Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 3, 451–461

THE NORM AND ALMOST EVERYWHERE CONVERGENCE OF APPROXIMATE IDENTITY AND FEJÉR MEANS OF TRIGONOMETRIC AND VILENKIN SYSTEMS

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this paper we investigate some very general approximation kernels with special properties, called an approximate identity, and prove norm and almost everywhere convergences of these general methods with respect to the trigonometric system. Investigations of these summation methods can be used also to obtain norm convergence of Fejér means with respect to the Vilenkin system, but they are not useful to study a.e. convergence in this case due to some special properties of the kernels of Vilenkin–Fejér means. Despite these different properties, we give alternative methods to prove a.e. convergence of Vilenkin–Fejér means.

1. Introduction

Let us define Fourier coefficients, partial sums, Fejér means and kernels with respect to the Vilenkin and trigonometric systems of any integrable function in the usual manner:

$$\widehat{f}^w(k) := \int f\overline{w}_k d\mu \qquad (k\in\mathbb{N},\quad w=\psi \text{ or } w=T),$$

$$S_n^w f := \sum_{k=0}^{n-1}\widehat{f}(k)\,\psi_k \qquad (n\in\mathbb{N}_+,\ S_0f:=0,\quad w=\psi \text{ or } w=T),$$

$$\sigma_n^w f := \frac{1}{n}\sum_{k=0}^{n-1} S_k^w f \qquad (n\in\mathbb{N}_+\ ),$$

$$K_n^w := \frac{1}{n}\sum_{k=0}^{n-1} D_k^w \qquad (n\in\mathbb{N}_+,\quad w=\psi \text{ or } w=T),$$

where $\mathbb{N}_+$ denotes the set of positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$.

It is well-known (for details see, e.g., [1, 4] and [19]) that the Fejér means

$$\sigma_n^w f \quad (w=\psi \text{ or } w=T),$$

where σ_n^ψ and σ_n^T are, respectively, the Vilenkin–Fejér and trigonometric-Fejér means converging to the function f in L_p norm, that is,

$$\|\sigma_n^w f - f\|_p \to 0, \text{ as } n\to\infty \quad (w=\psi \text{ or } w=T)$$

for any $f\in L_p$, where $1\le p<\infty$. Moreover, (see, e.g., [2] and [3]), if we consider the maximal operator of Fejér means with respect to Vilenkin and trigonometric systems defined by

$$\sigma^{*,w} f := \sup_{n\in\mathbb{N}} |\sigma_n^w f| \quad (w=\psi \text{ or } w=T),$$

then the weak type inequality

$$\mu\left(\sigma^{*,w} f>\lambda\right)\le \frac{c}{\lambda}\|f\|_1 \quad (f\in L_1(G_m),\quad \lambda>0),$$

2020 *Mathematics Subject Classification.* 42C10.

Key words and phrases. Vilenkin system; Trigonometric system, Fejér means; Almost everywhere convergence; Norm convergence.

was proved in Zygmund [23] for the trigonometric series, in Schipp [16] for the Walsh series and in Pál, Simon [13] (see also [14, 18, 21, 22]) for the bounded Vilenkin system.

The research in this paper is also related to the important contribution of Vakhtang Kokilashvili, see e.g. [9, 10], and references therein. It follows that the Fejér means with respect to trigonometric and Vilenkin systems of any integrable function converge a.e. to this function.

Very general approximation kernels with special properties, called an approximate identity consisting of a class of summability methods such as Fejér means, were investigated in [4, 12] and [15].

In this paper, we investigate more general summability methods which are called the approximation identities consisting of a class of summability methods and provide the norm and a.e. convergence of these summability methods with respect to the trigonometric system. Investigations of these summations can be used to obtain the norm convergence of Fejér means with respect to the Vilenkin system also, but these methods are not useful to study a.e. convergence in this case, because of some special properties of the kernels of the Vilenkin–Fejér means. Despite these different properties, we give alternative methods to prove almost everywhere convergence of Fejér means with respect to the Vilenkin systems.

This paper is organized as follows: in order not to disturb our discussions later on, some definitions and notations are presented in Sections 2 and 3. Moreover, to prove the main results, we will need some auxiliary Lemmas, some of them are new and of independent interest. These results are also presented in Sections 2 and 3. The main result with the proof is given in Sections 4 and 5.

