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THE REGULARITY OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS IN ORLICZ–MORREY SPACES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The obtained estimates in the generalized Orlicz–Morrey spaces are used to study the global regularity of the solution of the Dirichlet problem for nonlinear elliptic equations in divergence form over a bounded non-smooth domain. Towards this end, we apply the Calderon–Zygmund theory.

1. Introduction

In order to study the local behavior of solutions to elliptic and parabolic partial differential equations introduced in [32], we introduce the classical Morrey spaces $L_{p,\lambda}$. There is the inclusion between the Morrey and Hölder spaces that permits to obtain the regularity of solutions to elliptic and parabolic boundary problems. In [9], Chiarenza and Frasca show the boundedness of the Hardy–Littlewood maximal operator in $L_{p,\lambda}(R^n)$ that allows them to prove the continuity of fractional and classical Calderon–Zygmund operators in Morrey spaces. Calderon–Zygmund operators appear in the representation formulae of the solutions of elliptic and parabolic equations. Thus the continuity of Calderon–Zygmund integrals implies the regularity of the solutions in the corresponding spaces. In [31], Mizuhara gives a generalization of Morrey spaces by considering a function $\omega(x,r) : R^n \times R_+ \to R_+$ instead of r^λ. He studied the continuity in those spaces of some integral operators. In [33], Nakai extends the result of Chiarenza and Frasca to these type spaces by imposing certain integral and doubling conditions on ω. Taking the weight $\omega = \varphi^p r^n$, the Mizuhara-Nakai conditions become

$$\int_r^\infty \varphi^p(x,t)\frac{dt}{t} \le C\varphi^p(x,r), \qquad C^{-1} \le \frac{\varphi(x,t)}{\varphi(x,r)} \le C, \qquad \forall r \le t \le 2r,$$

where the constants do not depend on t, r and $x \in R^n$.

In [18], Guliev studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators. He extends the result of Nakai to the Morrey type spaces with the weight $\omega = \varphi r^n$. This result is given by the following

Theorem 1.1. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty \varphi_1(x,r)\frac{dr}{r} \le C\varphi_2(x,t), \tag{1.1}$$

where C does not depend on x and t. Then the maximal operator M and Calderon–Zygmund integral operators K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to the weak space WM_{p,φ_2}.

Later, this result was extended to the spaces with a weaker condition on the pair (φ_1, φ_2) (see [21]). For more recent results on the boundedness and continuity of singular integral operators in generalized Morrey and fractional spaces and their application in the theory of different order partial differential equations see $[1, 3, 4, 11, 15, 16, 23–26, 30, 34, 36, 38, 41, 42, 46]$.

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In this paper, we consider a nonlinear elliptic equation in divergence form in a bounded non-smooth domain in Orlicz–Morrey spaces. The problem for nondivergence second order linear elliptic equations with VMO coefficients was treated in [20], and for higher order linear elliptic equations it was considered in [16].

We also recall papers [2, 5, 8, 10, 11, 13, 14, 19, 22].

2. Preliminaries on Orlicz and Orlicz–Morrey Spaces

Definition 2.1. A function $\Phi : [0, +\infty] \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(\infty) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, +\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \leq C_a\Phi(t)$ for all $t > 0$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0,$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition, but does not satisfy the ∇_2 - condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2 - condition, but does not satisfy the Δ_2-condition.

The following two indices

$$q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \qquad p_\Phi = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}$$

of Φ, where $\varphi(t)$ is the right-continuous derivative of Φ, are well known in the theory of Orlicz spaces. As is known,

$$p_\Phi < \infty \Leftrightarrow \Phi \in \Delta_2,$$

and the function Φ is strictly convex if and only if $q_\Phi > 1$. If $0 < q_\Phi \leq p_\Phi < \infty$, then $\frac{\Phi(t)}{t^{q_\Phi}}$ is increasing and $\frac{\Phi(t)}{t^{p_\Phi}}$ is decreasing on $(0, \infty)$.

