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SOME CHARACTERIZATIONS OF BMO SPACES VIA COMMUTATORS OF FRACTIONAL MAXIMAL OPERATOR IN ORLICZ SPACES OVER SPACES OF HOMOGENEOUS TYPE

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. We give the necessary and sufficient conditions for the boundedness of the commutators of the fractional maximal operator $[b, M_\eta]$ in Orlicz spaces $L^\Phi(X)$ over spaces of homogeneous type $X = (X, d, \mu)$ when b belongs to $BMO(X)$ spaces. We obtain some new characterizations for certain subclasses of $BMO(X)$ spaces.

1. Introduction

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e., X is a topological space endowed with a quasi-distance d and a positive measure μ. The fractional maximal function $M_\eta f$ is defined by

$$M_\eta f(x) = \sup_{B \ni x} \mu(B)^{\eta-1} \int\limits_B |f(y)| d\mu(y), \ 0 \le \eta < 1,$$

where the supremum is taken over all balls $B \subset X$ containing x.

The fractional maximal commutator $M_{b,\eta}$ generated by $b \in L^1_{\rm loc}(X)$ and the operator M_η, is defined by

$$M_{b,\eta}(f)(x) = \sup_{B \ni x} \mu(B)^{\eta-1} \int\limits_B |b(x) - b(y)||f(y)| d\mu(y), \ 0 \le \eta < 1.$$

If $\eta = 0$, then we get the maximal commutator $M_{b,0} \equiv M_b$.

The commutator $[b, M_\eta]$ generated by a function b and the operator M_η, is defined by

$$[b, M_\eta](f)(x) = b(x) M_\eta(f)(x) - M_\eta(bf)(x).$$

If $\eta = 0$, then we get the commutator of maximal operator $[b, M] = [b, M_0]$.

$M_{b,\eta}$ and $[b, M_\eta]$ essentially differ from each other since $M_{b,\eta}$ is positive and sublinear and $[b, M_\eta]$ is neither positive, nor sublinear. The operators M_η, $[b, M_\eta]$ and $M_{b,\eta}$ play an important role in real and harmonic analysis and applications [4, 8, 10, 20–22, 32, 34].

The aim of this paper is to study the boundedness of commutators $[b, M_\eta]$ of the fractional maximal operator in Orlicz spaces $L^\Phi(X)$ over the spaces of homogeneous type $X = (X, d, \mu)$. We characterize the commutator functions b, involved in the boundedness in Orlicz spaces of the commutator $[b, M_\eta]$ of the fractional maximal operator (Theorems 4.3 and 4.6).

It is well known that the commutator estimates play an important role in many applications in harmonic analysis and partial differential equations [5, 15, 25, 31, 32]. The mapping properties of $M_{b,\eta}$ and $[b, M_\eta]$ have been studied extensively by many authors (see [1, 2, 6, 13, 17–19, 22, 25, 33, 34]). In the study of commutators of singular integral operators with BMO symbols the use is made of the operator $M_b := M_{b,0}$ (see [13,24,31]). Note that the boundedness of the operator M_b on L^p spaces was proved by Garcia–Cuerva et al. in [13]. The nonlinear commutator $[b, M]$ of the maximal operator is used in studying the product of a function in H_1 and a function in BMO (see [3]). In [2], Bastero et

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al. studied the necessary and sufficient conditions for the boundedness of $[b, M]$ on L^p spaces. In [33], Zhang and Lu considered the same problem for $[b, M_\eta]$ (see also [34]).

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C, independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e., X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$d(x,y) \ge 0;\ d(x,y) = 0 \text{ if and only if } \ x = y;\ d(x,y) = d(y,x),$$
$$d(x,y) \le \kappa\,(d(x,z) + d(z,y)).$$

The balls $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) \le K\,\mu(B(x,r)) < \infty, \tag{2.1}$$

where $\kappa, K \ge 1$ are the constants, independent of $x, y, z \in X$ and $r > 0$. As usual, the dilation of a ball $B = B(x,r)$ will be denoted by $\lambda B = B(x, \lambda r)$ for every $\lambda > 0$. Note that (2.1) implies that for all $\lambda \ge 1$.

Macias and Segovia showed that on any space of homogeneous type $X = (X, d, \mu)$ there exists an equivalent quasi-metric ρ such that the quasi-metric balls with respect to ρ are open. Therefore we could have assumed from the outset that our σ-algebra is the Borel algebra and that μ is a positive Borel measure which is doubling.

Now we recall the definition of Young functions.

Definition 2.1. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and due to the fact that $\Phi(0) = 0$, it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. Let $\mathcal{Y}$ be the set of all Young functions Φ such that $0 < \Phi(r) < \infty$ for $0 < r < \infty$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself. For a measurable set $\Omega \subset X$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = \mu(\{x \in \Omega : |f(x)| > t\})$. In the case $\Omega = X$, we shortly denote it by $m(f,\ t)$.