2. Fejér Means with Respect to the Vilenkin Systems

Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers, not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. In this paper, we discuss the bounded Vilenkin groups only, that is, $\sup_{n\in\mathbb{N}} m_n < \infty$.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

The elements of G_m are represented by the sequences

$$x := (x_0, x_1, \dots, x_k, \dots) \quad (\, x_k \in Z_{m_k}) .$$

It is easy to give a base for the neighbourhood of G_m, namely,

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m,\, n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

Let $e_n := (0, \dots, 0, x_n = 1, 0, \dots) \in G_m$ $(n \in \mathbb{N})$. If we define the so-called generalized number system based on m in the form

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be expressed uniquely as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$, and only a finite number of n_j's differ from zero. Let $|n| := \max\{j \in \mathbb{N};\, n_j \neq 0\}$.

If we define $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$, and

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}, \dots), & \text{for} \quad k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, x_{k+1} = 0, \dots, x_{N-1} = 0, x_N, \dots), & \text{for} \quad l = N, \end{cases}$$

then

$$\overline{I_N} = \Big(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \Big) \bigcup \Big(\bigcup_{k=0}^{N-1} I_N^{k,N} \Big). \tag{1}$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First, define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, as

$$r_k(x) := \exp(2\pi\imath x_k/m_k) \qquad \left(\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

By a Vilenkin polynomial we mean a finite linear combination of Vilenkin functions. We denote the collection of Vilenkin polynomials by $\mathcal{P}$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (for details see, e.g., [1, 17, 20]). Specially, we call this system the Walsh-Paley one if $m \equiv 2$ (for details see [7] and [17]).

Recall that (for details see, e.g., [1, 5] and [6]) if $n > t$, $t, n \in \mathbb{N}$, then

$$K_{M_n}^{\psi}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \backslash I_{t+1}, \quad x - x_t e_t \in I_n, \\ \frac{M_n+1}{2}, & x \in I_n, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and

$$n\left|K_n^{\psi}\right| \le c \sum_{l=0}^{|n|} M_l \left|K_{M_l}^{\psi}\right|. \tag{3}$$

By using these two properties of Fejér kernels, we obtain the following

Lemma 1. *For any n, $N \in \mathbb{N}_+$, we have*

$$\int_{G_m} K_n^{\psi}(x) d\mu(x) = 1, \tag{4}$$

$$\sup_{n\in\mathbb{N}} \int_{G_m} \left|K_n^{\psi}(x)\right| d\mu(x) \le c < \infty, \tag{5}$$

$$\int_{\overline{I_N}} \left|K_n^{\psi}(x)\right| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for any} \quad N \in \mathbb{N}_+, \tag{6}$$

where c is an absolute constant.

Proof. According to the orthonormality of Vilenkin systems, we immediately get the proof of (4). It is easy to prove that

$$\int_{G_m} \left|K_{M_n}^{\psi}(x)\right| d\mu(x) \le c < \infty.$$

Combining (2) and (3), we can conclude that

$$\int_{G_m} \left|K_n^{\psi}(x)\right| d\mu(x) \le \frac{1}{n} \sum_{l=0}^{|n|} M_l \int_{G_m} \left|K_{M_l}^{\psi}(x)\right| d\mu(x) \le \frac{1}{n} \sum_{l=0}^{|n|} M_l < c < \infty,$$

so, (5) is proved, as well.

Let $x \in I_N^{k,l}$, $k = 0, \ldots, N-2$, $l = k+1, \ldots, N-1$. Using again (2) and 3, we get

$$\left|K_n^{\psi}(x)\right| \le \frac{c}{n} \sum_{s=0}^{l} M_s \left|K_{M_s}^{\psi}(x)\right| \le \frac{c}{n} \sum_{s=0}^{l} M_s M_k \le \frac{c M_l M_k}{n}. \tag{7}$$

Let $x \in I_N^{k,N}$, where $x \in I_{q+1}^{k,q}$, for some $N \le q < |n|$, i.e.,

$$x = \left(x_0 = 0, \ldots, x_{k-1} = 0,\ x_k \ne 0, \ldots, x_{N-1} = 0,\ x_q \ne 0,\ x_{q+1} = 0, \ldots, x_{|n|-1}, \ldots\right),$$

then

$$\left|K_n^{\psi}(x)\right| \leq \frac{c}{n}\sum_{i=0}^{q-1} M_i M_k \leq \frac{cM_k M_q}{n}. \tag{8}$$

