Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 3, 491–493

SOME PROPERTIES OF MAZURKIEWICZ TYPE SETS

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In the present note, some results about Mazurkiewicz type sets are discussed in the context of their measurability and the Baire property.

In 1914, Mazurkiewicz proved that there exists a set $X \subset \mathbb{R}^2$ such that every straight line in $\mathbb{R}^2$ meets X at exactly two points (see [15]). Such a set X is called a Mazurkiewicz set. The above-mentioned theorem of Mazurkiewicz gave rise to a number of research works related to the existence of certain versions of Mazurkiewicz type sets, and the study of different aspects of such sets was continued. The analysis of the mentioned types of sets started from different points of view: geometrical, topological, and measure-theoretical. Many interesting results have been obtained in these directions (see, for instance, [1–6, 8, 10–12, 14, 16]).

Below, we list some of known results that are related to the statements presented in this paper.

Fact 1 (see, e.g., [7])**.** There exists a Mazurkiewicz set of Lebesgue measure zero.

Fact 2. If a Mazurkiewicz set is Lebesgue measurable, then its Lebesgue measure is zero.

Fact 3 (see [7])**.** There exists a Mazurkiewicz set which meets every set in $\mathbb{R}^2$ of positive Lebesgue measure. Any such set is Lebesgue non-measurable.

Later on, the following definition was introduced.

Definition 1. Let $n > 1$ be a natural number. A set $B \subset \mathbb{R}^2$ is called a Mazurkiewicz-type n-set with respect to the family of all straight lines (or n-point set) if $\operatorname{card}(B \cap l) = n$ for any straight line l lying in $\mathbb{R}^2$.

In connection with Definition 1, the following statement due to Sierpinski holds:

Fact 4 (see [1, 16])**.** For every natural number $n \geq 2$, there exists a set $X \subset \mathbb{R}^2$ such that every straight line in $\mathbb{R}^2$ meets X at exactly n points.

It should be noted that the following statement is very useful for proving Fact 4.

Lemma (see [7])**.** *If a subset $X \subset \mathbb{R}^2$ of the Euclidian plane has a common subset of cardinality continuum c with every straight line in the plane, then X has a Masurkiewicz-type n-subset with respect to the family of all straight lines* (*for every natural number $n \geq 2$*)*.*

Definition 2. We say that a set $X \subset \mathbb{R}^2$ of the Eucliduan plane is a Mazurkiewicz type set with respect to the family of all straight lines if there exists a natural number $n > 1$ such that $\operatorname{card}(X \cap l) = n$ for any straight line l lying in $\mathbb{R}^2$.

Similar statements to Fact 1, Fact 2 and Fact 3 are valid for Mazurkiewicz type sets with respect to straight lines, in terms of the Baire property and Baire category.

Further development of the presented results looks as follows:

Definition 3. Let $k > 2$ be a natural number. A set $B \subset \mathbb{R}^2$ is called a Mazurkiewicz-type k-set with respect to the family of all circles if $\operatorname{card}(B \cap C) = k$ for any circle C lying in $\mathbb{R}^2$.

In connection with Definition 3, the following statement holds.

2020 *Mathematics Subject Classification.* 03E75, 03E25, 28A05.

Key words and phrases. Mazurkiewicz type set; Lebesgue measure; Baire category; Transfinite induction.

Fact 5 (see [9,13])**.** For any natural number $k \geq 3$, there exists a set $X \subset \mathbb{R}^2$ such that every circle in $\mathbb{R}^2$ meets X at exactly k points.

Naturally, in order to prove this fact, one can use the principle of transfinite induction and a well-ordering of the continuum.

Definition 4. We say that a set $X \subset \mathbb{R}^2$ of the Eucliduan plane is a Mazurkiewicz type set with respect to the family of all circles if there exists a natural number $k > 2$ such that $\mathrm{card}(X \cap C) = k$ for any circle C lying in $\mathbb{R}^2$.

It turns out that the structure of the Mazurkiewicz type sets is sufficiently complex.

Fact 6 (see [9,13])**.** There exists a set $X \subset \mathbb{R}^2$ which is a Mazurkiewicz type set with respect to the family of all straight lines and is not a Mazurkiewicz type set with respect to the family of all circles. Also, there exists a set $Y \subset \mathbb{R}^2$ which is a Mazurkiewicz type set with respect to the family of all circles and is not a Mazurkiewicz type set with respect to the family of all straight lines.

Fact 7 (see [9,13])**.** For any two natural numbers $p \geq 2$ and $q \geq 3$, there exists a set $X \subset \mathbb{R}^2$ such that X is a Mazurkiewicz type p-set with respect to the family of all straight lines and X is a Mazurkiewicz type q-set with respect to the family of all circles.

The following statements related to the measurability of Mazurkiewicz type sets with respect to the family of all circles are valid:

Theorem 1. *There exists a Mazurkiewicz type set with respect to the family of all circles which has Lebesgue measure zero and is of the first category.*

Theorem 2. *If a Mazurkiewicz type set $X \subset \mathbb{R}^2$ with respect to the family of all circles is Lebesgue measurable* (*possesses the Baire property*)*, then X is of Lebesgue measure zero* (*is of the first category*)*.*

Theorem 3. *There exists a Mazurkiewicz type set $X \subset \mathbb{R}^2$ with respect to the family of all circles which is thick with respect to the Lebesgue measure* (*that is, $X \cap Y \neq \emptyset$ for every Lebesque measurable $Y \subset \mathbb{R}^2$ with a positive measure*)*.*

Theorem 4. *Let r_1 and r_2 be any two distinct positive real numbers. Let F_1 be the family of all circumferences in the Euclidean plane with radius r_1, and let F_2 be the family of all circumferences in the Euclidean plane with radius r_2. Then there exists a subset S of the plane such that S is a Mazurkiewicz type set with respect to the family F_1 and is not a Mazurkiewicz type set with respect to the family F_2.*

Remark 1. In ZF theory, there exists a subset of the Euclidian plane which has a common countable infinite subset with every straight line in the plane and has not a Mazurkiewicz-type subset with respect to the family of all straight lines. For example, such is the point set $\cup(F_1 \cup F_2)$, where F_1 is a countable infinite family of vertical pairwise distinct straight lines in the Euclidean plane and F_2 is a countable infinite family of horizontal pairwise distinct straight lines in this plane.

Remark 2. Suppose that some straight line in the Euclidian plane has a subset of cardinality α, where $\omega < \alpha < c$. Then there exists a subset of the Euclidian plane which has a common subset of cardinality α with every straight line in the plane, but X has no Mazurkiewicz-type subset with respect to the family of all straight lines. For example, such is the point set $\cup(F_1 \cup F_2 \cup F_3)$, where F_1, F_2, F_3 are the families of pairwise distinct straight lines in the plane, each family is of cardinality α and

$$\cap F_1 = \{A\}, \qquad \cap F_2 = \{B\}, \qquad \cap F_3 = \{C\},$$

where A, B, C are some non-collinear points in the plane.

At the end of this note, let us mention that the following theorem related to the measurability of Mazurkiewicz type sets with respect to the family of all straight lines is valid (see [11]).

If a Mazurkiewicz type set with respect to the family of all straight lines is measurable under a σ-finite and translation quasi-invariant measure μ on $\mathbb{R}^2$, then this set is a μ-null set.

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(Received 18.07.2023)

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