The Orlicz spaces and the weak Orlicz spaces on spaces of homogeneous type are defined as follows.

Definition 2.2. For a Young function Φ,

$$L^\Phi(X) = \left\{ f \in L^1_{\rm loc}(X) : \int_X \Phi(\epsilon|f(x)|)d\mu(x) < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{L^\Phi} \equiv \|f\|_{L^\Phi(X)} = \inf\left\{\lambda > 0 : \int_X \Phi\Big(\frac{|f(x)|}{\lambda}\Big)d\mu(x) \le 1\right\},$$

$$WL^\Phi(X) := \left\{ f \in L^1_{\rm loc}(X) : \sup_{r>0} \Phi(r) m(r, \epsilon f) < \infty \ \text{ for some } \ \epsilon > 0\right\},$$

$$\|f\|_{WL^\Phi} \equiv \|f\|_{WL^\Phi(X)} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1\right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{t>0} \Phi(t) m(\Omega,\ f,\ t) = \sup_{t>0} t\, m(\Omega,\ f,\ \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega,\ \Phi(|f|),\ t)$$

and

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\Big)dx \le 1, \qquad \sup_{t>0} \Phi(t) m\Big(\Omega,\ \frac{f}{\|f\|_{WL^\Phi(\Omega)}},\ t\Big) \le 1,$$

where $\|f\|_{L^\Phi(\Omega)} = \|f\chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f\chi_\Omega\|_{WL^\Phi}$.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for} \quad 0 \le r < \infty.$$

We also note that

$$\|\chi_B\|_{WL^\Phi} = \|\chi_B\|_{L^\Phi} = \frac{1}{\Phi^{-1}\left(\mu(B)^{-1}\right)}, \tag{2.2}$$

where B is a μ-measurable set in X with $\mu(B) < \infty$ and χ_B is the characteristic function of B.

A Young function Φ is said to satisfy the Δ_2-condition denoted by $\Phi \in \Delta_2$, if $\Phi(2r) \le k\Phi(r)$ for $r > 0$ for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if $\Phi(r) \le \frac{1}{2k}\Phi(kr)$, $r \ge 0$ for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition, but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition, but does not satisfy the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty), \\ \infty, & r = \infty. \end{cases}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = \infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.3}$$

Note that by the convexity of Φ and concavity of Φ^{-1}, we have the following properties:

$$\begin{cases} \Phi(\eta t) \le \eta\Phi(t), & \text{if} \quad 0 \le \eta \le 1 \\ \Phi(\alpha t) \ge \alpha\Phi(t), & \text{if} \quad \alpha > 1 \end{cases} \quad \text{and} \quad \begin{cases} \Phi^{-1}(\eta t) \ge \eta\Phi^{-1}(t), & \text{if} \quad 0 \le \eta \le 1 \\ \Phi^{-1}(\alpha t) \le \alpha\Phi^{-1}(t), & \text{if} \quad \alpha > 1. \end{cases} \tag{2.4}$$

The following analogue of Hölder's inequality

$$\int\limits_X |f(x)g(x)|d\mu(x) \le 2\|f\|_{L^\Phi}\|g\|_{L_{\widetilde{\Phi}}}$$

is known. In proving our main estimates we have used the following lemma which follows from Hölder's inequality, (2.2) and (2.3).

Lemma 2.3. *Let (X,d,μ) be a space of homogeneous type. For a Young function Φ and $B = B(x,r)$, the inequality*

$$\|f\|_{L^1(B)} \le 2\,\mu(B)\,\Phi^{-1}\left(\mu(B)^{-1}\right)\,\|f\|_{L^\Phi(B)}$$

is valid.

3. Fractional Maximal Commutator in Orlicz Spaces

We recall the boundedness property of M in Orlicz spaces since it will be used later.

Theorem 3.1 ([14])**.** *Let (X,d,μ) be a space of homogeneous type and Φ be a Young function.*

(i) *The operator M is bounded from $L^\Phi(X)$ to $WL^\Phi(X)$ and the inequality*

$$\|Mf\|_{WL^\Phi} \le C_0\|f\|_{L^\Phi}$$

holds with the constant C_0, independent of f.

(ii) *The operator M is bounded on $L^\Phi(X)$, and the inequality*

$$\|Mf\|_{L^\Phi} \le C_0\|f\|_{L^\Phi}$$

holds with the constant C_0, independent of f if and only if $\Phi \in \nabla_2$.

The following result completely characterizes the boundedness of M_η in Orlicz spaces.

Theorem 3.2 ([7]). *Let $0<\eta<1$, Φ,Ψ be the Young functions and $\Phi\in\mathcal{Y}$. The condition*

$$\mu(B)^\eta\,\Phi^{-1}\big(\mu(B)^{-1}\big)\le C\,\Psi^{-1}\big(\mu(B)^{-1}\big) \tag{3.1}$$

for all balls $B\subset X$, where $C>0$ does not depend on B, is necessary and sufficient for the boundedness of M_η from $L^\Phi(X)$ to $WL^\Psi(X)$. Moreover, if $\Phi\in\nabla_2$, condition (3.1) is necessary and sufficient for the boundedness of M_η from $L^\Phi(X)$ to $L^\Psi(X)$.

