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THE OPIAL TYPE NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF DIFFERENCE SCHEMES FOR THE INITIAL PROBLEM FOR LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The Opial type necessary and sufficient (as well as effective sufficient) conditions are established for the convergence of difference schemes of the Cauchy problem for linear systems of ordinary differential equations.

1. Statement of the Problem and Basic Notation

Consider the initial problem

$$\frac{dx}{dt} = P_0(t)\,x + q_0(t) \quad \text{for } \ t \in I, \tag{1.1}$$

$$x(t_0) = c_0, \tag{1.2}$$

where $P_0 \in L(I;\mathbb{R}^{n\times n})$, $q_0 \in L(I;\mathbb{R}^n)$, $t_0 \in I$ and $c_0 \in \mathbb{R}^n$, and $I = [a,b]$ is the closed interval, non-degenerated at the point.

Let $x_0 \in \mathrm{AC}(I;\mathbb{R}^n)$ be a unique solution of this initial problem.

Along with the problem, we consider the difference scheme

$$\Delta y(k-1) = G_{1m}(k)\,y(k) + G_{2m}(k-1)\,y(k-1) + g_{1m}(k) + g_{2m}(k-1) \ \ (k = 1,\dots,m), \tag{1.1m}$$

$$y(k_m) = \zeta_m \quad (m = 1,2,\dots), \tag{1.2m}$$

where G_{jm} and g_{jm} $(j = 1,2)$ are, respectively, the discrete $n\times n$-matrix- and n-vector-functions, $k_m \in \{0,\dots,m\}$ and $\zeta_m \in \mathbb{R}^n$.

In the paper, we wish to present the so-called Opial type necessary and sufficient (in particular, the effective sufficient) conditions for the convergence, in the definite sense, of solutions of difference scheme (1.1m), (1.2m) to x_0.

The numerical solvability of problem (1.1), (1.2) is classical. There are a lot of papers dealing with this problem (see, for example, [4, 7, 8, 10, 13, 16] and references therein). In our opinion, in these papers, with the exception of [4], only sufficient conditions are established for the numerical solvability of problem (1.1), (1.2). In the last paper, the criteria are obtained for the question under consideration, but the obtained results differ from the Opial type conditions. The goal of the present paper is to establish the conditions, analogous to the Opial type conditions.

In the paper, we use the following notation and definitions.

$\mathbb{N} = \{1,2,\dots\}$, $\widetilde{\mathbb{N}} = \{0,1,\dots\}$, $\mathbb{N}_l = \{1,\dots,l\}$, $\widetilde{\mathbb{N}}_l = \{0,\dots,l\}$ $(l \in \mathbb{N})$. $\mathbb{R} =]-\infty,+\infty[$.

$[t]$ is the integer part of $t \in \mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

$O_{n\times m}$ is the zero $n\times m$-matrix. 0_n is the zero n-vector.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

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$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of real column n-vectors $x = (x_i)_{i=1}^n$.

X^{-1} and $\det(X)$ are, respectively, the matrix, inverse to $X \in \mathbb{R}^{n\times n}$, and the determinant of X.

I_n is the identity $n \times n$-matrix;

δ_{ij} is the Kroneker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, etc., if each of its components is such.

$\bigvee\limits_a^b(X)$ is the sum of total variations of components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$; $V(X)(t) \equiv (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) \equiv \bigvee\limits_a^t(x_{ij})$;

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-) = X(a)$ and $X(b+) = X(b))$; $d_1X(t) = X(t)-X(t-)$, $d_2X(t) = X(t+)-X(t)$. $\|X\|_\infty = \sup\{\|X(t)\| : t \in I\}$. Somewhere else we use the designation $\|X\|_J = \sup\{\|X(t)\| : t \in J\}$, where $J \subset I$.

$\mathrm{BV}(I; \mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ with the norm $\|X\|_\infty$.

$C(I; \mathbb{R}^{n\times m})$ is the space of all continuous on I matrix-functions $X : I \to \mathbb{R}^{n\times m}$ with the standard norm $\|X\|_c = \max\{\|X(t)\| : t \in I\}$.

$\mathrm{AC}(I; \mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions.

s_0, s_1 and s_2 are the operators defined, respectively, as follows:

$$s_1(x)(a) = s_2(x)(a) = 0, \quad s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau), \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau);$$
$$s_0(x) = x(a), \quad s_0(x)(t) \equiv (t) - s_1(x)(t) - s_2(x)(t).$$

If $g \in \mathrm{BV}(I; \mathbb{R})$, $f : I \to \mathbb{R}$ and $s < t$, then we assume

$$\int\limits_s^t x(\tau)\, dg(\tau) = (L-S) \int\limits_{]s,t[} x(\tau)\, dg(\tau) + f(t)d_1g(t) + f(s)d_2g(s),$$

where $(L-S)\int\limits_{]s,t[} f(\tau)\, dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s, t[$. It is known (see [14, 17, 18]) that if the integral exists, then the right-hand side of the above integral equality coincides with the Kurzweil–Stieltjes integral $(K-S)\int\limits_s^t f(\tau)\, dg(\tau)$. So, $\int\limits_s^t f(\tau)\, dg(\tau) = (K-S)\int\limits_s^t f(\tau)\, dg(\tau)$.

If $G(t) = (g_{ik}(t))_{i,k=1}^{l,n}$ and $X(t) = (x_{kl}(t))_{k,l=1}^{n,m}$ for $t \in I$, then

$$S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 0, 1, 2), \quad \int\limits_a^b dG(\tau) \cdot X(\tau) = \left(\sum_{k=1}^n \int\limits_a^b x_{kl}(\tau)\, dg_{ik}(\tau) \right)_{i,l=1}^{l,m}.$$

Sometimes we use the designation $\int\limits_a^{\cdot} dG(s) \cdot X(s)$ for the integral $\int\limits_a^t dG(s) \cdot X(s)$ as the matrix-function to the variable t.

We introduce the following operators:

(a) if $X \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_jX(t)) \neq 0$ for $t \in I$ $(j = 1, 2)$, and $Y \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X, Y)(a) = O_{n\times m},$$
$$\mathcal{A}(X, Y)(t) \equiv Y(t) + \sum_{a<\tau\le t} d_1X(\tau)\,(I_n - d_1X(\tau))^{-1}\, d_1Y(\tau) - \sum_{a\le\tau<t} d_2X(\tau)\,(I_n + d_2X(\tau))^{-1}\, d_2Y(\tau).$$

(b) if $X \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ and $Y \in \mathrm{BV}(I;\mathbb{R}^{n\times m})$, then

$$\mathcal{B}(X,Y)(t) \equiv X(t)Y(t) - X(a)Y(a) - \int_a^t dX(\tau)\cdot Y(\tau);$$

(c) if $X \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $\det X(t)) \neq 0$, and $Y \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, then

$$\mathcal{I}(X,Y)(t) \equiv \int_a^t d\big(X(\tau) + \mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau).$$

$E(J,\mathbb{R}^{n\times m})$, where $J \subset \mathbb{N}$, is the space of all matrix-functions $Y : J \to \mathbb{R}^{n\times m}$ with the norm $\|Y\|_J = \max\left\{\|Y(k)\| : k \in J\right\}$.

Δ is the difference operator of the first order, i.e.,

$$\Delta Y(k-1) = Y(k) - Y(k-1) \quad \text{for} \quad Y \in E(\widetilde{\mathbb{N}}_l,\mathbb{R}^{n\times m}), \quad k \in \mathbb{N}_l.$$

If a matrix-function Y is defined on $\mathbb{N}_l$, or on $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0) = O_{n\times m}$, or $Y(l) = O_{n\times m}$, respectively, if necessary.

