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ON SOME PROPERTIES OF UNIFORM DISTRIBUTION SEQUENCES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. Some properties of uniform distribution sequences for invariant extensions of linear Lebesgue measures are considered.

For a real number x, let $x = x - [x]$ be a fractional part of x, where $[x]$ denotes the integer part of x, that is, the greatest integer which is less or equal to x. Let $\{x_n : n \in \mathbf{N}\}$ be a given sequence of real numbers. Notice that the fractional part of any real number is contained in the unit interval $I = [0, 1)$.

A sequence of real numbers $\{x_n : n \in \mathbf{N}\}$ is said to be uniformly distributed sequence modulo 1 (abbreviated u.d.s. mod 1) if for each a, b, with $0 \le a < b \le 1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{card}(\{x_k : 1 \le k \le n\} \cap [a, b))}{n} = b - a.$$

The above-mentioned equation can be written in the following form:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k\le n} \chi_{[a,b)}(\{x_k\}) = \int_0^1 \chi_{[a,b)}(x)dx,$$

where $\chi_{[a,b)}$ denotes the characteristic function on the interval $[a, b) \subset I$.

The following theorem is valid.

Theorem 1. *The sequence $\{x_n : n \in \mathbf{N}\}$ of real numbers is u.d.s. mod 1 if and only if for every real-valued continuous function f defined on the closed interval $\overline{I} = [0, 1]$; we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k\le n} f(\{x_k\}) = \int_0^1 (x)dx.$$

(*For the above definitions and theorem, see* [1–3, 6–8]).

In the present paper, an approach to some questions of the theory of uniform distribution sequences is discussed. Such an approach is suitable for certain situations, where the given [0,1] interval is equipped with the class of invariant extensions of the linear Lebesgue measure on [0,1], and in this case we consider the theorems, analogous to those due to E. Hlawka and H. Weyl (see, for example, [6]).

For our purpose, we will need some auxiliary notions and facts from the Measure Theory.

Throughout this article, we use the following standard notation:

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^\omega$);

λ is the linear Lebesgue measure on $\mathbf{R}$.

dom(μ) is the domain of a given measure μ;

$\mu_1 \supset \mu$ - a measure μ_1 is an extension of the given measure μ.

Let E be a nonempty set, G be a group of transformations of E, and let X be a subset of E.

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We say that X is an almost G-invariant set (in the set-theoretical sense) if for each transformation $g \in G$, we have

$$\mathrm{card}(g(X) \bigtriangleup X) < \mathrm{card}(E),$$

where the symbol $\bigtriangleup$ denotes the operation of symmetric difference of two sets.

Two measures μ_1 and μ_2 are called mutually singular if there exists a measurable set X such that $\mu_1(X) = 0$ and $\mu_2(E \setminus X) = 0$.

The next propositions are useful for our further consideration.

Lemma 1. *There exists a family* $\{X_i : i \in [0,1]\}$ *of subsets of the real line* $\mathbf{R}$ *such that:*

(1) $X_i \cap X_{i'} = \emptyset$.

(2) *If* F *is an arbitrary closed subset of the real line* $\mathbf{R}$ *such that* $\lambda(F) > 0$, *then* $\mathrm{card}(X_i \cap F) = \mathbf{c}$.

(3) $\cup_{i \in I'} X_i$ *is an almost* $\mathbf{R}$*-invariant set in* $\mathbf{R}$, *where* I' *is an arbitrary subset of* $[0,1]$.

Lemma 2. *There exists a family* $\{Y_i : i \in [0,1]\}$ *of subsets of the real line* $\mathbf{R}$ *such that:*

(a) *for any sequence* $\{i_k : k \in \mathbf{N}\} \subset [0,1]$, *the intersection*

$$\cap_{k \in \mathbf{N}} \overline{Y_{i_k}},$$

where

$$\overline{Y_{i_k}} = Y_{i_k} \vee \overline{Y_{i_k}} = \mathbf{R} \setminus Y_{i_k}$$

is an almost invariant set.

(b) *for any sequence* $\{i_k : k \in \mathbf{N}\} \subset [0,1]$ *and for any closed subset* F *of the real line* $\mathbf{R}$ *with* $\lambda(F) > 0$, *we have*

$$\mathrm{card}\Big(\big(\cap_{k \in \mathbf{N}} \overline{Y_{i_k}}\big) \cap F\Big) = \mathbf{c}.$$

(*For the proofs of Lemma* 1 *and Lemma* 2, *see* [4]).

According to the above-mentioned lemmas, we come to the following statement.

Lemma 3. *There exists a family* $\{\mu_t : t \in [0,1]\}$ *of measures defined on some shift-invariant* σ*-algebra* $S(\mathbf{R})$ *of subsets of the real axis* $\mathbf{R}$ *such that:*

1) *each measure* μ_t *is a shift-invariant extension of the linear Lebesgue measure* λ;

2) *measures* μ_t *and* $\mu_{t'}$ *are mutually singular,* $(t \neq t')$.

Moreover, $\mu_t(\mathbf{R} \setminus X_t) = 0$ *for each* $t \in [0,1]$, *where* $\{X_t : t \in [0,1]\}$ *follows from Lemma* 2.

Proof. For an arbitrary $t \in [0,1]$, we denote by K_t a shift-invariant σ-ideal generated by the set $\mathbf{R} \setminus X_t$. Applying Marczewski's method, we can extend the Lebesgue measure λ to the measure μ_t. We obtain the family $\{\overline{\mu_t} : t \in [0,1]\}$ of shift-invariant extensions of the Lebesgue measure λ.