Next, we define the lower index of Φ denoted by $i(\Phi)$ as follows:

$$i(\Phi) = \lim_{\lambda\to+0} \frac{\log(h_\Phi(\lambda))}{\log\lambda} = \sup_{0<\lambda<1} \frac{\log(h_\Phi(\lambda))}{\log\lambda},$$

where

$$h_\Phi(\lambda) = \sup_{r>0} \frac{\Phi(\lambda r)}{\Phi(r)}, \quad \lambda > 0.$$

For example, $i(\Phi) = q$ if $\Phi(r) = r^q$ with $q > 1$. In addition, the $\Delta_2 \cap \nabla_2$ - condition ensures that the Young function increases moderately. That is, there are two constants q_0 and q_1 with $1 < q_0 \leq q_1 < \infty$ such that

$$\frac{1}{c}\min\{\lambda^{q_0}, \lambda^{q_1}\}\Phi(r) \leq \Phi(\lambda r) \leq c\max\{\lambda^{q_0}, \lambda^{q_1}\}\Phi(r), \quad \lambda, r \geq 0, \tag{2.1}$$

where the constant c is independent of λ and r. In fact, the index number $i(\Phi)$ is equal to the supremum of q_0 satisfying (2.1). Let $\Phi \in \Delta_2 \cap \nabla_2$, $1 < i(\Phi) < \infty$. We note that this condition necessity for the type of regularity considered here (see [44]).

Lemma 2.1 ([27, Lemma 1.3.2]). *Let $\Phi \in \Delta_2$. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq b\frac{\Phi(t_1)}{t_1^p} \quad \textit{for} \quad 0 < t_1 < t_2.$$

Lemma 2.2 ([40, Proposition 62.20]). *Let Φ be a Young function with a canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, \quad t \geq 0.$$

(1) *Assume that* $\Phi \in \Delta_2$. *More precisely.* $\Phi(2t) \le A\Phi(t)$ *for some* $A \ge 2$. *If* $p > 1 + \log_2 A$, *then*

$$\int_1^t \frac{\varphi(s)}{s^p} ds \le C \frac{\Phi(t)}{t^p}, \quad t > 0.$$

(2) *Assume that* $\Phi \in \nabla_2$. *Then*

$$\int_0^t \frac{\varphi(s)}{s} ds \le C \frac{\Phi(t)}{t}, \quad t > 0.$$

Recall that a function Φ is said to be quasi-convex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \le \Phi(t) \le c\omega(ct), \qquad t \in [0, \infty).$$

Let Y be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty. \tag{2.2}$$

If $\Phi \in Y$, then Φ is absolutely continuous on every closed interval in $[0, +\infty)$ and bijective from $[0, +\infty)$ to itself .

Definition 2.2 (Orlicz space)**.** For a Young function Φ, the set

$$L_\Phi(R^n) = \left\{ f \in L_1^{\text{loc}}(R^n) : \int_{R^n} \Phi(k|f(x)|)dx < +\infty \ \text{ for some } \ k > 0 \right\}$$

is called Orlicz space. The space $L_\Phi^{\text{loc}}(R^n)$ endowed with the natural topology is defined as the set of all functions f such that $f_{\chi_B} \in L_\Phi(R^n)$ for all balls $B \subset R_n$.

Note that $L_\Phi(R^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf \left\{ \lambda > 0 : \int_{R^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\},$$

(see, for example, [39, Section 3, Theorem 10]) so,

$$\int_{R^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi}}\right) dx \le 1.$$

For a measurable set $\Omega \subset R^n$, a measurable function f and $t > 0$, let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = R^n$, we shortly denote it by $m(f, t)$.

Definition 2.3. The weak Orlicz space

$$WL_\Phi(R^n) = \left\{ f \in L^1_{\text{loc}}(R^n) : \|f\|_{WL_\Phi} < +\infty \right\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf \left\{ \lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right) \le 1 \right\}.$$

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \ \ (\inf \emptyset = +\infty).$$

If $\Phi \in Y$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \text{ for } 0 \le r < +\infty.$$

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty), \\ +\infty, & r = +\infty. \end{cases} \tag{2.3}$$

The complementary function $\widetilde{\Phi}(r)$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$, and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\Phi(r) = \frac{r^p}{p}$, then $\widetilde{\Phi}(r) = \frac{r^{p'}}{p'}$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$.

It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r, \quad \text{for} \quad r \ge 0.$$

Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$.

The following analogue of the Hölder inequality is known (see [45]).

Lemma 2.3 ([45])**.** *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality*

$$\|fg\|_{L_1(R^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}$$

is valid.

Note that Young functions satisfy the property

$$\Phi(\alpha t) \le \alpha \Phi(t),$$

for all $0 < \alpha < 1$ and $0 \le t < \infty$, which is a consequence of the convexity: $\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)t) \le \alpha\Phi(t) + (1-\alpha)\Phi(t) = \alpha\Phi(t)$.