Let $\tau_m = (b-a)\,m^{-1}$ and

$$\tau_{0m} = a,\ \tau_{km} = a + k\tau_m,\ I_{km} =]\tau_{k-1\,m},\tau_{km}[\ \ (k = 1,\dots,m;\ m = 1,2,\dots).$$

Let ν_m $(m = 1,2,\dots)$ be the functions defined by the equalities

$$\nu_m(t) \equiv \left[\frac{t-a}{b-a}\,m\right] \quad (m = 1,2,\dots).$$

It is evident that

$$\nu_m(\tau_{km}) = k \quad (k = 0,\dots,m;\ m = 1,2,\dots).$$

For each natural m, we introduce the following operators:

a) $p_m : \mathrm{BV}(I;\mathbb{R}^n) \to \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ and $q_m : \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n) \to \mathrm{BV}(I;\mathbb{R}^n)$ defined as follows:

$$p_m(x)(k) = x(\tau_{km}) \ \text{ for } \ x \in \mathrm{BV}(I;\mathbb{R}^n) \ \ (k = 0,\dots,m)$$

and

$$q_m(y)(t) = \begin{cases} y(k) & \text{for } \ t = \tau_{km} \ \ (k = 0,\dots,m), \\ (I_n - G_{1m}(k))y(k) + g_{1m}(k)) & \text{for } \ t \in I_{km} \ \ (k = 0,\dots,m); \end{cases}$$

b) operator $\mathcal{B}_m$, defined for $X \in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^{n\times n})$ and $Y, Z \in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^{n\times l})$, by

$$\mathcal{B}_m(X,Y,Z)(a) = O_{n\times n}, \ \ \mathcal{B}_m(X,Y,Z)(\tau_{km}) = \sum_{i=1}^{k} X(i)Y(i) + \sum_{i=1}^{k} X(i)Z(i-1),$$
$$\mathcal{B}_m(X,Y,Z)(t) = \mathcal{B}_m(X,Y,Z)(\tau_{km}) - X(k)Y(k) \ \text{ for } \ t \in I_{km} \ \ (k = 1,\dots,m);$$

c) operator $\mathcal{I}_m$, defined for $X, Y_1, Y_2 \in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^{n\times n})$, $Z \in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^{n\times l})$, by

$$\mathcal{I}_m(X,Z,Y_1,Y_2)(a) = O_{n\times n}, \ \ \mathcal{I}_m(X_1,X_2,Y_1,Y_2)(\tau_{km})$$
$$\equiv \sum_{i=1}^{k}\big(I_n - X(i)(I_n - Y_1(i))Z(i)\big) + \sum_{i=0}^{k-1}\big(X(i+1)(I_n + Y_2(i))Z(i) - In\big),$$
$$\mathcal{I}_m(X,Z,Y_1,Y_2)(t) = \mathcal{I}_m(X,Z,Y_1,Y_2)(\tau_{km}) - \big(I_n - X(k)(I_n - Y_1(k))Z(k)\big)$$
$$\text{for } \ t \in I_{km} \ \ (k = 1,\dots,m).$$

Let X_0, $X_0(t_0) = I_n$, be the fundamental matrix of the system

$$\frac{dx}{dt} = P_0(t)\,x.$$

Here, the use will be made of the following formulas from [18]:

$$\int_a^b f(t)\,dg(t) + \int_a^b f(t)\,dg(t) = f(b)g(b) - f(a)g(a) + \sum_{a<t\le b} d_1 f(t)\cdot d_1 g(t) - \sum_{a\le t<b} d_2 f(t)\cdot d_2 g(t)$$
$$\text{(integration-by-parts formula)}, \tag{1.3}$$

$$\int_a^b f(t) ds_1(g)(t) = \sum_{a<t\le b} f(t)\, d_1 g(t), \int_a^b f(t) ds_2(g)(t) = \sum_{a\le t<b} f(t) d_2 g(t), \tag{1.4}$$

$$\int_a^b f(t)\, d\bigg(\int_a^t g(s)dh(s)\bigg) = \int_a^b f(t)g(t)\,dh(t), \tag{1.5}$$

$$d_j\bigg(\int_a^t f(s)\,dg(s)\bigg) = f(t)\,d_j g(t) \quad \text{for} \quad t\in[a,b] \ \ (j=1,2). \tag{1.6}$$

1.1. **Formulation of the Results on the Numerical Solvability of Problem** (1.1)**,** (1.2)**.** Without loss of generality, we assume that

$$G_{1m}(0) = G_{2m}(m) = O_{n\times n}, \ \ g_{1m}(0) = g_{2m}(m) = 0_n \ \ (m=1,2,\dots).$$

Definition 1.1. We say that a sequence $(G_{1m}, G_{2m}, g_{1m}, g_{2m}; k_m)$ $(m=1,2,\dots)$ belongs to the set $\mathcal{CS}(P_0, q_0; t_0)$ if for every $c_0\in\mathbb{R}^n$ and the sequence $\zeta_m\in\mathbb{R}^n$ $(m=1,2,\dots)$, satisfying the condition

$$\lim_{m\to+\infty} \zeta_m = c_0, \tag{1.7}$$

the difference problem (1.1m), (1.2m) has a unique solution $y_m\in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ for any sufficiently large m and

$$\lim_{m\to+\infty} \|y_m - p_m(x_0)\|_{\widetilde{N}_m} = 0. \tag{1.8}$$

We assume that

$$\lim_{m\to+\infty} t_m = t_0, \tag{1.9}$$

where $t_m = a + k_m(b-a)m^{-1}$.

Somewhere else we need the condition

$$\det\big(I_n + (-1)^j G_{jm}(k)\big) \neq 0 \ \ (j=1,2;\ k\in\mathbb{N}_m;\ m\in\mathbb{N}). \tag{1.10}$$

Theorem 1.1. *The inclusion*

$$\big((G_{1m}, G_{2m}, g_{1m}, g_{2m}; k_m)\big)_{m=1}^{+\infty} \in \mathcal{CS}(P_0, q_0; t_0) \tag{1.11}$$

holds if and only if there exists a sequence of matrix-functions $H_{jm}\in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ $(j=1,2;\ m=1,2,\dots)$ *such that*

$$\lim_{m\to+\infty} \max\{\|H_{jm}(k) - I_n\| : k\in\widetilde{\mathbb{N}}_m\} = 0 \quad (j=1,2), \tag{1.12}$$

and the conditions

$$\begin{aligned} &\lim_{m\to+\infty}\bigg\{\|P_m(H_{1m},H_{2m})(t) - P_0(t)\|\bigg(1 + \bigg|\bigvee_a^t (P_m(H_{1m},H_{2m}) - P_0)\bigg|\bigg)\bigg\} = 0,\\ &\lim_{m\to+\infty}\bigg\{\|q_m(H_{1m})(t) - q_0(t)\|\bigg(1 + \bigg|\bigvee_a^t (P_m(H_{1m},H_{2m}) - P_0)\bigg|\bigg)\bigg\} = 0 \end{aligned} \tag{1.13}$$

are fulfilled uniformly on I*, where* $P_m(H_{1m},H_{2m})(t) \equiv \mathcal{I}_m(H_{1m},H_{2m},G_{1m},G_{2m})(t)$ *and* $q_m(H_{1m})(t) \equiv \mathcal{B}_m(H_{1m}, g_{1m}, g_{2m})(t)$.

Condition (1.13) is known as the Opial type condition which has been obtained by Z. Opial in [15], where the author investigated the well-posed question of the initial problem for ordinary differential systems. The analogous conditions are obtained in [4] for generalized ordinary differential (GOD) systems.

Let Y_m, $Y_m(0) = I_n$, for each natural m, be the fundamental matrix of the difference system

$$\Delta y(k-1) = G_{1m}(k)\, y(k) + G_{2m}(k-1)\, y(k-1) \quad (k \in \mathbb{N}_m).$$

Theorem 1.2. *Let condition* (1.10) *hold. Then inclusion* (1.11) *holds if and only if conditions*

$$\lim_{m\to+\infty} \mathcal{B}_m(Y_m^{-1}, Q_{1m}, Q_{2m})(t) = X_0^{-1}(t),$$
$$\lim_{m\to+\infty} \mathcal{B}_m(Y_m^{-1}, g_{1m}, g_{2m})(t) = \mathcal{B}(X_0^{-1}, f)(t)$$

are fulfilled uniformly on I, where $Q_{jm}(i) \equiv \big(I_n + (-1)^j G_{jm}(i)\big)^{-1} G_{jm}(i)$ $(j = 1,2;\ m = 1,2,\dots)$.