Remark 3.3. Note that Theorem 3.2 in the case $X=\mathbb{R}^n$ was proved in [19].

Suppose that $b\in L^1_{\rm loc}(X)$. Then b is said to be in $BMO(X)$ if the seminorm given by

$$\|b\|_*=\sup_B\frac{1}{\mu(B)}\int\limits_B|b(x)-b_B|d\mu(x)$$

is finite, where the supremum is taken over all balls $B\subset X$ and

$$b_B=\frac{1}{\mu(B)}\int\limits_B b(x)d\mu(x).$$

For any measurable set E with $\mu(E)<\infty$ and any suitable function f, the norm $\|f\|_{L(\log L),E}$ is defined by

$$|f\|_{L(\log L),E}=\inf\Big\{\lambda>0:\frac{1}{\mu(E)}\int\limits_E\frac{|f(x)|}{\lambda}\,\log\Big(2+\frac{|f(x)|}{\lambda}\Big)d\mu(x)\le 1\Big\}.$$

The norm $\|f\|_{\exp L,E}$ is defined by

$$|f\|_{\exp L,E}=\inf\Big\{\lambda>0:\frac{1}{\mu(E)}\int\limits_E\exp\Big(\frac{|f(x)|}{\lambda}\Big)d\mu(x)\le 2\Big\}.$$

Then for any suitable functions f and g, the generalized Hölder's inequality

$$\frac{1}{\mu(E)}\int\limits_E|f(x)||g(x)|d\mu(x)\lesssim\|f\|_{\exp L,E}\,\|g\|_{L(\log L),E} \tag{3.2}$$

holds (see [30]).

The following John-Nirenberg inequalities on spaces of homogeneous type come from [27, Propositions 6, 7].

Lemma 3.4. *Let $b\in BMO(X)$. Then there exist the constants $C_1,C_2>0$ such that for every ball $B\subset X$ and every $\alpha>0$, we have*

$$\mu\big(\{x\in B:|b(x)-b_B|>\alpha\}\big)\le C_1\,\mu(B)\exp\Big\{-\frac{C_2}{\|b\|_*}\,\alpha\Big\}.$$

By the generalized Hölder's inequality in Orlicz spaces (see [30, page 58]) and John-Nirenberg's inequality (see also [28, (2.14)]), we get

$$\frac{1}{|B|}\int\limits_B\big|b(x)-b_B\big||g(x)|d\mu(x)\lesssim\|b\|_*\,\|g\|_{L(\log L),B}.$$

For details on this space and properties we refer, for instance, to [26] and [29].

For the given ball B and $0\le\eta<1$, we define the following maximal function:

$$M_{\eta,B}f(x)=\sup_{B\supseteq B'\ni x}\ \mu(B')^{-1+\eta}\int\limits_{B'}|f(y)|d\mu(y),$$

where the supremum is taken over all balls B' such that $x\in B'\subseteq B$. Moreover, we denote $M_B=M_{0,B}$ when $\eta=0$.

For a function b defined on X, we denote

$$b^-(x) := \begin{cases} 0, & \text{if} \quad b(x) \geq 0, \\ |b(x)|, & \text{if} \quad b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

Before proving our theorems, we need the following lemmas and theorem.

Lemma 3.5 ([11])**.** *Let $b \in L^1_{\text{loc}}(X)$. Then the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
2. *There exists $s \in [1, \infty)$ such that*

$$\sup_B \frac{\left\| \left(b - \mu(B)^{-\eta} M_{\eta,B}(b)\right)\chi_B \right\|_{L^s(X)}}{\|\chi_B\|_{L^s(X)}} \leq C. \tag{3.3}$$

3. *For all $s \in [1, \infty)$, we have (3.4).*

Lemma 3.6 ([11])**.** *Let $b \in L^1_{\text{loc}}(X)$. Then the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
2. *There exists $s \in [1, \infty)$ such that*

$$\sup_B \frac{\left\| \left(b - M_B(b)\right)\chi_B \right\|_{L^s(X)}}{\|\chi_B\|_{L^s(X)}} \leq C. \tag{3.4}$$

3. *For all $s \in [1, \infty)$, we have (3.4).*

Lemma 3.7 ([11])**.** *Let $b \in L^1_{\text{loc}}(X)$. Then the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
2. *There exists $s \in [1, \infty)$ such that*

$$\sup_B \frac{\left\| \left(b - 2M^\sharp(b\chi_B)\right)\chi_B \right\|_{L^s(X)}}{\|\chi_B\|_{L^s(X)}} \leq C. \tag{3.5}$$