Let $x \in I_{|n|}^{k,|n|} \subset I_N^{k,N}$, i.e.,

$$x = \left(x_0 = 0, \ldots, x_{m-1} = 0,\, x_k \neq 0,\, x_{k+1} = 0, \ldots, x_N = 0, \ldots, x_{|n|-1} = 0, \ldots\right),$$

then

$$\left|K_n^{\psi}(x)\right| \leq \frac{c}{n}\sum_{i=0}^{|n|-1} M_i M_k \leq \frac{cM_k M_{|n|}}{n}. \tag{9}$$

Combining (8) and (9), we can conclude that

$$\begin{aligned}\int_{I_N^{k,N}} \left|K_n^{\psi}\right| d\mu &= \sum_{q=N}^{|n|-1} \int_{I_{q+1}^{k,q}} \left|K_n^{\psi}\right| d\mu + \int_{I_{|n|}^{k,|n|}} \left|K_n^{\psi}\right| d\mu \\ &\leq \sum_{q=N}^{|n|-1} \frac{cM_k}{n} + \frac{cM_k}{n} \\ &\leq \frac{c(|n|-N)M_k}{M_{|n|}}. \end{aligned} \tag{10}$$

Hence, if we apply (1), (7) and (10), we find that

$$\begin{aligned}&\int_{\overline{I_N}} \left|K_n^{\psi}\right| d\mu \\ =& \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0, j\in\{l+1,\ldots,N-1\}}^{m_{j-1}} \int_{I_N^{k,l}} \left|K_n^{\psi}\right| d\mu + \sum_{k=0}^{N-1}\int_{I_N^{k,N}} \left|K_n^{\psi}\right| d\mu \\ \leq& c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1} \frac{m_{l+1}\ldots m_{N-1}}{M_N}\frac{cM_l M_k}{n} + c\sum_{k=0}^{N-1}(|n|-N)M_k\frac{1}{M_{|n|}} \\ :=& I + II. \end{aligned}$$

It is evident that

$$\begin{aligned} I =& \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{M_k}{M_{|n|}} \leq c\sum_{k=0}^{N-2}\frac{(N-k)M_k}{M_{|n|}} \\ \leq& c\sum_{k=0}^{N-2}\frac{|n|-k}{2^{|n|-k}} = c\sum_{k=0}^{N-2}\frac{|n|-k}{2^{(|n|-k)/2}}\frac{1}{2^{(|n|-k)/2}} \\ \leq& \frac{c}{2^{(|n|-N)/2}}\sum_{k=0}^{N-2}\frac{|n|-k}{2^{(|n|-k)/2}} \leq \frac{C}{2^{(|n|-N)/2}} \to 0, \quad \text{as} \quad n\to\infty. \end{aligned}$$

Analogously, we see that

$$II \leq \frac{c(|n|-N)}{2^{|n|-N}} \to 0, \quad \text{as} \quad n \to \infty,$$

so, (6) holds also and thus the proof is complete. □

The next lemma is very important to prove almost everywhere convergence of the Vilenkin–Fejér means.

Lemma 2. *Let $n \in \mathbb{N}$. Then*

$$\int_{I_N} \sup_{n>M_N} \left|K_n^{\psi}\right| d\mu \le C < \infty,$$

where C is an absolute constant.

Proof. Let $n > M_N$ and $x \in I_N^{k,l}$, $k = 0, \dots, N-2$, $l = k+1, \dots, N-1$. Using (7) in the proof of Lemma 1, we get

$$\sup_{n>M_N} \left|K_n^{\psi}(x)\right| \le \frac{cM_l M_k}{M_N}.$$

Let $n > M_N$ and $x \in I_N^{k,N}$. Then, using (2), we find that $\left|K_n^{\psi}(x)\right| \le cM_k$, so,

$$\sup_{n>M_N} \left|K_n^{\psi}(x)\right| \le cM_k.$$

Hence, if we apply (1), we get

$$\begin{aligned}
&\int_{I_N} \sup_{n>M_N} \left|K_n^{\psi}\right| d\mu \\
=&\sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j\in\{l+1,\dots,N-1\}}^{m_{j-1}} \int_{I_N^{k,l}} \sup_{n>M_N} \left|K_n^{\psi}\right| d\mu \\
+&\sum_{k=0}^{N-1} \int_{I_N^{k,N}} \sup_{n>M_N} \left|K_n^{\psi}\right| d\mu \\
\le& c\sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \dots m_{N-1}}{M_N} \frac{M_l M_k}{M_N} + c\sum_{k=0}^{N-1} \frac{M_k}{M_N} \\
\le& \sum_{k=0}^{N-2} \frac{(N-k)M_k}{M_N} + c < C < \infty.
\end{aligned}$$