Remark 1.1. If condition (1.10) holds, then

$$Y_m(k) = \prod_{i=k}^{1} \big(I_n - G_{1m}(i)\big)^{-1} \big(I_n + G_{2m}(i-1)\big) \quad (k \in \mathbb{N}_m;\ m = 1,2,\dots).$$

Theorem 1.3. *Let condition* (1.7) *hold and the conditions*

$$\lim_{m\to+\infty} \left\{ \|P_m(t) - P_0(t)\| \left(1 + \left|\bigvee_a^t (P_m - P_0)\right|\right)\right\} = 0,$$
$$\lim_{m\to+\infty} \left\{ \|q_m(t) - q_0(t)\| \left(1 + \left|\bigvee_a^t (P_m - P_0)\right|\right)\right\} = 0$$

be fulfilled uniformly on I, where $P_m(t) \equiv \mathcal{B}_m(I_n, G_{1m}, G_{2m})(t)$ and $q_m(t) \equiv \mathcal{B}_m(I_n, g_{1m}, g_{2m})(t)$. Then the difference problem (1.1m), (1.2m) *has a unique solution $y_m \in \mathrm{E}(\widetilde{N}_m; \mathbb{R}^n)$ for any sufficiently large m and* (1.8) *holds.*

Proposition 1.1. *Let the conditions of Theorem* 1.3 *hold. Then there exists a positive r such that*

$$\|y_m - p_m(x_0)\|_{\widetilde{N}_m} \le r(\|\gamma_m - c_0\| + \varepsilon_m + \delta_m) \quad (m = 1,2,\dots),$$

where y_m is the solution of problem (1.1m), (1.2m),

$$\lim_{m\to+\infty} \varepsilon_m = 0 \quad \textit{and} \quad \lim_{m\to+\infty} \delta_m = 0, \tag{1.14}$$

here,

$$\varepsilon_m = \alpha_m(2 + 3\rho_0 + 3\gamma_m), \quad \delta_m = \beta_m(2 + 2\alpha_m + \gamma_m) + 3\varrho_0\alpha_m, \quad \alpha_m = \|\mathcal{B}_m(I_n, G_{1m}, G_{2m}) - A\|_\infty,$$

$$\beta_m = \|\mathcal{B}_m(I_n, g_{1m}, g_{2m}) - f\|_\infty, \quad \gamma_m = \|V(\mathcal{B}_m(I_n, G_{1m}, G_{2m}) - A)\|_\infty, \quad \rho_0 = \bigvee_a^b (A), \quad \varrho_0 = \bigvee_a^b (f).$$

Theorem 1.4. *Let $P_0^* \in L(I; \mathbb{R}^{n\times n})$, $q_0^* \in L(I; \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$ and x_0^* be a unique solution of the initial problem*

$$\frac{dx}{dt} = P_0^*(t)\, x + q_0^*(t) \quad \textit{for} \quad t \in I,$$
$$x(t_0) = c_0^*.$$

Let, moreover, the sequences G_{jm}, $H_{jm} \in \mathrm{E}(\widetilde{N}_m; \mathbb{R}^n)$; $g_{jm} \in \mathrm{E}(\widetilde{N}_m; \mathbb{R}^n)$ $(j = 1,2;\ m = 1,2,\dots)$ be such that conditions (1.12) *and*

$$\lim_{m\to+\infty} c_m^* = c_0^*, \tag{1.15}$$

hold, and the conditions

$$\lim_{m\to+\infty} \left\{ \|P_m(H_{1m}, H_{2m})(t) - P_0^*(t)\| \left(1 + \left|\bigvee_a^t (P_m(H_{1m}, H_{2m}) - P_0^*)\right|\right)\right\} = 0, \tag{1.16}$$

$$\lim_{m\to+\infty}\left\{\|q_m(H_{1m})(t)-q_0^*(t)\|\left(1+\left|\bigvee_a^t(P_m(H_{1m},H_{2m})-P_0^*)\right|\right)\right\}=0$$

are fulfilled uniformly on I, where the matrix- and vector-functions $P_m(H_{1m},H_{2m})$ and $q_m(H_{1m})$ are defined as in Theorem 1.1, *and $c_m^*=H_{2m}(k_m)\zeta_m$. Then condition* (1.8) *holds, where y_m is a unique solution of the difference initial problem* (1.1m), (1.2m) *for any sufficiently large m.*

Corollary 1.1. *Let sequences $G_{jm}\in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$, $g_{jm}\in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$, $\zeta_m\in\mathbb{R}^n$ and $k_m\in\widetilde{\mathbb{N}}_m$ ($j=1,2$; $m=1,2,\dots$) be such that conditions* (1.12) *and*

$$\lim_{m\to+\infty}(\zeta_m-\psi_m(k_m))=c_0$$

hold, and conditions (1.16) *and*

$$\lim_{m\to+\infty}\left\{\|\mathcal{B}_m(H_{1m},g_{1m}-\psi_m,g_{2m}-\psi_m)(t)+\mathcal{I}_m(H_{1m},H_{2m}^{-1}\psi_m,G_{1m},G_{2m})(t)-f_0(t)\|\right.$$
$$\left.\times\left(1+\left|\bigvee_a^t(P_m(H_{1m},H_{2m})-P_0)\right|\right)\right\}=0$$

are fulfilled uniformly on I, where $H_{jm}\in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$, $\psi_m\in \mathrm{E}(\widetilde{N}_m;\mathbb{R}^n)$ ($j=1,2$; $m=1,2,\dots$) and $P_m(H_{1m},H_{2m})$ is defined as in Theorem 1.1. *Then the difference problem* (1.1m), (1.2m) *has a unique solution y_m for any sufficiently large m and*

$$\lim_{m\to+\infty}\|y_m-\psi_m-p_m(x_0)\|_{\widetilde{N}_m}=0.$$

2. The Well-Posedness of Initial Problem for Generalized Ordinary Differential Systems

The proofs of the results given in Section 1 are based on the following concept.

We rewrite the initial problem under consideration for ordinary and difference systems as the initial one for the following type of the so-called GOD system

$$dx=dA(t)\cdot x+df(t)\quad\text{for}\quad t\in I, \tag{2.1}$$

$$x(t_0)=c_0, \tag{2.2}$$

where $A\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $A(a)=O_{n\times n}$; $f\in BV(I;\mathbb{R}^n)$, $f(a)=0_n$.

The theory of GOD equations has been introduced by J. Kurzweil in [12], where he investigated the well-posedness of the initial problem for ordinary differential equations. Some questions of the theory have been investigated in [1–6, 9, 11, 12, 14, 17, 18] (see also references therein).

A vector-function $x\in \mathrm{BV}(I;\mathbb{R}^n)$ is said to be a solution of system (2.1) if

$$x(t)-x(s)=\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\quad\text{for}\quad s<t;\quad s,t\in I.$$

It is evident that system (1.1) is equivalent to (2.1), where

$$A(t)\equiv\int_a^t P_0(s)ds\quad\text{and}\quad f(t)\equiv\int_a^t q_0(s)ds.$$

So, in this case, A and f are the continuous matrix- and vector-functions.

Let a vector-function $x_0\in \mathrm{BV}(I;\mathbb{R}^n)$ be the solution of problem (2.1), (2.2) and let X_0, $X_0(t_0)=I_n$, be the fundamental matrix of the system

$$dx=dA(t)\cdot x.$$

Along with the Cauchy problem (2.1), (2.2), we consider the sequence of the Cauchy problems

$$dx(t)=dA_m(t)\cdot x(t)+df_m(t) \tag{2.1m}$$

$$x(t_m)=c_m \tag{2.2m}$$

$(m = 1, 2, \dots)$, where $A_m \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_m \in BV(I; \mathbb{R}^n)$, $t_m \in I$ and $c_m \in \mathbb{R}^n$.

Concerning system (2.1), if

$$\det\big(I_n + (-1)^j d_j A(t)\big) \neq 0 \quad \text{for} \quad t \in I \quad (j = 1, 2), \tag{2.3}$$

then problem (2.1), (2.2) has a unique solution (see [14, 17, 18]).

We assume that $A_0(t) \equiv A(t)$, $f_0(t) \equiv f(t)$ and condition (1.9) holds.

We rewrite the discrete problems (1.1m), (1.2m) ($m \in \mathbb{N}$) as the initial problems for GOD systems of type (1.1). So, the discrete systems (1.1m) ($m \in \mathbb{N}$) are, really, the generalized systems. Therefore the convergence of the difference scheme (1.1m), (1.2m) ($m \in \mathbb{N}$) to the solution of problem (1.1), (1.2) is equivalent to the well-possed question of the initial problems for the GOD systems.