3. *For all $s \in [1, \infty)$, we have (3.5).*

Lemma 3.8 ([23])**.** *Let $b \in BMO(X)$ and Φ be a Young function with $\Phi \in \Delta_2$, then*

$$\|b\|_* \approx \sup_B \Phi^{-1}\left(\mu(B)^{-1}\right) \left\| \left(b - b_B\right)\chi_B \right\|_{L^\Phi}. \tag{3.6}$$

From Lemmas 3.5 and 3.8, we get

Lemma 3.9. *Let $b \in L^1_{\text{loc}}(X)$ and Φ be a Young function. Then the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
2. *There exists $\Phi \in \Delta_2$ such that*

$$\sup_B \Phi^{-1}\left(\mu(B)^{-1}\right) \left\| \left(b - \mu(B)^{-\eta} M_{\eta,B}(b)\right)\chi_B \right\|_{L^\Phi} < \infty. \tag{3.7}$$

3. *For all $\Phi \in \Delta_2$, we have (3.7).*

From Lemmas 3.6 and 3.8, we get

Lemma 3.10. *Let $b \in L^1_{\text{loc}}(X)$ and Φ be a Young function. Then the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
2. *There exists $\Phi \in \Delta_2$ such that*

$$\sup_B \Phi^{-1}\left(\mu(B)^{-1}\right) \left\| \left(b - M_B(b)\right)\chi_B \right\|_{L^\Phi} < \infty. \tag{3.8}$$

3. *For all $\Phi \in \Delta_2$, we have (3.8).*

From Lemmas 3.7 and 3.8, we get

Lemma 3.11. *Let $b \in L^1_{\mathrm{loc}}(X)$ and Φ be a Young function. Then the following statements are equivalent:*

1. $b \in BMO(X)$ *and* $b^- \in L^\infty(X)$.
2. *There exists* $\Phi \in \Delta_2$ *such that*

$$\sup_B \Phi^{-1}\big(\mu(B)^{-1}\big) \left\| \big(b - 2M^\sharp(b\,\chi_B)\big)\chi_B \right\|_{L^\Phi} < \infty. \tag{3.9}$$

3. *For all* $\Phi \in \Delta_2$, *we have* (3.9).

The known boundedness statements for the commutator operator M_b in Orlicz spaces run as follows (see [17, Theorem 1.9 and Corollary 2.3]). Note that a more general case of multi-linear commutators was studied in [12].

Theorem 3.12 ([12]). *Let $b \in BMO(X)$ and Φ be a Young function with $\Phi \in \nabla_2 \cap \nabla_2$. Then the operator M_b is bounded on $L^\Phi(X)$ and the inequality*

$$\|M_b f\|_{L^\Phi} \le C_0 \|b\|_* \|f\|_{L^\Phi}$$

holds with the constant C_0, independent of f.

We say that (X, d, μ) is Ahlfors regular (Q-homogeneous) if there exist the constants $C_1, C_2, Q > 0$ such that for every $x \in X$ and r,

$$C_1^{-1}\, r^Q \le \mu(B(x,r)) \le C_2\, r^Q. \tag{3.10}$$

The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous. Thanks to (3.10) and (2.4), we have

$$\Phi^{-1}(\mu(B(x,r))^{-1}) \approx \Phi^{-1}(r^{-Q}).$$

Theorem 3.13 ([7]). *Let $0 \le \eta < 1$, $b \in L^1_{\mathrm{loc}}(X)$, Φ, Ψ be Young functions with $\Phi \in \Delta_2 \cap \nabla_2$, $\Psi \in \Delta_2$ and $\Psi^{-1}\big(\mu(B)^{-1}\big) \approx \mu(B)^\eta\, \Phi^{-1}\big(\mu(B)^{-1}\big)$. If the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln \frac{t}{r}\Big) \mu(B(x,t))^\eta\, \Phi^{-1}\big(\mu(B(x,t))^{-1}\big) \le C \mu(B(x,r))^\eta\, \Phi^{-1}\big(\mu(B(x,r))^{-1}\big) \tag{3.11}$$

holds, then the condition $b \in BMO(X)$ is necessary and sufficient for the boundedness of $M_{b,\eta}$ from $L^\Phi(X)$ to $L^\Psi(X)$.

4. Commutators of Fractional Maximal Operator in Orlicz Spaces

In this section, we find the necessary and sufficient conditions for the boundedness of the commutator $[b, M_\eta]$ of the fractional maximal operator M_η in Orlicz spaces $L^\Phi(X)$ over the spaces of homogeneous type $X = (X, d, \mu)$.

The following relations between $[b, M_\eta]$ and $M_{b,\eta}$ are valid.

Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\mathrm{loc}}(X)$ and $x \in X$, we have the following inequality:

$$\begin{aligned} \big|[b, M_\eta] f(x)\big| &= \big|b(x) M_\eta f(x) - M_\eta(bf)(x)\big| \\ &= \big|M_\eta(b(x)f)(x) - M_\eta(bf)(x)\big| \le M_\eta(|b(x) - b|f)(x) \le M_{b,\eta}(f)(x). \end{aligned}$$

If b is any locally integrable function on X, then

$$|[b, M_\eta] f(x)| \le M_{b,\eta}(f)(x) + 2b^-(x) M_\eta f(x), \quad x \in X \tag{4.1}$$

holds for all $f \in L^1_{\mathrm{loc}}(X)$ (see [8, 34]).

