the space l_2.

The convergence rate is determined by the inequality

$$\|a-\varphi_0^{-1}\overline{a}^N\|_{l_2} \leq C_1\left[\sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty}|\widetilde{L}_{nm}|^2\right]^{1/2} + C_2\left(\frac{\sum\limits_{n=N+1}^{\infty}\widetilde{g}_n^2}{\sum\limits_{n=1}^{\infty}\widetilde{g}_n^2}\right)^{1/2},$$

where $a=\{a_n\}_{n\geq 1}=(a_1,a_2,\dots,a_n,\dots)$ is the solution of system (17), $\overline{a}^N=(a_1^N,a_2^N,\dots,a_N^N)$ is the solution of system (19), $\varphi_0^{-1}\overline{a}^N=(a_1^N,a_2^N,\dots,a_N^N,0,0,\dots)$.

Considering the expression for $\widetilde{L}_{nm}$, we have

$$C_1\bigg[\sum_{n=N+1}^{\infty}\sum_{m=1}^{\infty}|\widetilde{L}_{nm}|^2\bigg]^{1/2}\leq C_1^*\bigg[\sum_{n=1}^{\infty}\frac{1}{(n+N)^4}\bigg]^{1/2}=C_1^*[\zeta(4,N)]^{1/2},$$

$$C_2\left(\frac{\sum\limits_{n=N+1}^{\infty}\widetilde{g}_n^2}{\sum\limits_{n=1}^{\infty}\widetilde{g}_n^2}\right)^{1/2}\leq C_2^*\Big(\sum_{n=1}^{\infty}\frac{1}{(n+N)^2}\Big)^{1/2}\leq C_2^*[\zeta(2,N)]^{1/2}$$

where $\zeta(s,N)$ is the known generalized Zeta-function.

Using the asymptotic formula for the generalized Zeta-function [3, p. 62], we obtain

$$\|a-\varphi_0^{-1}\overline{a}^N\|_{l_2}\leq CN^{-1/2}.$$

Thus the solutions of systems (12) and (17) can be constructed by the reduction method with any accuracy [8, 9].

Theorem. *The infinite systems of linear algebraic equations* (12) *and* (17) *are quasi-completely regular in the space* l_2*. Accordingly, problem* (9)*,* (10) *has the unique solution in the form* (11)*.*

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