Definition 2.4. (Generalized Orlicz–Morrey space). Let $\varphi(x,r)$ be a positive measurable function on $R^n \times (0,\infty)$ and Φ be any Young function. We denote by $M_{\Phi,\varphi}(R^n)$ the generalized Orlicz–Morrey space, the space of all functions $f \in L_\Phi^{\mathrm{loc}}(R^n)$ with finite quasi-norm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in R^n, r>0} \varphi^{-1}(x,r)\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L_\Phi(B(x,r))}.$$

Also, by $WM_{\Phi,\varphi}(R^n)$ we denote the weak generalized Orlicz–Morrey space of all functions $f \in WL_\Phi^{\mathrm{loc}}(R^n)$ for which

$$\|f\|_{WM_{\Phi,\varphi}} = \sup_{x\in R^n, r>0} \varphi^{-1}(x,r)\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{WL_\Phi(B(x,r))} < \infty,$$

where $WL_\Phi(B(x,r))$ denotes the weak L_Φ space of measurable functions f for which

$$\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(R^n)}.$$

According to this definition,we recover the spaces $M_{p,\varphi}$ and $WM_{\Phi,\varphi}$ under the choise $\Phi(r) = r^p$

$$M_{p,\varphi} = M_{\Phi,\varphi}|_{\Phi(r)=r^p}, WM_{\Phi,\lambda} = WM_{\Phi,\varphi}|_{\Phi(r)=r^p}.$$

We give an assumption to the domain $\Omega \subset R^n$, $n \ge 2$. For a measure of deviation of $\partial\Omega$ from being an $(n-1)$ -dimensional affine space for each scale $\rho > 0$, we use the following so-called "Reifenberg flatness".

Definition 2.5. A bounded domain Ω is said to be (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $\rho \in (0, R]$, there exists a coordinate system $\{y_1, \ldots, y_n\}$, which may depend on ρ and x such that $x = 0$ in this coordinate system and that

$$B_\rho(0) \cap \{y_n > \delta\rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta\rho\}, \tag{2.4}$$

where $B_\rho(y) = \{x \in R^n : |x-y| < \rho\}$ denotes the open ball on R^n centered at $y \in R^n$, of radius $\rho > 0$. Next, $|E|$ denotes the n-dimensional Lebesgue measure of a set $E \subset R^n$.

The above definition warrants a few comments. Since our main problem has a scaling invariance property, the constant R can be taken as 1, or as any other constant, greater than 1. However, the constant δ is small positive and still invariant under such a scaling. In fact, the Reifenberg flatness (2.4) is meaningful when $0 < \delta < \frac{1}{8}$ (see [43]) and with such small δ, these flatness conditions imply that the deviation of $\partial\Omega$ from being an $(n-1)$-dimensional affine space is small enough for each scale $\rho > 0$. By (2.4), for all $y \in \Omega$ and $\rho \in (0, R)$, we obtain the following measure density condition:

$$|\Omega \cap B_\rho(y)| \ge \left(\frac{1-\delta}{2}\right)^n |B_\rho(y)| \ge \left(\frac{3}{10}\right)^n |B_\rho(y)|.$$

We give the definitions of functional spaces to which the coefficients and the data of the problem belong.

Definition 2.6. Let $\varphi : \Omega \times R_+ \to R_+$ be a measurable function and $1 \le p < \infty$. The generalized Orlicz–Morrey space $M_{\Phi,\varphi}(\Omega)$ consists of all $f \in L_\Phi^{\mathrm{loc}}(\Omega)$,

$$\|f\|_{M_{\Phi,\varphi}(\Omega)} = \sup_{x\in\Omega, r>0} \varphi^{-1}(x,r)\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L_\Phi(\Omega\cap B(x,r))}.$$

For any bounded domain Ω, we define $M_{\Phi,\varphi}(\Omega)$ taking $f \in L_\Phi(\Omega)$ and Ω_r instead of $B(x,r)$ in the norm above.

The generalized Sobolev–Orlicz–Morrey space $W_{2,\Phi,\varphi}(\Omega)$ consists of all Sobolev functions $u \in W_{2,\Phi}(\Omega)$ with distributional derivatives $D^s u \in M_{\Phi,\varphi}(\Omega)$, endowed with the norm

$$\|u\|_{W_{2,\Phi,\varphi}(\Omega)} = \sum_{0\le s\le 2} \|D^s u\|_{M_{\Phi,\varphi}(\Omega)}.$$

The space $W_{2,\Phi,\varphi}(\Omega) \cap W^0_{1,\varphi}$ consists of all functions $u \in W_{2,\Phi}(\Omega) \cap \overset{\circ}{W}_{1,\Phi}(\Omega)$ with $D^s u \in M_{\Phi,\varphi}(\Omega)$, and is endowed with the same norm. Recall that $\overset{\circ}{W}_{1,\varphi}$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W_{1,\varphi}$.