By Theorem 3.13, we have

Corollary 4.1. *Let $0 \le \eta < 1$, $b \in L^1_{\mathrm{loc}}(X)$, Φ, Ψ be the Young functions with $\Phi \in \Delta_2 \cap \nabla_2$, $\Psi \in \Delta_2$ and $\Psi^{-1}\big(\mu(B)^{-1}\big) \approx \mu(B)^\eta\, \Phi^{-1}\big(\mu(B)^{-1}\big)$. If condition (3.11) holds, then the operator $[b, M_\eta]$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.*

Theorem 4.2. *Let $0 \le \eta < 1$, $b \in L^1_{\mathrm{loc}}(X)$, Φ, Ψ be the Young functions with $\Phi \in \nabla_2 \cap \mathcal{Y}$ and $\Psi^{-1}\big(\mu(B)^{-1}\big) \approx \mu(B)^\eta\, \Phi^{-1}\big(\mu(B)^{-1}\big)$. Then the following statements are equivalent:*

1. *$b \in BMO(X)$ and $b^- \in L^\infty(X)$.*
2. *$[b, M_\eta]$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.*
3. *There exists $\Phi \in \Delta_2$ such that*
$$\sup_B \Phi^{-1}\big(\mu(B)^{-1}\big)\, \big\|b - \mu(B)^{-\eta} M_{\eta,B}(b)\big\|_{L^\Phi(B)} < \infty. \tag{4.2}$$
4. *There exists a constant $C > 0$ such that*
$$\sup_B \mu(B)^{-1}\, \big\|b - \mu(B)^{-\eta} M_{\eta,B}(b)\big\|_{L^1(B)} \le C. \tag{4.3}$$

Proof. Since the implication "(1) ⇒ (2)" follows readily by Corollary 4.1 and the equivalence of (1) and (4) follows from Lemma 3.9, we only need to prove the implications "(2) ⇒ (3)" and "(3) ⇒ (4)".

(2) ⇒ (3). From the definition of $M_{\eta,B}$, it is not difficult to check that $M_{\eta,B}\chi_B(x) = \mu(B)^\eta$ for all $x \in B$.

Note that for any $x \in B$, $M_\eta\big(b\chi_B\big)(x) = M_{\eta,B}(b)(x)$ (see [33]) and then $M_\eta\big(\chi_B\big)(x) = M_{\eta,B}\chi_B(x) = \mu(B)^\eta$.

Then for any $x \in B$,
$$\begin{aligned} b(x) - \mu(B)^{-\eta} M_{\eta,B}(b)(x) &= \mu(B)^{-\eta}\big(b(x)\mu(B)^\eta - M_{\eta,B}(b)(x)\big) \\ &= \mu(B)^{-\eta}\big(b(x)M_\eta\big(\chi_B\big)(x) - M_\eta\big(b\chi_B\big)(x)\big) = \mu(B)^{-\eta}[b, M_\eta]\big(\chi_B\big)(x). \end{aligned}$$

Since $[b, M_\eta]$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$, we get
$$\begin{aligned} I_1 &= \Psi^{-1}\big(\mu(B)^{-1}\big)\, \big\|b - \mu(B)^{-\eta} M_{\eta,B}(b)\big\|_{L^\Psi(B)} \\ &= \Psi^{-1}\big(\mu(B)^{-1}\big)\mu(B)^{-\eta}\, \big\|[b, M_\eta]\big(\chi_B\big)\big\|_{L^\Psi(B)} \\ &\le C\, \Psi^{-1}\big(\mu(B)^{-1}\big)\mu(B)^{-\eta}\, \|\chi_B\|_{L^\Phi} \le C, \end{aligned} \tag{4.4}$$
where at the last step we have applied (2.2) and the hypothesis $\Psi^{-1}\big(\mu(B)^{-1}\big) \approx \mu(B)^\eta\, \Phi^{-1}\big(\mu(B)^{-1}\big)$.