We give some results from [4], concerning the well-posedness of problem (2.1), (2.2), where the necessary and sufficient, as well as the efficient sufficient conditions are established for the Cauchy problem (2.1m), (2.2m) to have a unique solution x_m for every sufficiently large m and

$$\lim_{m\to+\infty} x_m(t) = x_0(t) \quad \text{uniformly on} \quad I. \tag{2.4}$$

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate linear ordinary differential, impulsive and difference equations from a unified point of view.

Along with systems (2.2m) $(m = 1, 2, \dots)$, we consider the corresponding homogeneous systems

$$dx = dA_m(t) \cdot x. \tag{2.1mo}$$

Definition 2.1. We say that the sequence $(A_m, f_m; t_m)$ $(m = 1, 2, \dots)$ belongs to the set $\mathcal{S}(A_0, f_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$ such that

$$\lim_{m\to+\infty} c_m = c_0, \tag{2.5}$$

problem (2.1m), (2.2m) has a unique solution x_m for any sufficiently large m and condition (2.4) holds.

Theorem 2.1. *The inclusion*

$$((A_m, f_m; t_m))_{m=1}^{+\infty} \in \mathcal{S}(A_0, f_0; t_0) \tag{2.6}$$

holds if and only if there exists a sequence of matrix-functions $H_m \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$ such that the conditions

$$\lim_{m\to+\infty} H_m = I_n, \tag{2.7}$$

$$\lim_{m\to+\infty} \Big\{ \big\|\mathcal{I}(H_m, A_m)(t) - A_0(t)\big\| \Big(1 + \bigvee_a^t (\mathcal{I}(H_m, A_m) - A_0)\Big) \Big\} = 0, \tag{2.8}$$

$$\lim_{m\to+\infty} \Big\{ \big\|\mathcal{B}(H_m, f_m)(t) - f_0(t)\big\| \Big(1 + \bigvee_a^t (\mathcal{I}(H_m, A_m) - A_0)\Big) \Big\} = 0 \tag{2.9}$$

are fulfilled uniformly on I.

Theorem 2.2. *Let condition* (2.3) *hold. Then inclusion* (2.6) *holds if and only if the conditions*

$$\lim_{m\to+\infty} X_m^{-1}(t) = X_0^{-1}(t),$$
$$\lim_{m\to+\infty} \mathcal{B}(X_m^{-1}, f_m)(t) = \mathcal{B}(X_0^{-1}, f)(t)$$

are fulfilled uniformly on I, where X_m is the fundamental matrix of system (2.1mo) *for any m.*

Theorem 2.3. *Let $A_0 \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I; \mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$ and let the sequences of matrix- and vector-functions $A_m \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$ and $f_m \in \mathrm{BV}(I; \mathbb{R}^n)$ $(m = 1, 2, \dots)$ and the sequence of constant vectors $c_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$ be such that condition* (2.5) *holds and the conditions*

$$\lim_{m\to+\infty} \Big\{ \|A_m(t) - A_0(t)\| \Big(1 + \Big|\bigvee_a^t (A_m - A_0)\Big|\Big) \Big\} = 0, \tag{2.10}$$

$$\lim_{m\to+\infty}\left\{\|f(t)-f_0\|\Big(1+\Big|\bigvee_a^t(A_m-A_0)\Big|\Big)\right\}=0 \tag{2.11}$$

are fulfilled uniformly on I. Then the initial problem (2.1m), (2.2m) *has the unique solution x_m for any sufficiently large m and* (2.4) *holds.*

Proposition 2.1. *Let the conditions of Theorem* 2.3 *be satisfied. Then there exists a positive number r such that*

$$\|x_m-x_0\|_\infty\le r(\|c_m-c_0\|+\varepsilon_m+\delta_m)\quad (m=1,2,\dots) \tag{2.12}$$

and condition (1.14) *holds, where x_m is the solution of problem* (2.1m), (2.2m),

$$\varepsilon_m=\alpha_m(2+3\rho_0+3\gamma_m),\quad \delta_m=\beta_m(2+2\alpha_m+\gamma_m)+3\varrho_0\alpha_m,$$

$$\rho_0=\bigvee_a^b(A),\quad \varrho_0=\bigvee_a^b(f),\quad \alpha_m=\|A_m-A\|_\infty,\quad \beta_m=\|f_m-f\|_\infty,\quad \gamma_m=\sup_{t\in[a,b]}\bigvee_a^t(A_m-A).$$

Theorem 2.4. *Let $A_0^*\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ and $f_0^*\in BV(I;\mathbb{R}^n)$ be continuous and let $c_0^*\in\mathbb{R}^n$ be such that the problem*

$$dx=dA_0^*(t)\cdot x+df_0^*(t), \tag{2.13}$$

$$x(t_0)=c_0^* \tag{2.14}$$

has a unique solution x_0^*. Let, moreover, there exist the sequences $H_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, f_m, $h_m\in BV(I;\mathbb{R}^n)$ and $c_m\in\mathbb{R}^n$ $(m=1,2,\dots)$ such that condition (1.15) *holds and the conditions* (2.7),

$$\lim_{m\to+\infty}\left\{\|A_m^*(t)-A_0^*(t)\|\left(1+\left|\bigvee_a^t(A_m^*-A_0^*)\right|\right)\right\}=0 \tag{2.15}$$

and $$\lim_{m\to+\infty}\left\{\|f_m^*(t)-f_0^*\|\left(1+\left|\bigvee_a^t(A_m^*-A_0^*)\right|\right)\right\}=0 \tag{2.16}$$

are fulfilled uniformly on I, where

$$A_m^*(t)=\mathcal{I}(H_m,A_m)(t),\quad f_m^*(t)=h_m(t)-h_m(a)+\mathcal{B}(H_m,f_m)(t)-\int_a^t dA_m^*(s)\cdot h_m(s),$$

$$c_m^*=H_m(t_m)\,c_m+h_m(t_m)\quad (m=1,2,\dots).$$

Then problem (2.1m), (2.2m) *has a unique solution x_m for any sufficiently large m and*

$$\lim_{m\to+\infty}\|H_m(t)\,x_m(t)+h_m(t)-x_0^*(t)\|=0\quad \textit{uniformly on}\quad I. \tag{2.17}$$

Remark 2.1. In Theorem 2.4, the vector-function $x_m^*(t)=H_m(t)\,x_m(t)+h_m(t)$ is a solution of the problem

$$dx=dA_m^*(t)\cdot x+df_m^*(t), \tag{2.13m}$$

$$x(t_m)=c_m^* \tag{2.14m}$$

for every sufficiently large m.

Corollary 2.1. *Let $A_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_m\in BV(I;\mathbb{R}^n)$, $c_m\in\mathbb{R}^n$ and $t_m\in I$ $(m=0,1,\dots)$ be such that*

$$\lim_{k\to+\infty}(c_m-\varphi_m(t_m))=c_0, \tag{2.18}$$

and conditions (2.7), (2.8) *and*

$$\lim_{k\to+\infty}\left\{\left\|\mathcal{I}(H_m,f_m-\varphi_m)(t)-f_0(t)+\int_a^t d\mathcal{I}(H_m,A_m)(\tau)\cdot\varphi_m(\tau)\right\|\right.$$

$$\times\left(1+\left|\bigvee_a^t(\mathcal{I}(H_m,A_m)-A_0)\right|\right)\Bigg\}=0 \tag{2.19}$$

are fulfilled uniformly on I, where $H_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ and $\varphi_m\in\mathrm{BV}(I;\mathbb{R}^n)$ $(m=0,1,\dots)$. Then problem (2.1m), (2.2m) *has a unique solution x_m for any sufficiently large m and*

$$\lim_{m\to+\infty}(x_m(t)-\varphi_m(t))=x_0(t)\quad \textit{uniformly on}\quad I. \tag{2.20}$$

2.1. **Proofs of the results concerning the well-posedness of the generalized initial problem** (2.1)**,** (2.2)**.** For the completeness, we present here the proofs of the given results in brief (the full version can be found in [1,2,4]).