Denote by $S(\mathbf{R})$ the shift-invariant σ-algebra of subsets of the real line $\mathbf{R}$, generated by the union

$$\mathbf{L}(\mathbf{R}) \cup \mathbf{F}(\mathbf{R}) \cup \{X_t : t \in [0,1]\},$$

where $\mathbf{L}(\mathbf{R})$ denotes a class of all Lebesgue measurable subsets of the real line $\mathbf{R}$ and

$$\mathbf{F}(\mathbf{R}) = \{X : X \subset \mathbf{R}, \mathrm{card}(X) < \mathbf{c}\}.$$

For each $t \in [0,1]$, we assume that

$$\mu_t = \overline{\mu_t}|_{S(\mathbf{R})}.$$

The family of measures $\{\mu_t : t \in [0,1]\}$ satisfies the conditions of Lemma 3. □

Remark 1. Let us consider the family $\{\mu_t : t \in [0,1]\}$ of shift-invariant extensions of the measure λ obtained from Lemma 3. Let λ_t denote the restriction of the measure μ_t to the class

$$S[0,1] = \{Y \cap [0,1] : Y \in S(\mathbf{R})\},$$

where $S(\mathbf{R})$ follows from Lemma 3. It is obvious that for each $t \in [0,1]$, the measure λ_t is concentrated on the set $Z_t = X_t \cap [0,1]$, provided that

$$\lambda_t([0,1] \setminus Z_t) = 0.$$

Consider the family of probability measures $\{\lambda_t : t \in [0,1]\}$ and the family $\{Z_t : t \in [0,1]\}$ of subsets of [0,1] which come from Remark 1 and let λ_t^∞ denote an infinite power of λ_t.

The next lemma is valid.

Lemma 4. *For $t \in [0,1]$, we denote by $\mathbf{L}([0,1],\lambda_t)$ the class of λ_t-integrable functions. Then for $f \in \mathbf{L}([0,1],\lambda_t)$, we have*

$$\lambda_t^\infty\left(\left\{\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty : \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} f(x_k)}{n} = \int\limits_0^1 f(x)d\lambda_t(x)\right\}\right) = 1.$$

(*For the proof of Lemma* 4*, see* [5]*).*

A sequence of real numbers $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ is said to be λ-uniformly distributed sequence (λ-u.d.s.) if for each c, d, with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d])}{n} = d - c.$$

A sequence of real numbers $\{x_k : k \in \mathbf{N}\} \in \mathbf{R}^\infty$ is said to be uniformly distributed module 1 if the sequence of its fractional parts $\{x_k : k \in \mathbf{N}\}$ is λ-u.d.s.

Remark 2. It is obvious that $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ is uniformly distributed module 1 if and only if $\{x_k : k \in \mathbf{N}\}$ is λ-u.d.s.

A sequence of real numbers $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ is said to be λ_t-uniformly distributed sequence (λ_t-u.d.s.) if for each c, d, with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n} = d - c.$$

We say that a family $\{f_k : k \in \mathbf{N}\}$ of elements of $\mathbf{L}([0,1],\lambda_t)$ defines a λ_t-u.d.s. on [0,1], if for each $\{x_n : n \in \mathbf{N}\} \subset [0,1]^\infty$, the validity of the condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k \le n} f_k(x_k = \int\limits_0^1 f_k(x)d\lambda_t(x)$$

for $k \in \mathbf{N}$ implies that $\{x_n : n \in \mathbf{N}\}$ is λ_t-u.d.s.

Notice that the indicator functions of the sets $[a,b] \cap Z_t$ with rational a, b is an example of such a family.

Theorem 2. *Let T_t be the set of all real-valued sequences from $[0,1]^\infty$ which are λ_t-u.d.s. Then $\lambda_t^\infty(T_t) = 1$.*

Proof. Let $\{f_k : k \in \mathbf{N}\}$ be a countable subclass of $\mathbf{L}([0,1],\lambda_t)$ that defines a λ_t-u.d.s. on [0,1]. For $k \in \mathbf{N}$, we set

$$B_k = \left\{\{x_k : k \in \mathbf{N}\} : \{x_k : k \in \mathbf{N}\} \in [0,1]^\infty, \lim_{n\to\infty} \frac{1}{n} \sum_{k \le n} f_k(x_k = \int\limits_0^1 f_k(x)d\lambda_t(x)\right\}.$$

By Lemma 4, we know that

$$\lambda_t^\infty(B_k) = 1$$

for $k \in \mathbf{N}$, which implies

$$\lambda_t^\infty\Big(\bigcap_{k\in\mathbf{N}} B_k\Big) = 1.$$

Hence we have

$$\lambda_t^\infty\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty : (\forall k)(k \in \mathbf{N}) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{k \le n} f_k\left(x_k = \int\limits_0^1 f_k(x)d\lambda_t(x)\right) = 1.$$

The latter relation means that λ_t^∞-almost every elements of $[0,1]^\infty$ are λ_t-u.d.s, or equivalently, $\lambda_t^\infty(T_t)=1$. $\square$

Theorem 3. *For $t\in[0,1]$, we put*

$$Z_t[0,1]=\big\{\widetilde{f}=f(x)\times\chi_{Z_t(x):fi\in C[0,1]}\big\}.$$

Then the sequence $\{x_n : n\in\mathbf{N}\}$ is λ_t-u.d.s. if and only if the condition

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k\le n}\widetilde{f}(x_k)=\int\limits_0^1\widetilde{f}(x)d\lambda_t(x)$$

holds for each $\widetilde{f}\in Z_t[0,1]$.

The proof of Theorem 2 is similar to that of H. Weyl's Theorem (see [6]).

Remark 3. Some results presented in the paper were accepted jointly by Professor Gogi Pantsulaia. Here is a modified version of our previous unpublished survey.

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