(3) ⇒ (1). Now, let us prove $b \in BMO(X)$ and $b^- \in L^\infty(X)$. For any ball B, let $E = \{y \in B : b(y) \le b_B\}$ and $E = \{y \in B : b(y) > b_B\}$. The following equality is true (see [2, page 3331]):
$$\int_E |b(y) - b_B| d\mu(y) = \int_F |b(y) - b_B| d\mu(y).$$
Since $b(y) \le b_B \le |b_B| \le \mu(B)^{-\eta} M_{\eta,B}(b)(y)$ for any $y \in E$, we obtain
$$|b(y) - b_B| \le \big|b(y) - \mu(B)^{-\eta} M_{\eta,B}(b)(y)\big|, \ y \in E.$$
Then from Lemma 2.3 and (4.4), we have
$$\begin{aligned} \frac{1}{\mu(B)} \int_B |b(y) - b_B| d\mu(y) &= \frac{2}{\mu(B)} \int_E |b(y) - b_B| d\mu(y) \\ &\le \frac{2}{\mu(B)} \int_E \big|b(y) - \mu(B)^{-\eta} M_{\eta,B}(b)(y)\big| d\mu(y) \\ &\le \frac{2}{\mu(B)} \int_B \big|b(y) - \mu(B)^{-\eta} M_{\eta,B}(b)(y)\big| d\mu(y) \\ &\lesssim \Psi^{-1}\big(\mu(B)^{-1}\big)\, \big\|b - \mu(B)^{-\eta} M_{\eta,B}(b)\big\|_{L^\Psi(B)} \le C. \end{aligned}$$

So, using the definition of $BMO(X)$, we have $b \in BMO(X)$.

Now, let us show that $b^- \in L^\infty(X)$. Observe that $0 \le b^+(y) \le |b(y)| \le M_B(b)(y)$ for $y \in B$, therefore for any $y \in B$, we get
$$0 \le b^-(y) \le M_B(b)(y) - b^+(y) + b^-(y) = M_B(b)(y) - b(y).$$

Then for any ball B, we have

$$\frac{1}{\mu(B)}\int\limits_B b^-(y)d\mu(y) \le \frac{1}{\mu(B)}\int\limits_B \big(M_B(b)(y)-b(y)\big)d\mu(y)$$
$$= \frac{1}{\mu(B)}\int\limits_B \big|b(y)-M_B(b)(y)\big|d\mu(y) \le C.$$

Let $\mu(B)\to 0$ with $x\in B$. Lebesgue's differentiation theorem assures that

$$0 \le b^-(x) = \lim_{\mu(B)\to 0}\frac{1}{\mu(B)}\int\limits_B b^-(y)d\mu(y) \le C.$$

Thus $b^-\in L^\infty(X)$.

(3) $\Rightarrow$ (4). We deduce (4.3) from (4.2). Assume (4.2) holds, then for any fixed balls B, it follows from Lemma 2.3 that

$$\mu(B)^{-1}\left\|b-\mu(B)^{-\eta}M_{\eta,B}(b)\right\|_{L^1(B)}$$
$$\le C\,\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|b-\mu(B)^{-\eta}M_{\eta,B}(b)\big\|_{L^\Psi(B)} \le C,$$

where the constant C is independent of B. So, we obtain (4.3). □

Theorem 4.3. *Let $0\le\eta<1$, $b\in L^1_{\rm loc}(X)$, Φ,Ψ be the Young functions with $\Phi\in\nabla_2\cap\mathcal{Y}$ and $\Psi^{-1}\big(\mu(B)^{-1}\big)\approx\mu(B)^\eta\,\Phi^{-1}\big(\mu(B)^{-1}\big)$. Then the following statements are equivalent:*

1. *$b\in BMO(X)$ and $b^-\in L^\infty(X)$.*
2. *$[b,M_\eta]$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.*
3. *There exists $\Phi\in\Delta_2$ such that*

$$\sup_B \Psi^{-1}\big(\mu(B)^{-1}\big)\,\|b-M_B(b)\|_{L^\Psi(B)}<\infty. \tag{4.5}$$

4. *There exists a constant $C>0$ such that*

$$\sup_B \mu(B)^{-1}\,\|b-M_B(b)\|_{L^1(B)}\le C. \tag{4.6}$$

Proof. Since the implication "(1) $\Rightarrow$ (2)" follows readily by Corollary 4.1 and the equivalence of (1) and (4) follows from Lemma 3.10, we only need to prove the implications "(2) $\Rightarrow$ (3)" and "(3) $\Rightarrow$ (4)".

(2) $\Rightarrow$ (3). We divide the proof into two cases according to the range of η.

Case 1. Assume $\eta=0$. For any fixed ball B and $x\in B$, we have

$$b(x)-M_B(b)(x)=b(x)M\big(\chi_B\big)(x)-M\big(b\chi_B\big)(x)=[b,M]\big(\chi_B\big)(x).$$

Since in this case we assume $\Psi^{-1}\big(\mu(B)^{-1}\big)\approx\mu(B)^0\,\Phi^{-1}\big(\mu(B)^{-1}\big)=\Phi^{-1}\big(\mu(B)^{-1}\big)$ and $[b,M]$ is bounded from $L^\Psi(X)$ to $L^\Psi(X)$, therefore by (2.2), we have

$$\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|b-M_B(b)\big\|_{L^\Psi(B)}=\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|[b,M]\big(\chi_B\big)\big\|_{L^\Psi(B)}$$
$$\le C\,\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|\chi_B\big\|_{L^\Psi(B)}=C,$$

which implies (4.5).