Proof of Theorem 2.3. By (2.10),

$$\lim_{m\to+\infty}\|A_m-A_0\|_\infty=0$$

and, therefore,

$$\lim_{m\to+\infty}d_jA_m(t)=0_n\quad\text{uniformly on}\quad I\quad(j=1,2)$$

So, according to Lemma 1.2.6 from [4], there exists a positive number r_0 such that

$$\det\big(I_n+(-1)^jd_jA_m(t)\big)\neq0\quad\text{for}\quad t\in I\quad(j=1,2)$$

and

$$\left\|\big(I_n+(-1)^jd_jA_m(t)\big)^{-1}\right\|\le r_0\quad\text{for}\quad t\in I\quad(j=1,2)$$

for every sufficiently large m.

Therefore, there exists a natural number m_0 such that problem (2.1m), (2.2m) has a unique solution x_m for every m, without loss of generality.

Let $z_m(t)\equiv x_m(t)-x_0(t)$ for every m.

Let ε be an arbitrarily small positive number.

It is not difficult to check that

$$z_m(t)=z_m(t_m)+\int_{t_m}^t dA_0(s)\cdot z_m(s)+\int_{t_m}^t d\overline{A}_m(s)\cdot x_m(s)+\overline{f}_m(t)-\overline{f}_m(t_m)\quad\text{for}\quad t\in I,$$

where

$$\overline{A}_m(t)=A_m(t)-A_0(t),\quad \overline{f}_m(t)=f_m(t)-f_0(t)\quad(m=0,1,\dots).$$

Using (1.6), we find

$$d_jx_m(t)=d_jA_m(t)\cdot x_m(t)+d_jf(t)\quad\text{for}\quad t\in I\quad(j=1,2).$$

Consequently, by the integration-by-parts formula (1.3), we conclude that

$$\int_{t_m}^t d\overline{A}_m(s)\cdot x_m(s)=\overline{A}_m(t)\,x_m(t)-\overline{A}_m(t_m)\,x_m(t_m)$$

$$-\int_{t_m}^t\overline{A}_m(s)dx_m(s)+\sum_{t_m<s\le t}d_1\overline{A}_m(s)\cdot d_1x_m(s)-\sum_{t_m\le s<t}d_2\overline{A}_m(s)\cdot d_2x_m(s)$$

$$=\overline{A}_m(t)\,x_m(t)-\overline{A}_m(t_m)\,x_m(t_m)-\int_{t_m}^t\overline{A}_m(s)\big(dA_m(s)\cdot x_m(s)+df_m(s)\big)$$

$$+\sum_{t_m<s\le t}d_1\overline{A}_m(s)\cdot\big(d_1A_m(s)\cdot x_m(s)+d_1f_m(s)\big)$$

$$-\sum_{t_m\le s<t}d_2\overline{A}_m(s)\cdot\big(d_2A_m(s)\cdot x_m(s)+d_2f_m(s)\big)\quad\text{for}\quad t\in I.$$

So,

$$z_m(t) = z_m(t_m) + \mathcal{J}_m(t,t_m) + \mathcal{Q}_m(t,t_m) + \int_{t_m}^{t} dA_0(s)\cdot z_m(s) \quad \text{for} \quad t\in I. \tag{2.21}$$

where

$$\mathcal{J}_m(t,\tau) = \overline{A}_m(t)\cdot x_m(t) - \overline{A}_{mj}(\tau)\cdot x_m(\tau) - \int_{\tau}^{t} \overline{A}_m(s)dA_m(s)\cdot x_m(s)$$

$$+\sum_{s\in]\tau,t]} d_1\overline{A}_m(s)\cdot d_1A_m(s)\cdot x_m(s) - \sum_{s\in[\tau,t[} d_2\overline{A}_m(s)\cdot d_2A_m(s)\cdot x_m(s) \quad \text{for} \quad \tau<t \quad (j=1,2),$$

$$\mathcal{J}_{mj}(t,t)\equiv 0 \quad (j=1,2) \quad \text{and} \quad \mathcal{J}_m(t,\tau) = -\mathcal{J}_m(\tau,t) \quad \text{for} \quad t<\tau \quad (j=1,2),$$

and

$$\mathcal{Q}_m(t,\tau)\equiv \overline{f}_m(t) - \overline{f}_m(\tau) - \mathcal{B}(\overline{A}_m,f_m)(t) + \mathcal{B}(\overline{A}_m,f_m)(\tau) \quad (j=1,2).$$

From (2.21), it follows that

$$\|z_m(t)\| \le \|z_m(t_m)\| + \|\mathcal{J}_m(t,t_m)\| + \|\mathcal{Q}_m(t,t_m)\| + \left|\int_{t_m}^{t} \|z_m(\tau)\|\, d\|V(A_0)(\tau)\|\right| \quad \text{for} \quad t\in I. \tag{2.22}$$

Further, let $\alpha_m,\beta_m,\gamma_m,\delta_m,\varepsilon_m$ $(m=1,2,\dots)$ and ρ_0,ϱ_0 be defined as in Proposition 2.1.

In view of the conditions $A_0\in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0\in \mathrm{BV}(I;\mathbb{R}^n)$, (2.10) and (2.11), we have

$$\lim_{k\to+\infty} \alpha_m(1+\gamma_m) = \lim_{m\to+\infty} \beta_m(1+\varrho_0+\gamma_m) = 0. \tag{2.23}$$

Moreover, by the inequalities

$$\left|\bigvee_{t_m}^{t}(A_m)\right| \le \left|\bigvee_{t_m}^{t}(A_m-A_0)\right| + \left|\bigvee_{t_m}^{t}(A_0)\right| \quad \text{for} \quad t\in I \quad (m=1,2,\dots),$$

we find

$$\|\mathcal{J}_m(t,t_m)\| \le 2\alpha_m\|x_m\|_m + \alpha_m(\gamma_m+\rho_0)\|x_m\|_m$$
$$+2\alpha_m\|x_m\|_m\Bigg(\sum_{t_m<s\le t}\Big(\big\|d_1(A_m(s)-A_0(s))\big\| + \|d_1A_0(s)\|\Big)$$
$$+\sum_{t_m\le s<t}\Big(\big\|d_2(A_m(s)-A_0(s))\big\| + \|d_2A_0(s)\|\Big)\Bigg)$$

and therefore,

$$\big\|\mathcal{J}_m(t,t_m)\big\| \le \varepsilon_m\|x_m\|_\infty \quad \text{for} \quad t\in I, \tag{2.24}$$

where $\varepsilon_m = \alpha_m(2+3\rho_0+3\gamma_m)$ $(m=1,2,\dots)$. In addition, if we take into account the fact that the operator $\mathcal{B}$ is linear with respect to every its variable and equals zero if the second variable is a constant function, then we conclude that

$$\big\|\mathcal{B}(\overline{A}_m,f_m)(t) - \mathcal{B}(\overline{A}_m,f_m)(t_m)\big\|$$
$$\le \big\|\mathcal{B}(\overline{A}_m,\overline{f}_m)(t) - \mathcal{B}(\overline{A}_m,\overline{f}_m)(t_m)\big\| + \big\|\mathcal{B}(\overline{A}_m,f_0)(t) - \mathcal{B}(\overline{A}_m,f_0)(t_k+\varepsilon)\big\| \quad \text{for} \quad t\in I.$$

By the definition of the operator $\mathcal{B}$, we have

$$\big\|\mathcal{B}(\overline{A}_m,\overline{f}_m)(t) - \mathcal{B}(\overline{A}_m,\overline{f}_m)(t_m)\big\| \le \beta_m(2\alpha_m+\gamma_m) \quad \text{for} \quad t\in I.$$

Using the integration-by-parts formula, we find

$$\big\|\mathcal{B}(\overline{A}_m,f_0)(t) - \mathcal{B}(\overline{A}_m,f_0)(t_m)\big\|$$
$$\le \alpha_m\bigvee_{t_m}^{t}(f_0) + 2\alpha_m\Bigg(\sum_{t_m<s\le t}\|d_1f_0(s)\| + \sum_{t_m\le s<t}\|d_2f_0(s)\|\Bigg) \quad \text{for} \quad t\in I$$

and therefore,
$$\big\|\mathcal{B}(\overline{A}_m, f_0)(t) - \mathcal{B}(\overline{A}_m, f_0)(t_m)\big\| \le 3\varrho_0\alpha_m \quad \text{for} \quad t \in I.$$
So,
$$\|\mathcal{Q}_m(t, t_m)\| \le \delta_m \quad \text{for} \quad t \in I, \tag{2.25}$$
where $\delta_m = \beta_m(2 + 2\alpha_m + \gamma_m) + 3\varrho_0\alpha_m$.