The proof is complete. □

3. Fejér Means with Respect to the Trigonometric System

If we consider the Fejér kernels with respect to the trigonometric system $\{(1/2\pi)e^{inx}, n = 0, \pm 1, \pm 2, \dots\}$, for $x \in [-\pi, \pi]$, we have $K_n^T(x) \ge 0$ and

$$K_n^T(x) = \frac{1}{n}\Big(\frac{\sin((nx)/2)}{\sin(x/2)}\Big)^2.$$

Moreover, the Fejér kernel $K_n^T (n \in \mathbb{N}_+)$ with respect to the trigonometric system has an upper envelope

$$0 \le K_n^T(x) \le \min(n, \pi(n|x|^2)^{-1}). \tag{11}$$

It also follows that every Fejér kernels have one integrable upper envelope

$$\sup_{n\in\mathbb{N}} K_n^T(x) \le \pi|x|^{-2}.$$

Lemma 3. *Let $n \in \mathbb{N}$. Then, for any $n, N \in \mathbb{N}_+$, we have*

$$\int_{[-\pi,\pi]} \left|K_n^T(x)\right| d\mu(x) = \int_{[-\pi,\pi]} K_n^T(x) d\mu(x) = 1, \tag{12}$$

$$\int_{[-\pi,\pi]\setminus[-\varepsilon,\varepsilon]} \left|K_n^T(x)\right| d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for any} \quad \varepsilon > 0. \tag{13}$$

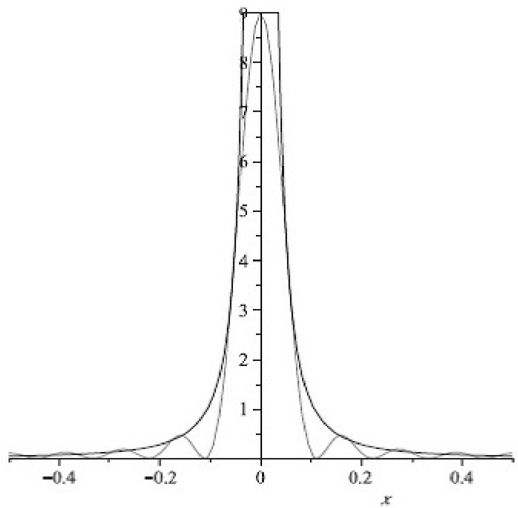

FIGURE 1. Fejér kernel and the upper envelope $\min(n, \pi(n|x|^2)^{-1})$.

Moreover,

$$\lim_{n\to\infty} \sup_{[-\pi,\pi]\backslash[-\varepsilon,\varepsilon]} \left|K_n^T(x)\right| = 0, \quad \textit{for any} \;\; \varepsilon > 0. \tag{14}$$

Proof. According to the property $K_n^T(x) \geq 0$ and the orthonormality of trigonometric system, we immediately get the proof of (12). On the other hand, (13) and (14) follow estimate (11) so, we leave out the details. □

4. Approximate Identity

The properties established in Lemma 1 and Lemma 3 ensure that the kernel of the Fejér means $\{K_N^w\}_{N=1}^{\infty}$ $(w = \psi$ or $w = T)$, with respect to Vilenkin and trigonometric systems, forms the so-called approximation identity. To unify the proofs for trigonometric and Vilenkin systems we mean that I denotes G_m or $[-\pi, \pi]$ and I_N denotes $I_N(0)$ or $[-1/2^N, 1/2^N]$ for $N \in \mathbb{N}_+$.

Definition 1. The family $\{\Phi_n\}_{n=1}^{\infty} \subset L_\infty(I)$ forms an approximate identity provided that

(A1) $\int\limits_I \Phi_n(x)d(x) = 1,$

(A2) $\sup\limits_{n\in\mathbb{N}} \int\limits_I |\Phi_n(x)|\, d\mu(x) < \infty,$

(A3) $\int\limits_{I\backslash I_N} |\Phi_n(x)|\, d\mu(x) \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for any} \quad N \in \mathbb{N}_+.$

The term "approximate identity" is used due to the fact that $\Phi_n * f \to f$ as $n \to \infty$ in any reasonable sense.