Also, we can give a definition of the generalized weak Morrey space $WM_{\Phi,\varphi}(\Omega)$.

Let the nonlinearity $a = a(x,\xi) : R^n \to R^n \times R^n$ be measurable in x for all $\xi \in R^n$ and continuous in ξ for almost all $x \in R^n$. We give the regularity assumption on the nonlinearity $a(x,\xi)$. First, we set

$$\theta(a; B_\rho(y))(x) = \sup_{\xi\in R^n\ \{0\}} \frac{|a(x,\xi) - \bar{a}_{B_\rho(y)}(\xi)|}{|\xi|},$$

where

$$\bar{a}_{B_\rho(y)}(\xi) = \int\limits_{B_\rho(y)} a(x,\xi)dx = \frac{1}{|B_\rho(y)|}\int\limits_{B_\rho(y)} a(x,\xi)dx$$

is the integral average of $a(x,\xi)$ in the variable x over $B_\rho(y)$ for fixed $\xi \in R^n$. The function $\theta(a; B_\rho(y))$ provides the measurement of the oscillation of $\frac{a(x,\xi)}{|\xi|}$ in the variable x over $B_\rho(y)$, uniformly in ξ.

Definition 2.7. A vector field a is said to be (δ, R)-vahishing if

$$\sup_{0<\rho\le R}\sup_{y\in R^n} \int\limits_{B_\rho(y)} \theta(a; B_\rho(y))(x)dx \le \delta.$$

3. Statement of the Problem

Let Ω be a bounded domain in R^n, $n \ge 2$, with its non-smooth boundary $\partial\Omega$. We suppose the domain Ω to be (δ, R)-Reifenberg flat. Let $f = f(x) : \Omega \to R^n$ be a given vector-valued function at least in $L^2(\Omega, R^n)$ and consider the following nonlinear elliptic equation

$$\begin{cases} \operatorname{div} a(x,Du) = \operatorname{div} f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where the nonlinearity $a(x,\xi)$ is as in Section 2. Here, assume the monotonicity and growth conditions on $a = a(x,\xi)$ as follows:

$$\begin{cases} c_0|\xi-\eta|^2 \le [a(x,\xi) - a(x,\eta)] \le (\xi - \eta) \\ |a(x,\xi)| + |D_\xi a(x,\xi)|\,|\xi| \le c_1|\xi|, \end{cases} \tag{3.2}$$

for all $\xi, \eta \in R^n$ and for almost every $x \in R^n$ and c_0, c_1 some positive constants.

We consider a weak solution $\overset{\circ}{W}{}^1_2(\Omega)$, which means that for any $\varphi \in \overset{\circ}{W}{}^1_2(\Omega)$, the integral formula

$$\int\limits_\Omega a(x,Du)D\varphi dx = \int\limits_\Omega fD\varphi dx$$

holds. The existence and uniqueness of a weak solution to problem (3.1) can be obtained by the Minty-Browder method for monotone operators (see [17, 28], also [29]) under the assumption $f \in L^2(\Omega, R^n)$, with the estimate

$$\||Du|^2\|_{L^1(\Omega,R^n)} \le C\||f|^2\|_{L^1(\Omega,R^n)},$$

the constant C is independent of u and f.

We are now ready to state the main result.

Theorem 3.1. *Let $u \in \overset{\circ}{W}{}^1_2(\Omega)$ be a weak solution of (3.1) and $|f|^2 \in M_{\Phi,\varphi}(\Omega)$ with $\Phi \in \Delta_2 \cap \nabla_2$ and $\varphi : \Omega \times R_+ \to R_+$ be a measurable function such that for all $x \in R^n$ and $r > 0$ satisfying*

$$\int_r^\infty \left(\sup_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}(s^{-n})} \right) \Phi^{-1}(t^{-n}) \frac{dt}{t} \le C\varphi(x,r). \tag{3.3}$$

Let there exist a small positive constant $\delta = \delta(c_0, c_1, n, \Phi, \varphi)$ such that if $a(x,\xi)$ is (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat, then $|Du|^2 \in M_{\Phi,\varphi}(\Omega)$ with the estimate

$$\||Du|^2\|_{M_{\Phi,\varphi}(\Omega)} \le C\||f|^2\|_{M_{\Phi,\varphi}(\Omega)}, \tag{3.4}$$

where the constant C depends on c_0, c_1, n, R, Φ, φ and Ω.