From (2.22), by (2.24) and (2.25), we find
$$\|z_m(t)\| \le r_1\bigg(\|z_m(t_m)\| + \varepsilon_m\|x_m\|_\infty + \delta_m + \int\limits_{t_m}^{t} \|z_m(\tau)\|\, d\|V(A_0)(\tau)\|\bigg) \quad \text{for} \quad t \in I.$$

Hence, according to Gronwall's inequality (see [18, Theorem I.4.30]),
$$\begin{aligned}\|z_m(t)\| &\le r_1\big(\|z_m(t_m)\| + \varepsilon_m\|x_m\|_\infty + \delta_m\big) \exp\big(r_1\|V(A_0)(t) - V(A_0)(t_m)\|\big)\\ &\le r_1\big(\|z_m(t_m)\| + \varepsilon_m\|x_m\|_\infty + \delta_m\big) \exp(\rho_0 r_1) \quad \text{for} \quad t \in I.\end{aligned}$$

Now, passing to the limit as $\varepsilon \to 0$ in the last inequality, we conclude that
$$\|z_m\|_\infty \le r_1\Big(\|z_m(t_m)\| + \varepsilon_m\|x_m\|_\infty + \delta_m\Big) \exp(\rho_0 r_1). \tag{2.26}$$

Due to (2.23), we have
$$\lim_{m\to+\infty} \varepsilon_m = 0. \tag{2.27}$$
Hence there exists a natural m_1 such that
$$r_1\varepsilon_m \exp(\rho_0 r_1) < \frac{1}{2} \quad \text{for} \quad m > m_1,$$
whence, owing to (2.26), it follows that
$$\|x_m\|_\infty \le \|x_0\|_\infty + \|z_m\|_\infty \le \|x_0\|_\infty + \frac{1}{2}\,\|x_m\|_\infty + r_1\big(\|z_m(t_m)\| + \delta_m\big) \exp(\rho_0 r_1).$$
Therefore
$$\|x_m\|_\infty \le \Big(\|x_0\|_\infty + r_1\big(\|z_m(t_m)\| + \delta_m\big) \exp(\rho_0 r_1)\Big)$$
for $m > m_1$, which, due to (2.5), implies that the sequence $\|x_m\|_\infty$ $(m = 1, 2, \dots)$ is bounded.

In view of conditions (2.10) and (2.11),
$$\lim_{m\to+\infty} \delta_m = 0. \tag{2.28}$$
On the other hand, using (2.5), (2.27) and (2.28), it follows from (2.26) that
$$\lim_{m\to+\infty} \|z_m\|_\infty = 0.$$
The theorem is proved. □

Proof of Proposition 2.1. Estimate (2.12) immediately follows from estimate (2.26). In addition, by (2.27) and (2.28), condition (1.14) holds. The proposition is proved. □

Proof of Theorem 2.4. Analogously to the proof of Theorem 2.3, we show that the initial problem (2.13m), (2.14m) has the unique solution x^*_m for every sufficiently large m. Moreover, according to Lemma 1.2.2 from [4], the mapping $x \to x^*$, $x^* = H_m\, x + h_m$, ensures a one-to-one correspondence between the solutions of problem (2.1m), (2.2m) and those of the initial problem (2.13m), (2.14m) for every such m. Thus problem (2.1m), (2.2m) has the unique solution x_m and
$$x^*_m(t) \equiv H_m(t)x_m + h_m(t),$$
for every sufficiently large m.

Conditions (2.15), (2.16) guarantee the fulfilment of the conditions of Theorem 2.3 for the initial problem (2.13), (2.14) and for the sequence of the initial problems (2.13m), (2.14m) $(m = 1, 2, \dots)$. Thus, owing to Theorem 2.3,
$$\lim_{m\to+\infty} x^*_m(t) = x^*_0(t) \quad \text{uniformly on} \quad I.$$

So, condition (2.17) holds. The theorem is proved. □

Remark 2.1 immediately follows from the proof of Theorem 2.4.

Proof of Corollary 2.1. Verify the conditions of Theorem 2.4. From (2.7), it follows that

$$\lim_{m\to+\infty} H_m^{-1}(t) = I_n \tag{2.29}$$

holds uniformly on I.

Put

$$h_m(t) \equiv -H_m(t)\varphi_m(t) \quad (m=1,2,\dots).$$

Due to (2.7), we get

$$\lim_{m\to+\infty} H_m(t_m) = I_n,$$

and in view of the above and by (2.18), condition (1.15) is fulfilled for $c_0^* = c_0$.

Moreover, by (2.8) and (2.19), conditions (2.15) and (2.16) hold uniformly on I, where

$$h_m(t) \equiv -H_m(t)\varphi_m(t), \quad A_m^*(t) \equiv \mathcal{I}(H_m,A_m)(t) - \mathcal{I}(H_m,A_m)(t_m) \quad (m=0,1,\dots);$$

$$f_0^*(t) \equiv f_0(t) - f_0(t_0),$$

$$f_m^*(t) \equiv \mathcal{B}(H_m, f_m-\varphi_m)(t) - \mathcal{B}(H_m, f_m-\varphi)(t_m) + \int_{t_m}^{t} d\mathcal{I}(H_m,A_m)(s)\cdot\varphi_m(s) \quad (m=1,2,\dots).$$

Owing to the described above Lemma 1.2.2, from [4], it is evident that problem (2.13), (2.14) has the unique solution $x_0^*(t) \equiv x_0(t)$.

By Theorem 2.4 and Remark 2.1, we have

$$\lim_{m\to+\infty} \|H_m(t)x_m(t) - H_m(t)\varphi_m(t) - x_0^*(t)\| = 0$$

uniformly on I. Therefore, owing to (2.7) and (2.29), condition (2.20) holds uniformly on I. □

Proof of Theorem 2.1. The sufficiency follows from Corollary 2.1 if we assume $\varphi_m(t) \equiv 0$ ($m = 1,2,\dots$) therein.

Let us show the necessity. Let inclusion (2.6) hold and $c_m \in \mathbb{R}^n$ ($m=0,1,\dots$) be an arbitrary sequence of constant vectors satisfying condition (2.5).

In view of (2.6), we may assume that problem (2.1m), (2.2m) has a unique solution x_m for every natural m, without loss of generality.

For any $m \in \widetilde{\mathbb{N}}$ and $j \in \{1,\dots,n\}$, let us denote $z_{mj}(t) \equiv x_m(t) - x_{mj}(t)$, where x_{mj} is the unique solution of system (2.1m) under the initial condition $x(t_m) = c_m - e_j$; here, $e_j = (\delta_{ij})_{i=1}^n$, and δ_{ij} is the Kronecker symbol. Moreover, let $X_m(t)$ be the matrix-function with the columns $z_{m1}(t),\dots,z_{mn}(t)$.

If $\sum\limits_{j=1}^{n} \alpha_j z_{mj}(t) \equiv 0$ for some $m \in \widetilde{\mathbb{N}}$ and $\alpha_1,\dots,\alpha_n \in \mathbb{R}$, then by the equalities $z_{mj}(t_m) = e_j$ ($j=1,\dots,n$; $m=0,1,\dots$), we have $\sum\limits_{j=1}^{n} \alpha_j e_j = 0$ and, therefore, $\alpha_1 = \dots = \alpha_n = 0$, i.e., X_m ($X_0(t) \equiv X(t)$) is the fundamental matrix of the homogeneous system (2.1mo).

We may assume without loss of generality that $X_m(a) = I_n$ ($m \in \widetilde{\mathbb{N}}$).

Further, due to (2.6), we have

$$\lim_{m\to+\infty} \|X_m - X_0\|_\infty = 0,$$

which in view of Lemma 2 in [1] implies that

$$\lim_{m\to+\infty} \|X_m^{-1} - X_0^{-1}\|_\infty = 0.$$

Put $H_m(t) \equiv X_0(t)\,X_m^{-1}(t)$ ($m \in \widetilde{\mathbb{N}}$) and verify conditions (2.7), (2.8), (2.9) of the theorem. Due to the last equality, condition (2.7) holds uniformly on I.