Next, we prove an important result, which will be used to obtain the norm convergence of some well-known and general summability methods.

Theorem 1. *Let $f \in L_p(I)$, where $1 \leq p < \infty$ and the family $\{\Phi_n\}_{n=1}^{\infty} \subset L_\infty(I)$ forms an approximate identity. Then*

$$\|\Phi_n * f - f\|_p \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. Let $\varepsilon > 0$. Using the continuity of L_p norm and $(A2)$ condition, we get

$$\sup_{t\in I_N} \|f(x-t) - f(x)\|_p \sup_{n\in\mathbb{N}} \|\Phi_n\|_1 < \varepsilon/2.$$

Applying now Minkowski's integral inequality and $(A1)$ and $(A3)$ conditions, we find that

$$\begin{aligned}\|\Phi_n * f - f\|_p &= \left\| \int\limits_I \Phi_n(t)(f(x-t)-f(x))d\mu(t) \right\|_p \\ &\le \int\limits_I |\Phi_n(t)| \, \|f(x-t)-f(x)\|_p \, d\mu(t) \\ &= \int\limits_{I_N} |\Phi_n(t)| \, \|f(x-t)-f(x)\|_p \, d\mu(t) \\ &+ \int\limits_{I\setminus I_N} |\Phi_n(t)| \, \|f(x-t)-f(x)\|_p \, d\mu(t) \\ &\le \sup_{t\in I_N} \|f(x-t)-f(x)\|_p \sup_{n\in\mathbb{N}} \|\Phi_n\|_1 \\ &+ \sup_{t\in I} \|f(x-t)-f(x)\|_p \int\limits_{I\setminus I_N} |\Phi_n(t)| \, d\mu(t) < \varepsilon/2 + \varepsilon/2 < \varepsilon.\end{aligned}$$

The proof is complete. □

According to Lemma 1 and Lemma 3, we immediately get that the following results hold.

Corollary 1. *Let $f \in L_p(I)$, where $1 \le p < \infty$. Then*

$$\|\sigma_n^w f - f\|_p \to 0, \quad as \quad n \to \infty \quad (w = \psi \quad or \quad w = T),$$

where σ_n^ψ and σ_n^T are the Vilenkin–Fejér and trigonometric–Fejér means, respectively.

Theorem 2. *Suppose that $f \in L_1(I)$ and the family $\{\Phi_n\}_{n=1}^\infty \subset L_\infty(I)$ forms an approximate identity. In addition, let*

$$(A4) \qquad \lim_{n\to\infty} \sup_{I\setminus I_N} |\Phi_n(x)| = 0, \quad \text{for any} \quad N \in \mathbb{N}_+.$$

a) *If the function f is continuous at t_0, then*

$$\Phi_n * f(t_0) \to f(t_0) \quad as \quad n \to \infty.$$

b) *If the functions $\{\Phi_n\}_{n=1}^\infty$ are even and the left and right limits $f(t_0-0)$ and $f(t_0+0)$ do exist and are finite, then*

$$\Phi_n * f(t_0) \to L, \quad as \quad n \to \infty,$$

where

$$L =: \frac{f(t_0+0)+f(t_0-0)}{2}. \tag{15}$$

Proof. It is evident that

$$\begin{aligned}|\Phi_n * f(t_0) - f(t_0)| &= \left| \int\limits_I \Phi_n(t)(f(t_0-t)-f(t_0))d\mu(t) \right| \\ &\le \left| \int\limits_{I_N} \Phi_n(t)(f(t_0-t)-f(t_0))d\mu(t) \right| \\ &+ \left| \int\limits_{I\setminus I_N} \Phi_n(t) f(t_0-t) d\mu(t) \right| + \left| \int\limits_{I\setminus I_N} \Phi_n(t) f(t_0) d\mu(t) \right| \\ &=: I + II + III.\end{aligned}$$

Let f be continuous at t_0. For any $\varepsilon > 0$, there exists N such that

$$I \le \sup_{t\in I_N} |f(t_0+t) - f(t_0))| \sup_{n\in\mathbb{N}} \|\Phi_n\|_1 < \varepsilon/2.$$

Using (A4) condition, we get

$$II \leq \sup_{t\in I\setminus I_N} |\Phi_n(t)| \, \|f\|_1 \to 0, \quad \text{as} \quad n \to \infty.$$

We conclude from (A3) that

$$III \leq |f(t_0)| \int\limits_{I\setminus I_N} |\Phi_n(t)| d\mu(t) \to 0, \quad \text{as} \quad n \to \infty.$$

Thus part a) is proved.

