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STEADY VIBRATION PROBLEMS IN THE COUPLED THEORY OF VISCOELASTICITY FOR MATERIALS WITH TRIPLE POROSITY

MAIA M. SVANADZE

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this paper, the coupled linear theory of viscoelasticity for materials with a triple porosity is developed and the basic boundary value problems (BVPs) of steady vibrations are investigated. The governing systems of equations of motion and steady vibrations are presented, the fundamental solution of the system of steady vibration equations is constructed and the basic properties of the potentials (surface and volume) are given. Green's identities for bounded and unbounded domains are established and the existence and uniqueness theorems for classical solutions of the foregoing BVPs are proved by using the potential method and the theory of singular integral equations.

1. Introduction

Determining the mechanical properties of materials, subject to the influence of viscosity, is one of the most important issues of engineering, technology and continuum mechanics (see, e.g., Amendola et al. [1], Lakes [20], Brinson and Brinson [3]). Viscosity effects are observed in natural and manufactured materials such as: polymers, metals, alloys, rock, wood, soil, piezoelectric and biological materials.

The foundations of the theory of viscoelasticity are presented in the papers by Coleman and Noll [8], Gurtin and Sternberg [12]. Moreover, the basic classical models of viscoelastic materials are analyzed and several mathematical problems are studied in the books by Flügge [11], Christensen [7], Pipkin [23], Fabrizio and Morro [10].

In [2], M.A. Biot established the basic equations of the deformation of a viscoelastic porous solid under the most general assumptions of anisotropy. Since this work, many research papers have been published involving the theory of viscoelasticity of porous materials. Namely, in the last two decades, Biot's theory has been generalized by Schanz and Cheng [24] to the poroviscoelasticity by applying the elastic-viscoelastic correspondence principle in the Laplace domain. A visco-poroelastic theory for polymeric gels is developed by Wang and Hong [34]. In [27, 28], the quasi-static and steady vibration problems of the linear theory of viscoelasticity for Kelvin–Voigt materials with double porosity are investigated by the potential method (boundary integral equation method). The theories of viscoelastic and thermoviscoelastic porous mixtures are presented by Ieşan [13], Ieşan and Quintanilla [17]. The steady vibration problems in these theories of porous mixtures are studied by Svanadze [29, 30].

Moreover, by virtue of the volume fraction concept, the theory of thermoviscoelasticity for Kelvin–Voigt materials with voids is developed by Ieşan [14]. Then, this theory was extended and the basic problems for materials with single and double voids were investigated in a series of papers (see, e.g., [4–6, 9, 15, 16, 18, 25, 26, 33] and references therein).

Recently, in [31], a mathematical model of viscoelastic single-porosity materials is presented in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is considered, and steady vibration problems of this model are also investigated. More recently, in [32], based on this coupled phenomenon of the two above concepts, a mathematical model of viscoelastic double-porosity materials was developed and the basic BVPs of steady vibrations of the model were studied. Note that the basic BVPs of quasi-static linear coupled theory of elasticity and thermoelasticity for single-porosity materials are considered by Mikelashvili [21, 22].

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In this paper, the coupled linear theory of viscoelasticity for Kelvin–Voigt materials with triple porosity is considered and the basic BVPs of steady vibrations are investigated. This work is articulated as follows. In Section 2, the governing equations of motion and steady vibrations in the theory under consideration are presented. The system of governing equations is expressed in terms of the displacement vector field, the changes of volume fractions for three (macro, meso and micro) levels of porosity, and the changes of the fluid pressures in these three pore networks. In Section 3, the fundamental solution of the system of steady vibration equations is constructed explicitly by means of elementary functions and its basic properties are established. In Section 4, the basic BVPs of steady vibrations are formulated and Green's identities are established. Then, in Section 5, the uniqueness theorems for the regular (classical) solutions of the above-mentioned BVPs are proved. In Section 6, the surface (single-layer and double-layer) and volume potentials are constructed and their basic properties are given. In addition, the basic properties of some singular integral operators are established. Finally, in Section 7, using the potential method and the theory of singular integral equations, the existence theorems for classical solutions of the BVPs of steady vibrations are proved.

2. Governing Equations

We consider a material with three levels of porosity in which the skeleton is a homogeneous and isotropic viscoelastic Kelvin–Voigt solid with pores on a macro-scale, on a much smaller meso-scale and on a micro-scale. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point in the Euclidean three-dimensional space $\mathbb{R}^3$ and t be the time variable, $t \geq 0$, and let the dot denote differentiation with respect to t. Repeated Latin indices are summed over the ranges (1,2,3). In the sequel, the point- and time-dependent functions will be denoted by hats.