The following lemma ensures that for each $f(x)$ with $|f|^2 \in M_{\Phi,\varphi}(\Omega)$, problem (3.1) has a unique weak solution.

Lemma 3.2. *Let $\Phi \in \Delta_2 \cap \nabla_2$ and $\varphi : \Omega \times R_+ \to R_+$ be a measurable function. If $|f|^2 \in M_{\Phi,\varphi}(\Omega)$, then $|f|^2 \in L^1(\Omega)$, and*

$$\int_\Omega |f(x)|^2 dx \le C\left[\left(\int_\Omega \Phi(|f|^2) dx \right)^{\frac{1}{q_0}} + \left(\int_\Omega \Phi(|f|^2) dx \right)^{\frac{1}{q_1}} \right],$$

where q_0 and q_1 are defined in (2.1).

Proof. Set $F(x) = |f(x)|^2$. With a direct calculation,

$$\int_{\{\Omega:|F|\ge 1\}} |F(x)| dx \le \left(\int_{\{\Omega:|F|\ge 1\}} |F(x)|^{i(\Phi)-\varepsilon_0} dx \right)^{\frac{1}{i(\Phi)-\varepsilon_0}} \cdot |\Omega|.$$

By the property of Young's function Φ that

$$|F(x)|^{i(\Phi)-\varepsilon_0} \le \frac{C}{\Phi(1)} \Phi(|F(x)|) \quad \text{if} \quad |F(x)| \ge 1,$$

we have

$$\int_{\{\Omega:|F|\ge 1\}} |F(x)| dx \le C\left(\int_\Omega \Phi(|F(x)|) dx \right)^{\frac{1}{i(\Phi)-\varepsilon_0}}.$$

On the other hand, it follows from $i(\Phi) \le q_1$ that in view of (2.1),

$$|F(x)|^{q_1} \le \frac{C}{\Phi(1)} \Phi(|F(x)|) \quad \text{if} \quad |F(x)| \ge 1,$$

and so,

$$\int_{\{\Omega:|F|\le 1\}} |F(x)| dx \le C\left(\int_\Omega \Phi(|F(x)|) dx \right)^{\frac{1}{q_1}}.$$

Since ε_0 is small enough, we get

$$\int_\Omega |F(x)| dx \le C\left[\left(\int_\Omega \Phi(|F(x)|) dx \right)^{\frac{1}{q_0}} + \left(\int_\Omega \Phi(|F(x)|) dx \right)^{\frac{1}{q_1}} \right].$$

Thus the lemma is complete. □

4. Auxiliary Results

We present the following invariance property under normalization and scaling, some result needed from the measure theory in the Orlicz–Morrey space and one version of the Calderon–Zygmund type covering lemma. These results will be used to prove the main theorem.

Lemma 4.1. *Let* $u \in \overset{\circ}{W}{}^1_2(\Omega)$ *be a weak solution to problem* (3.1), *the nonlinearity* $a(x,\xi)$ *satisfy* (3.2), *and also, let* (δ, R) *be vanishing. For each* $\lambda > 1$ *and* $0 < r < 1$, *define the rescaled maps*

$$\tilde{a}(x,\xi) = \frac{a(rx, \lambda\xi)}{\lambda},$$

$$\widetilde{\Omega} = \Big\{\frac{1}{r}x : x \in \Omega\Big\}, \qquad \tilde{u}(x) = \frac{u(rx)}{\lambda r},$$

$$\tilde{f}(x) = \frac{f(rx)}{\lambda_0}.$$

Then:

(1) $\tilde{u} \in \overset{\circ}{W}{}^1_2(\tilde{\Omega})$ *is the weak solution of*

$$\operatorname{div} \tilde{a}(x, D\tilde{u}) = \operatorname{div} \tilde{f} \ \text{ in } \ \tilde{\Omega};$$

(2) $\tilde{a}(x.\xi)$ *satisfies the conditions* (3.2) *with the same constants* c_0 *and* c_1*;*

(3) $\tilde{a}$ *is* $\big(\delta, \frac{R}{r}\big)$*-vanishing and* $\tilde{\Omega}$ *is* $\big(\delta, \frac{R}{r}\big)$*- Reifenberg flat.*

Proof. The proof follows by direct computations (see also, for example, [7]).