Since the functions $\{\Phi_n\}_{n=1}^{\infty}$ are even, for the proof of part b), we first note that

$$(\Phi_n * f)(t_0) - L = \int\limits_{I} \Phi_n(t) \Big(\frac{f(t_0 - t) + f(t_0 + t)}{2} - \frac{f(t_0 - 0) + f(t_0 + 0)}{2} \Big) d\mu(t).$$

Thus if we use part a), we immediately get the proof of part b) so, the proof is complete. □

Corollary 2. *Let $f \in L_1[-\pi,\pi]$. Then the following statements hold true.*

a) *If the function f is continuous at t_0, then*

$$\sigma_n^T f(t_0) \to f(t_0) \quad as \quad n \to \infty.$$

b) *Let the left and right limits $f(t_0-0)$ and $f(t_0+0)$ do exist and are finite. Then*

$$\sigma_n^T f(t_0) \to L \quad as \quad n \to \infty,$$

where L is defined by (15).

Remark 1. Conditions (A4) and (11) do not hold for the Vilenkin–Fejér kernels. Indeed, by using (2), for any $k \in \mathbb{N}_+$ and for any $e_0 \in I_n(e_0) \subset G_m \setminus I_n$, $(n \in \mathbb{N}_+)$, we get

$$|K_{M_k}^{\psi}(e_0)| = \Big| \frac{M_0}{1 - r_0\,(e_0)} \Big| = \Big| \frac{M_0}{1 - \exp\,(2\pi\imath/m_0)} \Big| = \frac{1}{2\sin(\pi/m_0)} \geq \frac{1}{2},$$

so,

$$\lim_{k\to\infty} \sup_{I_n(e_0)\subset G_m\setminus I_n} \left| K_{M_k}^{\psi}(x) \right| \geq \lim_{k\to\infty} \left| K_{M_k}^{\psi}(e_0) \right| \geq \frac{1}{2} > 0, \text{ for any } n \in \mathbb{N}_+.$$

Hence (A4) and (11) are not true for the Fejér kernels with respect to the Vilenkin system. However, in some publications one can find that some researchers use such an estimate (for details see [8]).

Moreover, for any $x \in I_k \setminus I_{k-1}$, we have

$$|K_{M_k}^{\psi}(x)| = \Big| \frac{M_{k-1}}{1 - \exp\,(2\pi\imath/m_{k-1})} \Big| = \frac{M_{k-1}}{2\sin(\pi/m_{k-1})} \geq \frac{M_k}{2\pi}\,,$$

and it follows that the Fejér kernels with respect to the Vilenkin system have no one integrable upper envelope. In particular, the following lower estimate:

$$\sup_{n\in\mathbb{N}} |K_n^{\psi}(x)| \geq (2\pi\lambda|x|)^{-1}, \quad \text{where} \quad \lambda := \sup_{n\in\mathbb{N}} m_n,$$

holds.

This remark shows that there is an essential difference between the Vilenkin–Fejér kernels and the Fejér kernels with respect to trigonometric system. Moreover, Theorem 2 is useless to prove almost everywhere convergence of Vilenkin–Fejér means.

5. Almost Everywhere Convergence of Vilenkin–Fejér Means

The next theorem is very important to study almost everywhere convergence of the Vilenkin–Fejér means.

Theorem 3. *Suppose that the sigma sub-linear operator V is bounded from L_{p_1} to L_{p_1} for some $1 < p_1 \leq \infty$ and*

$$\int_{\overline{I}} |Vf| \, d\mu \leq C \, \|f\|_1$$

for $f \in L_1(G_m)$ and Vilenkin interval $I \subset G_m$ which satisfies

$$\operatorname{supp} f \subset I, \qquad \int_{G_m} f d\mu = 0. \tag{16}$$

Then the operator V is of weak-type $(1,1)$, i.e.,

$$\sup_{y>0} y\mu\left(\{Vf > y\}\right) \leq \|f\|_1 \, .$$

Theorem 4. *Let $f \in L_1(G_m)$. Then*

$$\sup_{y>0} y\mu\left\{\sigma^{*,\psi} f > y\right\} \leq \|f\|_1 \, .$$

Proof. By Theorem 3, we find that the proof will be complete if we show that

$$\int_{\overline{I}} \left|\sigma^{*,\psi} f\right| d\mu \leq \|f\|_1,$$

for every function f which satisfies conditions in (16), where I denotes the support of the function f.