We denote the three-component displacement vector field for the skeleton of a porous material by $\hat{\mathbf{u}}$, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$. Let $\hat{\varphi}_1$, $\hat{\varphi}_2$ and $\hat{\varphi}_3$ be the changes of the volume fractions from the reference configuration for the macro-, meso- and micro-pore (the first, second and third levels of pores) networks, respectively. In addition, $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$ are the changes of the fluid pressures in the macro-, meso- and micro-pores, respectively.

Suppose that the volume fractions of the three levels of pores and the fluid pressures in these pores change along with the deformation of the body skeleton. Then, following [32], the governing equations in the coupled linear dynamical theory of viscoelastic Kelvin–Voigt materials with a triple porosity can be written as:

1. *Equations of motion:*

$$\hat{t}_{lj,j} = \rho(\ddot{\hat{u}}_l - \hat{F}'_l), \qquad \hat{\sigma}^{(l)}_{j,j} + \hat{\xi}^{(l)} = \rho_l \ddot{\hat{\varphi}}_l - \rho \hat{s}_l \quad (\text{no sum by } l), \quad l = 1, 2, 3, \tag{1}$$

where $\hat{t}_{lj}$ is the component of total stress tensor, $\hat{\mathbf{F}}' = (\hat{F}'_1, \hat{F}'_2, \hat{F}'_3)$ is the body force per unit mass; $\hat{\sigma}^{(l)}_j$, $\hat{\xi}^{(l)}$, $\hat{s}_l$ and ρ_l are the components of the equilibrated stress, the intrinsic equilibrated body force, the extrinsic equilibrated body force and the coefficients of the equilibrated inertia, associated to the l-th pore networks, respectively; ρ is the reference mass density, $\rho > 0$, $\rho_l > 0$,

$$\hat{\xi}^{(l)} = -\tilde{\gamma}_l \, \hat{e}_{rr} - \tilde{\zeta}_{lj} \hat{\varphi}_j + m_{lj} \hat{p}_j, \tag{2}$$

$\hat{e}_{lj}$ is the component of strain tensor,

$$\hat{e}_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \qquad l, j = 1, 2, 3. \tag{3}$$

2. *Constitutive equations:*

$$\hat{t}_{lj} = 2\tilde{\mu}\hat{e}_{lj} + \tilde{\lambda}\, \hat{e}_{rr}\delta_{lj} + (\tilde{b}_r \hat{\varphi}_r - \beta_r \hat{p}_r)\delta_{lj}, \quad \hat{\sigma}^{(l)}_l = \tilde{\alpha}_{lr} \hat{\varphi}_{r,j}, \quad l, j = 1, 2, 3, \tag{4}$$

where δ_{lj} is Kronecker's delta.

3. *Equations of fluid mass conservation:*

$$\hat{v}^{(l)}_{j,j} + a_{lj} \dot{\hat{p}}_j + \beta_l \dot{\hat{e}}_{rr} + m_{lj} \dot{\hat{\varphi}}_j + \hat{q}_l = 0, \tag{5}$$

where

$$\hat{q}_l = \sum_{j=1;j\neq l}^{3} \gamma_{l+j-2}(\hat{p}_l - \hat{p}_j) = d_{lj}\hat{p}_j, \quad d_{11} = \gamma_1 + \gamma_2, \quad d_{22} = \gamma_1 + \gamma_3, \quad d_{33} = \gamma_2 + \gamma_3,$$
$$d_{12} = d_{21} = -\gamma_1, \quad d_{13} = d_{31} = -\gamma_2, \quad d_{23} = d_{32} = -\gamma_3, \quad l = 1, 2, 3. \tag{6}$$

4. *Darcy's extended law*

$$\hat{v}_j^{(l)} = -\frac{\kappa_{lr}}{\mu''}\hat{p}_{r,j} - \rho_{l+3}\hat{s}_j^{(l)} \quad (\text{no sum by } l), \qquad l, j = 1, 2, 3. \tag{7}$$