Now, we give the Hardy–Littlewood maximal function and its basic properties. Let g be a locally integrable function on R^n. Then the Hardy–Littlewood maximal function is given by

$$(My)(x) = \sup_{\rho>0} \int\limits_{B_\rho(x)} |g(y)|dy = \sup_{\rho>0} \frac{1}{|B_\rho(x)|} \int\limits_{B_\rho(x)} |g(y)|dy.$$

If g is defined only on a bounded subset of R^n, then we define it as $Mg = M\bar{g}$, where $\bar{g}$ is the zero extension of g in R^n. This maximal function holds for the so-called weak (1.1) inequality. More specifically, there exists a positive constant $c = c(n)$ such that for any $\lambda > 0$,

$$|\{x \in R^n : (Mg)(x) > \lambda\}| \le \frac{C}{\lambda} \int\limits_{R^n} |g(x)|dx. \tag{4.1}$$

The proof is complete. □

We next state that the Hardy–Littlewood maximal operators are bounded from the Orlicz–Morrey space $M_{\Phi,\varphi}(R^n)$ to themselves.

Lemma 4.2 (see [20])**.** *Assume that there is a positive constant* C *such that for any fixed* $x \in R^n$, $r > 0$, $\Phi \in \Delta_2 \cap \nabla_2$, *the inequality*

$$\frac{\sup\limits_{s<\sigma<\infty} \varphi(B_\sigma(x))\sigma^{\frac{n}{p}}}{S^{\frac{n}{p}}} < C\varphi(B_r(x)) \tag{4.2}$$

holds. Then there is a constant $C_p > 0$ *such that*

$$\|f\|_{M_{\Phi,\varphi}(R^n)} \le \|Mf\|_{M_{\Phi,\varphi}(R^n)} \le C_p \|f\|_{M_{\Phi,\varphi}(R^n)},$$

$\forall f \in M_{\Phi,\varphi}(R^n)$ *with compact support in* R^n.

We use the following version of the Vitali covering Lemma.

Lemma 4.3 (see [12])**.** *Assume that* Ω *is a* $(\delta, 1)$*-Reifenberg flat domain for some small* $\delta > 0$ *and a Young function* $\Phi \in \Delta_2 \cap \nabla_2$. *Let* E *and* D *be measurable sets with* $E \subset D \subset \Omega$. *Suppose that there exists small* $\varepsilon > 0$ *such that:*

(1) *for any* $y \in \Omega$, $\ |E \cap B_1(y)| < \varepsilon |B_1(y)|$;

(2) *for each* $y \in \Omega$ *and* $r \in (0,1)$, *if* $|E \cap B_r(y)| \geq \varepsilon |B_r(y)|$, *then* $B_r(y) \cap \Omega \subset D$.

Then

$$|E| \leq C \cdot \varepsilon |D|,$$

where the constant C *depends only on* n, Φ, *and the constant* $\dfrac{1}{1-\delta}$.

Proof. The proof of this lemma can also be found in [6, Lemma 5.4], or [37, Lemma 3.4] with a slight modification. □

We also use the following standard arguments of the measure theory.

Lemma 4.4 (see [20])**.** *Let a Young function* $\Phi \in \Delta_2 \cap \nabla_2$, f *be a nonnegative and measurable function defined on a bounded domain* $\Omega \subset R^n$ *and* φ *be a weight satisfying* (4.2) *and, in addition, a kind of monotonicity condition*

$$\varphi(B_r(y))r^n \leq \varphi(B_s(z))s^n \quad \text{for all} \quad B_r(y) \subset B_s(z,), \tag{4.3}$$

and let $\theta > 0$, $\lambda > 1$ *be the constants. Then* $f \in M_{\Phi,\varphi}(\Omega)$ *if and only if*

$$= \sup_{y \in \Omega, r>0} \sum_{k \geq 1} \frac{\Phi(\lambda^k)|\{x \in \Omega : f(x) > \theta\lambda^k\}|}{\varphi(B_r(y))r^n} < \infty$$

and

$$\frac{1}{C} S \leq \|f\|_{M_{\Phi,\varphi}(\Omega)} \leq C(1+\delta),$$

where $C > 0$ *is a constant, depending only on* $\theta, \lambda, \varphi, \Phi$.

Lemma 4.5. *Assume that* $u \in \overset{\circ}{W}{}_2^1(\Omega)$ *is the weak solution of* (3.1). *Then there exists a constant* $N = N(c_0, c_1, n) > 1$ *such that for each* $\varepsilon \in (0,1)$ *fixed, one can select small* $\delta = \delta(\varepsilon, c_0, c_1, n, \Phi, \varphi) \in \left(0, \frac{1}{8}\right)$ *such that if* $a(x,\xi)$ *is* $(\delta,1)$*-vanishing,* Ω *is* $(\delta,1)$*-Reifenberg flat, and if for* $0 < r < 1$ *and* $y \in \Omega$, $B_r(y)$ *satisfies*

$$|\{x \in \Omega : M(|Du|^2) > N^2\} \cap B_r(y)| \geq C|B_r(y)|.$$

Then we have

$$B_r(y) \cap \Omega \subset \{x \in \Omega : M(|Du|^2) > 1\} \cup \{x \in \Omega : M(|f|^2) > \delta^2\}.$$

Proof. The proof of this lemma is based on the same method as in the proof in [29, Theorem 4.10]. □

5. Proof the Main Theorem

Now, we are ready to prove the main theorem.