Case 2. Assume $0\le\eta<1$. For any fixed balls B,

$$\Psi^{-1}\big(\mu(B)^{-1}\big)\,\|b-M_B(b)\|_{L^\Psi(B)}\le\Psi^{-1}\big(\mu(B)^{-1}\big)\,\big\|b-\mu(B)^{-\eta}M_{\eta,B}(b)\big\|_{L^\Psi(B)}$$
$$+\,\Psi^{-1}\big(\mu(B)^{-1}\big)\,\big\|M_B(b)(\cdot)-\mu(B)^{-\eta}M_{\eta,B}(b)\big\|_{L^\Psi(B)}:=I_1+I_2. \tag{4.7}$$

First, we consider I_1. From (4.4), we get

$$I_1=\Psi^{-1}\big(\mu(B)^{-1}\big)\,\big\|b-\mu(B)^{-\eta}M_{\eta,B}(b)\big\|_{L^\Psi(B)}\le C.$$

Next, we estimate I_2. For any $x\in B$, $M_B\big(\chi_B\big)(x)=\chi_B(x)$ (see [33]) and then $M\big(\chi_B\big)(x)=\chi_B(x)$ and $M\big(b\chi_B\big)(x)=M_B(b)(x)$ for any $x\in B$. Then

$$\begin{aligned}
&\big|\mu(B)^{-\eta}M_{\eta,B}(b)(x)-M_B(b)(x)\big|=\mu(B)^{-\eta}\big|M_{\eta,B}(b)(x)-\mu(B)^{\eta}M_B(b)(x)\big|\\
&=\mu(B)^{-\eta}\big|M_\eta\big(b\chi_B\big)(x)-M_\eta\big(\chi_B\big)(x)M\big(b\chi_B\big)(x)\big|\\
&=\mu(B)^{-\eta}\big|M_\eta\big(b\chi_B\big)(x)-|b(x)|M_\eta\big(\chi_B\big)(x)\big|\\
&+\mu(B)^{-\eta}\big||b(x)|M_\eta\big(\chi_B\big)(x)-M_\eta\big(\chi_B\big)(x)M\big(b\chi_B\big)(x)\big|\\
&=\mu(B)^{-\eta}\big|M_\eta\big(|b|\chi_B\big)(x)-|b(x)|M_\eta\big(\chi_B\big)(x)\big|\\
&+\mu(B)^{-\eta}M_\eta\big(\chi_B\big)(x)\big||b(x)|M\big(\chi_B\big)(x)-M\big(b\chi_B\big)(x)\big|\\
&=\mu(B)^{-\eta}\big|[|b|,M_\eta]\big(\chi_B\big)(x)\big|+\big|[|b|,M]\big(\chi_B\big)(x)\big|.
\end{aligned}\tag{4.8}$$

Note that $b\in BMO(X)$ implies $|b|\in BMO(X)$.

From (4.8), for any $x\in B$, we obtain

$$\big|\mu(B)^{-\eta}M_{\eta,B}(b)(x)-M_B(b)(x)\big|\le\mu(B)^{-\eta}\big|[|b|,M_\eta]\big(\chi_B\big)(x)\big|+\big|[|b|,M]\big(\chi_B\big)(x)\big|.$$

Then it follows from (2.2) that

$$\begin{aligned}
I_2&=\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|\mu(B)^{-\eta}M_{\eta,B}(b)(\cdot)-M_B(b)(\cdot)\big\|_{L^\Psi(B)}\\
&\lesssim\Psi^{-1}\big(\mu(B)^{-1}\big)\mu(B)^{-\eta}\big\|[|b|,M_\eta]\big(\chi_B\big)\big\|_{L^\Psi(B)}+\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|[|b|,M]\big(\chi_B\big)\big\|_{L^\Psi(B)}\\
&\lesssim\|b\|_*\Psi^{-1}\big(\mu(B)^{-1}\big)\mu(B)^{-\eta}\,\|\chi_B\|_{L^\Phi}+\|b\|_*\Psi^{-1}\big(\mu(B)^{-1}\big)\,\|\chi_B\|_{L^\Psi}\\
&\lesssim\|b\|_*.
\end{aligned}\tag{4.9}$$

By (4.7), (4.4) and (4.9), we get

$$\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|b-M_B(b)\big\|_{L^\Psi(B)}\lesssim\|b\|_*,$$

which leads us to (4.5) since B is arbitrary.

(3) $\Rightarrow$ (4). We deduce (4.6) from (4.5). Assume (4.5) holds, then for any fixed balls B, it follows from Lemma 2.3 and (4.5) that

$$\mu(B)^{-1}\,\|b-M_B(b)\|_{L^1(B)}\le C\,\Psi^{-1}\big(\mu(B)^{-1}\big)\big\|b-M_B(b)\big\|_{L^\Psi(B)}\le C,$$

where the constant C is independent of B. So, we obtain (4.6).