By the general equality $\mathcal{B}(GH,B)(t) \equiv \mathcal{B}(G,\mathcal{B}(H,B)(t)$ (see Lemma 2.1 in [2]) and the equality

$$X_m^{-1}(t) = X_m^{-1}(s) - \mathcal{B}(X_m^{-1},A_m)(t) + \mathcal{B}(X_m^{-1},A_m)(s) \quad \text{for} \quad t,s \in I$$

$(m=0,1,\dots)$ (see Proposition 1.1.4 from [4]), we have

$$H_m(t)+\mathcal{B}(H_m,A_m)(t)=X_0(t)X_m^{-1}(t)+\mathcal{B}(X_0,\mathcal{B}(X_m^{-1},A_m))(t)$$
$$=X_0(t)X_m^{-1}(t)+\mathcal{B}(X_0,I_n-X_m^{-1})(t)=\int\limits_a^t dX_0(s)\cdot X_m^{-1}(s)\quad\text{for}\quad t\in I\quad(m\in\widetilde{\mathbb{N}}).$$

Hence, due to (1.5),

$$\mathcal{I}(H_m,A_m)(t)=\int\limits_a^t dX_0(s)\cdot X_m^{-1}(s)\,H_m^{-1}(s)=\int\limits_a^t dX_0(s)\cdot X_0^{-1}(s)$$
$$=\int\limits_a^t dA_0(s)\cdot X_0(s)\,X_0^{-1}(s)=A_0(t)\quad\text{for}\quad t\in I\quad(m\in\widetilde{\mathbb{N}}).$$

So, condition (2.8) is fulfilled uniformly on I.

On the other hand, by (2.1), the described above Lemma 2.1 of [2] and the definition of the solutions of system (2.1), we have

$$\mathcal{B}(H_m,f_m)(t)=\mathcal{B}(H_m,x_m)(t)-\mathcal{B}\Big(H_m,\int\limits_a^{\cdot} dA_m(s)\cdot x_m(s)\Big)(t)$$
$$=\mathcal{B}(H_m,x_m)(t)-\mathcal{B}(H_m,x_m)(a)-\int\limits_a^t d\mathcal{B}(H_m,A_m)(s)\cdot x_m(s)\quad\text{for}\quad t\in I\quad(m=0,1,\dots),$$

which yields

$$\mathcal{B}(H_m,f_m)(t)\equiv H_m(t)x_m(t)-H_m(a)x_m(a)-\int\limits_a^t dX_0(s)\cdot X_0(s)\,X_m^{-1}(s)\,x_m(s)\quad(m=0,1,\dots).$$

By this, if we take into account the fact that due to the necessity of the theorem condition (2.4) holds, we conclude that condition (2.9) holds uniformly on I, as well. The theorem is proved. □

Proof of Theorem 2.2. The theorem follows due to the proof of the necessity of Theorem 2.1. □

3. Auxiliary Propositions and Proofs of the Main Results

Consider now the difference problem (1.1m), (1.2m), where $m\in\mathbb{N}$.

Let the matrix-function $A_m\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$ and the vector-function $f_m\in\mathrm{BV}(I;\mathbb{R}^n)$ be defined, respectively, by the equalities

$$A_m(a)=A_m(\tau_{0m})=O_{n\times n},\quad A_m(\tau_{km})=\sum_{i=0}^k G_{1m}(i)+\sum_{i=1}^k G_{2m}(i-1),$$
$$A_m(t)=\sum_{i=0}^{k-1} G_{1m}(i)+\sum_{i=1}^k G_{2m}(i-1)\quad\text{for}\quad t\in I_{km}\quad(k\in\mathbb{N}_m);\tag{3.1}$$

$$f_m(a)=f(\tau_{0m})=0_n,\quad f_m(\tau_{km})=\sum_{i=0}^k g_{1m}(i)+\sum_{i=1}^k g_{2m}(i-1),$$
$$f_m(t)=\sum_{i=0}^{k-1} g_{1m}(i)+\sum_{i=1}^k g_{2m}(i-1)\quad\text{for}\quad t\in I_{km}\quad(k\in\mathbb{N}_m);\tag{3.2}$$

$$t_m=a+k_m\,\tau_m,\quad c_m=\gamma_m\quad(m=1,2,\dots).\tag{3.3}$$

Let

$$G_m(i) \equiv G_{1m}(i) + G_{2m}(i), \quad g_m(i) \equiv g_{1m}(i) + g_{2m}(i) \quad (m = 1, 2, \dots).$$

It is not difficult to verify that by (3.1)–(3.3) the defined matrix- and vector-functions A_m and f_m $(m = 1, 2, \dots)$ have the following properties:

$$\begin{aligned}
&d_1 A_m(\tau_{k\,m}) = G_{1m}(k), \quad d_2 A_m(\tau_{km}) = G_{2m}(k) \quad (k = 1, \dots, m),\\
&A_m(\tau_{k\,m}-) = \sum_{i=0}^{k-1} G_m(i), \quad A_m(\tau_{k\,m}+) = \sum_{i=0}^{k} G_m(i) \quad (k = 1, \dots, m),\\
&d_j A_m(t) = O_{n\times n} \quad \text{for} \quad t \in I \setminus \{\tau_{1m}, \dots, \tau_{km}\} \quad (j = 1, 2), \quad S_0(A_m)(t) \equiv O_{n\times n};
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&d_1 f_m(\tau_{k\,m}) = g_{1m}(k), \quad d_2 f_m(\tau_{km}) = g_{2m}(k) \quad (k = 1, \dots, m),\\
&f_m(\tau_{k\,m}-) = \sum_{i=0}^{k-1} g_m(i), \quad f_m(\tau_{k\,m}+) = \sum_{i=0}^{k} g_m(i) \quad (k = 1, \dots, m),\\
&d_j f_m(t) = 0_n \quad \text{for} \quad t \in I \setminus \{\tau_{1m}, \dots, \tau_{km}\} \quad (j = 1, 2), \quad S_0(f_m)(t) \equiv 0_n.
\end{aligned} \tag{3.5}$$

Moreover,

$$A_m(t) \equiv \mathcal{B}_m(I_n, G_{1m}, G_{2m})(t) \quad \text{and} \quad f_m(t) \equiv \mathcal{B}_m(I_n, g_{1m}, g_{2m})(t) \quad (m = 1, 2, \dots). \tag{3.6}$$

Lemma 3.1. *Let m be fixed. Then the discrete vector-function $y \in E(\widetilde{N}_m; \mathbb{R}^n)$ is a solution of problem* (1.1m), (1.2m) *if and only if the vector-function $x = q_m(y) \in \mathrm{BV}(I; \mathbb{R}^n)$ is a solution of the generalized problem* (2.1), (2.2), *where the matrix-A_m and the vector-f_m functions are defined by* (3.1) *and* (3.2), *respectively, and t_m and c_m are defined by* (3.3).

The lemma is proved in [4, 6].

Remark 3.1. Due to this lemma, under condition (1.9), the convergence of the difference scheme (1.1m), (1.2m) $(m = 1, 2, \dots)$ is equivalent to the well-possed question for the corresponding initial problem (2.1), (2.2).

So, in view of Definitions 1.1 and 2.1, the following lemma is true.

Lemma 3.2. *Inclusion* (1.11) *holds if and only if inclusion* (2.6) *holds, where the matrix-functions A, A_m, the vector-functions f, f_m, the points t_m and the constant vectors c_n $(m = 1, 2, \dots)$ are defined by* (3.1)–(3.3).

Remark 3.2. In view of (3.1) and (3.2), we have $A_m(t) = \text{const}$ and $f_m(t) = \text{const}$ for $t \in I_{k\,m}$ $(k = 1, \dots, m;\ m = 1, 2, \dots)$, i.e., they are the break matrix- and vector-functions. Therefore all the solutions of systems (2.1m) $(m = 1, 2, \dots)$ have the same property.

In order to use Theorems 2.1–2.4, we have to establish the forms of operators applied to the results for a particular case which correspond to the matrix- and vector- functions defined by (3.1)–(3.3).