Proof of Theorem 3.1. By Lemma 4.1, it suffices to prove that

$$\||Du|^2\|_{M_{\Phi,\varphi}(\Omega)} \leq C, \tag{5.1}$$

under the assumption $\||f|^2\|_{M_{\Phi,\varphi}(\Omega)} \leq \delta^2$.

We take

$$u_1 = \frac{\delta u}{\sqrt{\||f|^2\|_{M_{\Phi,\varphi}(\Omega)} + \sigma}}, \qquad f_1 = \frac{\delta f}{\sqrt{\||f|^2\|_{M_{\Phi,\varphi}(\Omega)} + \sigma}};$$

in place of u and f, respectively, in problem (3.1), the estimate

$$\begin{aligned} &\frac{1}{c} \min\{\|g\|^{q_0}_{M_{\Phi,\varphi}(\Omega)}, \|g\|^{q_1}_{M_{\Phi,\varphi}(\Omega)}\} \\ &\leq \int_\Omega \Phi(|g(x)|)dx \leq c \max\{\|g\|^{q_0}_{M_{\Phi,\varphi}(\Omega)}, \|g\|^{q_1}_{M_{\Phi,\varphi}(\Omega)}\} \end{aligned} \tag{5.2}$$

holds.

Estimate (5.2) follows by the convexity of Φ and owing to estimate (2.1).

It follows from Lemma 3.2 and estimate (5.2) that

$$\||f_1|^2\|_{M_{\Phi,\varphi}(\Omega)} \le \delta^2, \qquad \int_\Omega |f_1|^2 dx \le C\delta^{2\tau_2}, \tag{5.3}$$

where $\tau_2 = \frac{q_0}{q_1}$. Therefore if (5.1) is obtained with Du_1 instead of Du, then the proof is completed after letting $\sigma \to 0$. However, in view of (5.2) and Lemma 4.2,

$$\begin{aligned}&\||Du|^2\|^{\alpha}_{M_{\Phi,\varphi}(\Omega)}\\ &\le C\int_\Omega \Phi(|Du|^2)dx \le C\int_\Omega \Phi(M(|Du|^2))dx,\end{aligned}$$

for some $\alpha > 0$. Consequently, it suffices to show that by Lemma 4.4,

$$S = \sup_{y\in\Omega, r>0} \sum_{k\ge 1} \frac{\Phi(N^{2k})|\{x\in\Omega : M(|Du|^2) > N^{2k}\}|}{\varphi(B_r(y))r^n} < \infty.$$

Now, we derive the decay estimates of the measure of the upper-level set $\{x \in \Omega : M(|Du|^2) > N^{2k}\}$ for $k = 1, 2, \ldots$. To apply Lemma 4.3, we first fix ε and take δ and N as given in Lemma 4.5. Then we define the sets

$$\begin{aligned}E &= \{x\in\Omega : M(|Du|^2) > N^2\},\\ D &= \{x\in\Omega : M(|Du|^2) > 1\} \cup \{x\in\Omega : M(|f|^2) > \delta^2\}.\end{aligned}$$

Check its hypotheses. It is clear that $E \subset D \subset \Omega$, and for each $y \in \Omega$,

$$\frac{|E\cap B_1(y)|}{|B_1(y)|} \le C\left(\frac{|E\cap B_1(y)|}{|B_1(y)|}\right)^{\tau_1} \stackrel{def}{=} A_1$$

for some constants $C > 1$ and $\tau_1 \in (0, 1)$ in case $E \subset B_1$. Here, the constants C and τ_1 depend only on n, p, but not on E and B_1. Then

$$\begin{aligned}&A_1 \le C|E|^{\tau_1} \le (\text{by estimate (4.1)})\\ &\le C\Big(\int_\Omega |Du|^2dx\Big)^{\tau_1} \le C\Big(\int_\Omega |f|^2dx\Big)^{\tau_1}\\ &\le (\text{by estimate(5.3)}) \le C\delta^{2\tau_1\tau_2} < \varepsilon,\end{aligned}$$

for δ small enough. Because the second condition of Lemma 4.3 is already checked in Lemma 4.5, we have

$$\begin{aligned}&|\{x\in\Omega : M(|Du|^2) > N^{2k}\}|\\ &\le C\varepsilon|\{x\in\Omega : M(|Du|^2) > 1\}| + C\varepsilon|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|.\end{aligned} \tag{5.4}$$