The proof of Theorem 4.3 is completed. □

Corollary 4.4. *Let $b\in L^1_{\rm loc}(X)$, Φ be a Young function with $\Phi\in\nabla_2\cap\mathcal{Y}$. Then the following statements are equivalent:*

1. *$b\in BMO(X)$ and $b^-\in L^\infty(X)$.*
2. *$[b,M]$ is bounded on $L^\Phi(X)$.*
3. *There exists $\Phi\in\Delta_2$ such that*

$$\sup_B\Phi^{-1}\big(\mu(B)^{-1}\big)\,\|b-M_B(b)\|_{L^\Phi(B)}<\infty.$$

4. *There exists a constant $C>0$ such that*

$$\sup_B\mu(B)^{-1}\,\|b-M_B(b)\|_{L^1(B)}\le C.$$

Remark 4.5. Note that in the case $\Phi(t)=t^p$, Corollary 4.4 for the case $\Phi(t)=t^p$, was proved in [11, Theorem 2.1].

Theorem 4.6. *Let $0\le\eta<1$, $b\in L^1_{\rm loc}(X)$, Φ,Ψ be the Young functions with $\Phi\in\nabla_2\cap\mathcal{Y}$ and $\Psi^{-1}\big(\mu(B)^{-1}\big)\approx\mu(B)^\eta\,\Phi^{-1}\big(\mu(B)^{-1}\big)$. Then the following statements are equivalent:*

1. *$b\in BMO(X)$ and $b^-\in L^\infty(X)$.*
2. *$[b,M_\eta]$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$.*

3. *There exists a constant* $C > 0$ *such that*

$$\sup_B \Psi^{-1}\big(\mu(B)^{-1}\big) \, \|b - b_B\|_{L^\Psi(B)} \le C. \tag{4.10}$$

4. *There exists a constant* $C > 0$ *such that*

$$\sup_B \mu(B)^{-1} \, \|b - b_B\|_{L^1(B)} \le C. \tag{4.11}$$

Proof. Part "(1) $\Leftrightarrow$ (2)" follows from Theorem 4.3, the implication "(1) $\Rightarrow$ (4)" follows readily from [19, Theorem 4.5] and Lemma 3.10, respectively. Since "(3) $\Rightarrow$ (4)" follows directly from Lemma 3.10, it suffices to prove the implication "(2) $\Rightarrow$ (3)".

(2) $\Rightarrow$ (3).

For any given ball B, we have

$$\begin{aligned}|b(x) - b_B| &\le \frac{1}{\mu(B)} \int_B \Big| |b(x) - b(y)| d\mu(y) \\ &\le \frac{1}{\mu(B)^\eta} \frac{1}{\mu(B)^{1-\eta}} \int_B |b(x) - b(y)| \chi_B(y) d\mu(y) \le \mu(B)^{-\eta} M_{b,\eta}\big(\chi_B\big)(x)\end{aligned}$$

for all $x \in B$. Since $M_{b,\eta}$ is bounded from $L^\Phi(X)$ to $L^\Psi(X)$, by applying Lemma 3.8 and noting that $\Psi^{-1}\big(\mu(B)^{-1}\big) \approx \mu(B)^\eta \, \Phi^{-1}\big(\mu(B)^{-1}\big)$, we have

$$\begin{aligned}\Psi^{-1}\big(\mu(B)^{-1}\big) \, \|b - b_B\|_{L^\Psi(B)} &\le \mu(B)^{-\eta} \, \Psi^{-1}\big(\mu(B)^{-1}\big) \, \big\|M_{b,\eta}\big(\chi_B\big)(\cdot)\big\|_{L^\Psi(B)} \\ &\le \mu(B)^{-\eta} \, \Psi^{-1}\big(\mu(B)^{-1}\big) \, \|\chi_B\|_{L^\Phi(B)} = \frac{\Psi^{-1}\big(\mu(B)^{-1}\big)}{\mu(B)^\eta \, \Phi^{-1}\big(\mu(B)^{-1}\big)} \le C\end{aligned}$$

which leads us to (4.10) since B is arbitrary and the constant C does not depend on B. □

Corollary 4.7. *Let* $b \in L^1_{\rm loc}(X)$, Φ *be a Young function with* $\Phi \in \nabla_2 \cap \mathcal{Y}$. *Then the following statements are equivalent:*

1. $b \in BMO(X)$ *and* $b^- \in L^\infty(X)$.
2. $[b, M]$ *is bounded on* $L^\Phi(X)$.
3. *There exists a constant* $C > 0$ *such that*

$$\sup_B \Phi^{-1}\big(\mu(B)^{-1}\big) \, \|b - b_B\|_{L^\Phi(B)} \le C.$$

4. *There exists a constant* $C > 0$ *such that*

$$\sup_B \mu(B)^{-1} \, \|b - b_B\|_{L^1(B)} \le C.$$

Remark 4.8. Note that in the case of Carnot groups Theorems 4.3 and 4.6 were proved in [16].

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