In these equations, $\hat{\mathbf{v}}^{(l)} = (\hat{v}_1^{(l)}, \hat{v}_2^{(l)}, \hat{v}_3^{(l)})$ is the fluid flux vector associated to the l-th pore network; γ_l is the internal transport coefficient, $\gamma_l \geq 0$, μ'' is the fluid viscosity, ρ_{l+3} and $\hat{\mathbf{s}}^{(l)} = (\hat{s}_1^{(l)}, \hat{s}_2^{(l)}, \hat{s}_3^{(l)})$ are the density of fluid and the external force (such as gravity) for the l-th pore network, respectively; $\rho_{l+3} > 0$,

$$\tilde{\lambda} = \lambda + \lambda^*\frac{\partial}{\partial t}, \quad \tilde{\mu} = \mu + \mu^*\frac{\partial}{\partial t}, \quad \tilde{b_l} = b_l + b_l^*\frac{\partial}{\partial t}, \quad \tilde{\gamma_l} = b_l + \gamma_l^*\frac{\partial}{\partial t},$$
$$\tilde{\alpha}_{lj} = \alpha_{lj} + \alpha_{lj}^*\frac{\partial}{\partial t}, \quad \tilde{\zeta}_{lj} = \zeta_{lj} + \zeta_{lj}^*\frac{\partial}{\partial t}, \quad l, j = 1, 2, 3,$$

where the values λ, μ, b_l, β_l, a_{lj}, α_{lj}, ζ_{lj}, m_{lj}, κ_{lj} are the constitutive coefficients associated to the elasticity and porosity of materials, but the values $\lambda^*, \mu^*, b_l^*, \gamma_l^*, \alpha_{lj}^*, \zeta_{lj}^*$ ($l = 1, 2,\ j = 1, 2, 3$) are the viscosity constitutive coefficients and $a_{lj} = a_{jl}$, $m_{lj} = m_{jl}$, $\kappa_{lj} = \kappa_{jl}$, $\alpha_{lj}^* = \alpha_{jl}^*$, $\zeta_{lj}^* = \zeta_{jl}^*$.

On the basis of relations (2)–(4), (6) and (7), from equations (1) and (5), we obtain the following system of dynamical equations in the coupled linear theory of viscoelasticity for Kelvin–Voigth materials with a triple porosity expressed in terms of the displacement vector $\hat{\mathbf{u}}$, the changes of the volume fractions $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{\varphi}_3$ and the changes of fluid pressures $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_2$:

$$\tilde{\mu}\Delta\hat{u}_l + (\tilde{\lambda} + \tilde{\mu})\hat{u}_{j,lj} + \tilde{b}_j\hat{\varphi}_{j,l} - \beta_j\hat{p}_{j,l} = \rho(\ddot{\hat{u}}_l - \hat{F}_l'),$$
$$\tilde{\alpha}_{lj}\Delta\hat{\varphi}_j - \tilde{\zeta}_{lj}\hat{\varphi}_j - \tilde{\gamma}_l\hat{u}_{j,j} + m_{lj}\hat{p}_j = \rho_l\ddot{\hat{\varphi}}_l - \rho\hat{s}_l, \tag{8}$$
$$k_{lj}\Delta\hat{p}_j - a_{lj}\dot{\hat{p}}_j - \beta_l\dot{\hat{u}}_{j,j} - m_{lj}\dot{\hat{\varphi}}_j - d_{lj}\hat{p}_j = -\rho_{l+3}\hat{s}_{j,j}^{(l)} \quad (\text{no sum by } l),$$

where Δ is the Laplacian operator and $k_{lj} = \dfrac{\kappa_{lj}}{\mu''}$.

In the steady vibrations case, the functions $\hat{u}_l, \hat{F}_l', \hat{\varphi}_l, \hat{p}_l, \hat{s}_l$ and $\hat{s}_j^{(l)}$ $(l, j = 1, 2, 3)$ have a harmonic time variation, that is,

$$\left\{\hat{u}_l, \hat{F}_l', \hat{\varphi}_l, \hat{p}_l, \hat{s}_l, \hat{s}_j^{(l)}\right\}(\mathbf{x}, t) = \mathrm{Re}\left[\left\{u_l, F_l', \varphi_l, p_l, s_l, s_j^{(l)}\right\}(\mathbf{x})\, e^{-i\omega t}\right].$$

Consequently, equations (8) reduce to the following system of equations of steady vibrations in the coupled linear theory of Kelvin–Voight viscoelastic materials with a triple porosity

$$(\mu'\Delta + \rho\omega^2)\mathbf{u} + (\lambda' + \mu')\nabla\mathrm{div}\mathbf{u} + \mathbf{b}'\nabla\boldsymbol{\varphi} - \boldsymbol{\beta}\nabla\mathbf{p} = \mathbf{F}^{(1)},$$
$$(\boldsymbol{\alpha}'\Delta + \mathbf{c}')\boldsymbol{\varphi} - \boldsymbol{\gamma}'\mathrm{div}\mathbf{u} + \mathbf{m}\mathbf{p} = \mathbf{F}^{(2)}, \tag{9}$$
$$(\mathbf{k}\Delta + \mathbf{a}')\mathbf{p} + \boldsymbol{\beta}'\mathrm{div}\mathbf{u} + \mathbf{m}'\boldsymbol{\varphi} = \mathbf{F}^{(3)},$$