On the other hand, the main problem (3.1) has the invariance property from the normalization Lemma 4.1 and therefore the same result (5.4) can be obtained for $\left(\frac{u}{N}, \frac{f}{N}\right), \left(\frac{u}{N^2}, \frac{f}{N^2}\right), \ldots$, inductively.

After this iteration, for $k = 1, 2, \ldots$, we obtain the following power decay estimates:

$$\begin{aligned}&|\{x\in\Omega : M(|Du|^2) > N^{2k}\}|\\ &\le \varepsilon_1^k|\{x\in\Omega : M(|Du|^2) > 1\}| + \sum_{i=1}^{k} \varepsilon_1^i|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|,\end{aligned}$$

where $\varepsilon_1 = C\varepsilon$. Then a direct computation yields

$$\begin{aligned} S &= \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{\Phi(N^{2k})|\{x\in\Omega : M(|Du|^2) > N^{2k}\}|}{\varphi(B_r(y))r^n} \\ &\leq \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{\Phi(N^{2k})\varepsilon_1^k|\{x\in\Omega : M(|Du|^2) > 1\}|}{\varphi(B_r(y))r^n} \\ &+ \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{\Phi(N^{2k})\sum_{i=1}^{k}\varepsilon_1^i|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\ &= S_1 + S_2. \end{aligned}$$

Recall the property of $\Phi \in \Delta_2$. There exists a constant ν_1, depending only on Φ, φ and N such that $\Phi(N^2) \leq \nu_1\Phi(1)$, and therefore

$$\Phi(N^{2k}) \leq \nu_1^k \Phi(1),$$

for $k = 1, 2, \ldots$. We estimate S_1 and S_2 as follows:

$$S_1 \leq \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{(\Phi(1)\nu_1^k\varepsilon_1^k|\Omega|)}{\varphi(B_r(y))r^n} \leq C\sum_{k\geq 1}(\nu_1\varepsilon_1)^k.$$

$$\begin{aligned} S_2 &= \sup_{y\in\Omega, r>0} \sum_{k\geq 1} \frac{\Phi(N^{2(k-i)}N^{2i})\sum_{i=1}^{k}\varepsilon_1^i|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\ &\leq \sup_{y\in\Omega, r>0} \sum_{i\geq 1}\sum_{k\geq i} \frac{\Phi(N^{2(k-i)})\nu_1^i\varepsilon_1^i|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\ &\leq C\sum_{i\geq 1}(\nu_1\varepsilon_1)^i \sum_{k\geq i} \frac{\Phi(N^{2(k-i)})|\{x\in\Omega : M(|f|^2) > \delta^2 N^{2(k-i)}\}|}{\varphi(B_r(y))r^n} \\ &\leq C\sum_{i\geq 1}(\nu_1\varepsilon_1)^i \sum_{j\geq 0} \frac{\Phi(N^{2j})|\{x\in\Omega : M(\left|\frac{f}{\delta}\right|^2) > N^{2j}\}|}{\varphi(B_r(y))r^n} \\ &\leq \text{ by Lemma 4.4} \leq C\sum_{i\geq 1}(\nu_1\varepsilon_1)^i \int_\Omega \Phi\Big(\left|\frac{f}{\delta}\right|^2\Big)dx \\ &\leq \text{ by Lemma 4.2 and inequality (5.2)} \leq C\sum_{i\geq 1}(\nu_1\varepsilon_1)^i \left\|\frac{|f|^2}{\delta^2}\right\|_{M_{\Phi,\varphi}(\Omega)}^{q_0} \\ &\leq \text{ by inequality (5.1)} \leq C\sum_{i\geq 1}(\nu_1\varepsilon_1)^i. \end{aligned}$$

Therefore

$$S \leq C\sum_{k\geq 1}(\nu_1\varepsilon_1)^k$$

where $\varepsilon_1 = C\varepsilon$ as in Lemma 4.3. First, taking sufficiently small $\varepsilon > 0$, we get

$$\nu_1\varepsilon_1 < 1.$$

Then one can select correspondingly small $\delta = \delta(c_0, c_1, n, \Phi, \varphi) > 0$ from Lemma 4.5. This completes the proof. □

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