Let H_m $(m \in \mathbb{N})$ be the matrix-functions appearing in Theorem 2.1. It follows from the proof of this theorem that the matrix-functions H_m $(m \in \mathbb{N})$ appearing in the proof have the property, analogous to the matrix-functions A_m $(m \in \mathbb{N})$. In particular, we may assume that $H_m(t) = I_n$ for $t \in I_{k\,m}$ $(k \in \mathbb{N}_m,\ m \in \mathbb{N})$. So, we have

$$H_m(\tau_{k-1\,m}+) = H_m(\tau_{km}-) \quad (k \in \mathbb{N}_m,\ m \in \mathbb{N}). \tag{3.7}$$

By the definition of the operator $\mathcal{B}$, integration-by-parts formula (1.3) and equalities (1.4), we have

$$\begin{aligned}
\mathcal{B}(H_m, A_m)(t) &= \int_a^t H_m(\tau) dA_m(\tau) - \sum_{a<\tau\le t} d_1 H_m(\tau) \cdot d_1 A_m(\tau) + \sum_{a\le\tau<t} d_2 H_m(\tau) \cdot d_2 A_m(\tau)\\
&= \sum_{a<\tau\le t} H_m(\tau)\, d_1 A_m(\tau) + \sum_{a\le\tau<t} H_m(\tau)\, d_2 A_m(\tau) - \sum_{a<\tau\le t} d_1 H_m(\tau) \cdot d_1 A_m(\tau)
\end{aligned}$$

$$+\sum_{a\le\tau<t} d_2H_m(\tau)\cdot d_2A_m(\tau) \quad \text{for} \quad t\in I \quad (m\in\mathbb{N}).$$

Therefore

$$\mathcal{B}(H_m,A_m)(t)\equiv\sum_{a<\tau_{im}\le t} H_m(\tau_{im}-)\,d_1A_m(\tau_{im})+\sum_{a\le\tau_{im}<t} H_m(\tau_{im}+)\,d_2A_m(\tau_{im}) \quad e(m\in\mathbb{N}). \tag{3.8}$$

Analogously, we show that

$$\mathcal{B}(H_m,f_m)(t)\equiv\sum_{a<\tau_{im}\le t} H_m(\tau_{im}-)\,d_1f_m(\tau_{im})+\sum_{a\le\tau_{im}<t} H_m(\tau_{im}+)\,d_2f_m(\tau_{im}) \quad (m\in\mathbb{N}). \tag{3.9}$$

Let

$$H_{1m}(k)=H_m(\tau_{km}-) \quad \textit{and} \quad H_{2m}(k)=H_m(\tau_{km}) \quad (k\in\mathbb{N}_m,\ m\in\mathbb{N}).$$

Then due to (3.7), we get

$$H_{1m}(k)\equiv H_m(\tau_{k-1\,m}+) \quad \text{and} \quad H_m(\tau_{km}+)\equiv H_{1m}(k+1) \quad (m\in\mathbb{N}).$$

From this and equalities (3.8) and (3.9), using equalities (3.4) and (3.5), for every natural m and $k\in\mathbb{N}_m$, we obtain

$$\begin{aligned}
\mathcal{B}(H_m,A_m)(t)&=\sum_{a<\tau_{im}\le t} H_{1m}(i)G_{1m}(i)+\sum_{a\le\tau_{im}<t} H_{1m}(i+1)G_{2m}(i),\\
&=\sum_{i=1}^{k-1} H_{1m}(i)G_{1m}(i)+\sum_{i=0}^{k-1} H_{1m}(i+1)G_{2m}(i) \quad \text{for} \quad t\in I_{km},\\
\mathcal{B}(H_m,A_m)(\tau_{km})&\equiv\sum_{i=1}^{k} H_{1m}(i)G_{1m}(i)+\sum_{i=0}^{k-1} H_{1m}(i+1)G_{2m}(i);\\
\mathcal{B}(H_m,f_m)(t)&=\sum_{a<\tau_{im}\le t} H_{1m}(i)g_{1m}(i)+\sum_{a\le\tau_{im}<t} H_{1m}(i+1)g_{2m}(i)\\
&=\sum_{i=1}^{k-1} H_{1m}(i)g_{1m}(i)+\sum_{i=0}^{k-1} H_{1m}(i+1)g_{2m}(i) \quad \text{for} \quad t\in I_{km},\\
\mathcal{B}(H_m,f_m)(\tau_{km})&\equiv\sum_{i=1}^{k} H_{1m}(i)g_{1m}(i)+\sum_{i=0}^{k-1} H_{1m}(i+1)g_{2m}(i).
\end{aligned}$$

So, for every m, we have the equalities

$$\mathcal{B}(H_m,A_m)(t)\equiv\mathcal{B}_m(H_{1m},G_{1m},G_{2m})(t), \quad \mathcal{B}(H_m,f_m)(t)\equiv\mathcal{B}_m(H_{1m},g_{1m},g_{2m})(t); \tag{3.10}$$

$$\begin{aligned}
&d_jH_m(t)=d_j\mathcal{B}(H_m,A_m)(t)=O_{n\times n}, \quad d_j\mathcal{B}(H_m,f_m)(t)=0_n\\
&\qquad\qquad \text{for} \quad t\in I\setminus\{\tau_{0m},\dots,\tau_{mm}\} \quad (j=1,2);
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&d_1H_m(\tau_{km})\equiv H_{2m}(k)-H_{1m}(k), \quad d_2H_m(\tau_{km})\equiv H_{1m}(k+1)-H_{2m}(k),\\
&d_1\mathcal{B}(H_m,A_m)(\tau_{km})\equiv H_{1m}(k)G_{1m}(k), \quad d_2\mathcal{B}(H_m,A_m)(\tau_{km})\equiv H_{1m}(k+1)G_{2m}(k).
\end{aligned} \tag{3.12}$$

Hence, by (3.10)–(3.12), using (1.4), for every m, we conclude that

$$\begin{aligned}
\mathcal{I}(H_m,A_m)(t)&=\int_a^t d\big(H_m(\tau)+\mathcal{B}(H_m,A_m)(\tau)\big)\cdot H_m^{-1}(\tau)\\
&=\sum_{a<\tau_{im}\le t} d_1\big(H_m(\tau_{im})+\mathcal{B}(H_m,A_m)(\tau_{im})\big)\cdot H_m^{-1}(\tau_{im})\\
&\quad+\sum_{a\le\tau_{im}<t} d_2\big(H_m(\tau_{im})+\mathcal{B}(H_m,A_m)(\tau_{im})\cdot H_m^{-1}(\tau_{im})
\end{aligned}$$

$$= \sum_{i=1}^{k-1} \big(H_{2m}(i) - H_{1m}(i) + H_{1m}(i)G_{1m}(i)\big) \cdot H_{2m}^{-1}(i)$$
$$+ \sum_{i=0}^{k-1} \big(H_{1m}(i+1) - H_{2m}(i) + H_{1m}(i+1)G_{2m}(i)\big) \cdot H_{2m}^{-1}(i) \quad \text{for} \quad t \in I_{km}.$$

Therefore

$$\mathcal{I}(H_m, A_m)(t) = \mathcal{I}_m(H_{1m}, H_{2m}^{-1}, G_{1m}, G_{2m})(t) \quad \text{for} \quad t \in I_{km} \quad (m = 1, 2, \dots). \tag{3.13}$$

Similarly, we show

$$\mathcal{I}(H_m, A_m)(\tau_{km}) \equiv \mathcal{I}_m(H_{1m}, H_{2m}^{-1}, G_{1m}, G_{2m})(\tau_{km}) \quad (m = 1, 2, \dots). \tag{3.14}$$

Let now $\varphi_m \in \mathrm{E}(\widetilde{N}_m; \mathbb{R}^n)$ and $\varphi_m(k) \equiv \varphi_m(\tau_{km})$ $(m = 1, 2, \dots)$. Then, as above, we conclude that

$$\int_a^t d\mathcal{I}(H_m, A_m)(\tau) \cdot \varphi_m(\tau) \equiv \mathcal{I}_m(H_{1m}, H_{2m}^{-1}\psi_m, G_{1m}, G_{2m})(t) \quad (m = 1, 2, \dots). \tag{3.15}$$

By (3.10), (3.13) and (3.14), the conditions of Theorem 2.1 coincide with the conditions of Theorem 1.1, respectively. In addition, according to Lemmas 3.1 and 3.2, Theorem 2.1 has the form of Theorem 1.1. So, Theorem 1.1 is proved.

Owing to Lemmas 3.1, 3.2 and equalities (3.6)–(3.15), we conclude that Theorems 2.2–2.4 have the forms of Theorems 1.2–1.4, and Corollary 2.1 has the forms of Corollary 1.1, respectively.

Proposition 1.1 is a realization of Proposition 2.1 for the considered difference case.

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