Without lost the generality, we may assume that f is a function with support I and $\mu\left(I\right) = M_N$. We may assume that $I = I_N$. It is easy to see that

$$\sigma_n^{\psi} f = \int_{I_N} K_n^{\psi}(x-t) f(t) d\mu\left(t\right) = 0, \quad \text{ for } \quad n \leq M_N.$$

Therefore, we may suppose that $n > M_N$. Hence

$$\begin{aligned}\left|\sigma^{*,\psi} f(x)\right| &\leq \sup_{n \leq M_N} \left| \int_{I_N} K_n^{\psi}(x-t) f(t) d\mu\left(t\right) \right| \\ &\quad + \sup_{n > M_N} \left| \int_{I_N} K_n^{\psi}(x-t) f(t) d\mu\left(t\right) \right| = \sup_{n > M_N} \left| \int_{I_N} K_n^{\psi}(x-t) f(t) d\mu\left(t\right) \right|.\end{aligned}$$

Let $t \in I_N$ and $x \in \overline{I_N}$. Then $x - t \in \overline{I_N}$ and if we apply Lemma 2, we get

$$\begin{aligned}\int_{\overline{I_N}} \left|\sigma^{*,\psi} f(x)\right| d\mu(x) &\leq \int_{\overline{I_N}} \sup_{n > M_N} \int_{I_N} \left|K_n^{\psi}\left(x-t\right) f(t)\right| d\mu\left(t\right) d\mu\left(x\right) \\ &\leq \int_{\overline{I_N}} \int_{I_N} \sup_{n > M_N} \left|K_n^{\psi}\left(x-t\right) f(t)\right| d\mu\left(t\right) d\mu\left(x\right) \\ &\leq \int_{I_N} \int_{\overline{I_N}} \sup_{n > M_N} \left|K_n^{\psi}\left(x-t\right) f(t)\right| d\mu\left(x\right) d\mu\left(t\right) \\ &\leq \int_{I_N} \left|f(t)\right| d\mu\left(t\right) \int_{\overline{I_N}} \sup_{n > M_N} \left|K_n^{\psi}\left(x-t\right)\right| d\mu\left(x\right)\end{aligned}$$

$$\leq \int_{I_N} |f(t)|\, d\mu(t) \int_{\overline{I_N}} \sup_{n>M_N} \left|K_n^{\psi}(x)\right| d\mu(x)$$
$$= \|f\|_1 \int_{\overline{I_N}} \sup_{n>M_N} \left|K_n^{\psi}(x)\right| d\mu(x)$$
$$\leq c\, \|f\|_1 .$$

The proof is complete. □

Theorem 5. *Let $f \in L_1(G_m)$. Then*

$$\sigma_n^{\psi} f \to f \quad a.e., \quad as \quad n \to \infty.$$

Proof. Since

$$S_n^{\psi} P = P, \quad \text{for every} \quad P \in \mathcal{P},$$

according to the regularity of Fejér means, we obtain

$$\sigma_n^{\psi} P \to P \qquad \text{a.e., as } n \to \infty,$$

where $P \in \mathcal{P}$ is dense in the space L_1 (for details see, e.g., [1]).

On the other hand, using Theorem 4, we obtain that the maximal operator σ^* is bounded from the space L_1 to the space $weak - L_1$, that is,

$$\sup_{y>0} y\mu \left\{ x \in G_m : \left|\sigma^{*,\psi} f(x)\right| > y \right\} \leq \|f\|_1 .$$

According to the usual density argument (see Marcinkiewicz and Zygmund [11]), we obtain almost everywhere convergence of Fejér means

$$\sigma_n^{\psi} f \to f \quad \text{a.e.,} \quad \text{as} \quad n \to \infty.$$

The proof is complete. □

Acknowledgement

The authors would like to thank the referee for helpful suggestions. The research was supported by Shota Rustaveli National Science Foundation (SRNSF) grant no. FR-21-2844.

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(Received 22.02.2023)

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