where ∇ is the gradient operator, ω is the oscillation frequency, $\omega > 0$, $\mathbf{F}^{(1)} = (-\rho F_1', -\rho F_2', -\rho F_3')$, $\mathbf{F}^{(2)} = (-\rho s_1, -\rho s_2, -\rho s_3)$, $\mathbf{F}^{(3)} = (-\rho_4 s_{j,j}^{(1)}, -\rho_5 s_{j,j}^{(2)}, -\rho_6 s_{j,j}^{(3)})$,

$$\lambda' = \lambda - \omega\lambda^*, \qquad \mu' = \mu - i\omega\mu^*, \quad \mathbf{b}' = (b_1', b_2', b_3'), \quad b_l' = b_l - i\omega b_l^*,$$
$$\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3), \quad \boldsymbol{\beta}' = i\omega\boldsymbol{\beta}, \quad \mathbf{k} = (k_{lj})_{3\times3}, \quad \mathbf{m} = (m_{lj})_{3\times3},$$
$$\mathbf{m}' = i\omega\mathbf{m}, \quad \boldsymbol{\gamma}' = (\gamma_1', \gamma_2', \gamma_3'), \quad \gamma_l' = b_l - i\omega\gamma_l^*, \quad \mathbf{a}' = (a_{lj}')_{3\times3}, \tag{10}$$
$$a_{lj}' = i\omega a_{lj} - d_{lj}, \quad \mathbf{c}' = (c_{lj}')_{3\times3}, \quad c_{lj}' = \rho_l\omega^2\delta_{lj} - \zeta_{lj} + i\omega\zeta_{lj}^* \quad (\text{no sum by } l),$$
$$\boldsymbol{\alpha}' = (\alpha_{lj}')_{3\times3}, \quad \alpha_{lj}' = \alpha_{lj} - i\omega\alpha_{lj}^*, \quad l = 1, 2, \qquad j = 1, 2, 3.$$

Now we introduce the following second order matrix differential operator

$$\mathbf{A}(\mathbf{D_x}) = (A_{lj}(\mathbf{D_x}))_{9\times 9}\,, \quad A_{lj} = (\mu'\Delta + \rho\omega^2)\delta_{lj} + (\lambda' + \mu')\frac{\partial^2}{\partial x_l \partial x_j},$$
$$A_{l;j+3} = b_j' \frac{\partial}{\partial x_l}, \quad A_{l;j+6} = -\beta_j \frac{\partial}{\partial x_l}, \quad A_{l+3;j} = -\gamma_l' \frac{\partial}{\partial x_j}, \quad A_{l+3;j+3} = \alpha_{lj}'\Delta + c_{lj}',$$
$$A_{l+3;j+6} = m_{lj}, \quad A_{l+6;j} = i\omega\beta_l \frac{\partial}{\partial x_j}, \qquad A_{l+6;j+3} = i\omega m_{lj}, \quad A_{l+6;j+6} = k_{lj}\Delta + a_{lj}',$$
$$\mathbf{D_x} = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big), \quad l, j = 1, 2, 3.$$

Obviously, we can rewrite system (9) into the following form

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{11}$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ and $\mathbf{F} = (\mathbf{F}^{(1)}, \mathbf{F}^{(2)}, \mathbf{F}^{(3)})$ are the nine-component vector functions, $\mathbf{x} \in \mathbb{R}^3$.

In what follows, we assume that the constitutive coefficients satisfy the condition

$$\mu^* > 0, \qquad \mathbf{a},\ \mathbf{k},\ \boldsymbol{\alpha}^*,\ \boldsymbol{\zeta}^*,\ \boldsymbol{\vartheta}^* \text{ are positive definite matrices,} \tag{12}$$

where

$$\boldsymbol{\vartheta}^* = (\vartheta_{lj}^*)_{4\times 4}, \quad \vartheta_{11}^* = \frac{1}{3}(3\lambda^* + 2\mu^*), \quad \vartheta_{1;j+1}^* = \vartheta_{j+1;1}^* = \frac{1}{2}(b_j^* + \gamma^*),$$
$$\vartheta_{l+1;l+1}^* = \zeta_{ll}^* \text{ (no sum)}, \quad \vartheta_{l+1;j+1}^* = \frac{1}{2}\zeta_{lj}^*, \quad l, j = 1, 2, 3.$$

The purpose of this work is as follows: to investigate the internal and external BVPs for system (9) by using the potential method and the theory of singular integral equations. In particular, let us prove the existence and uniqueness theorems of classical solutions of these problems. To achieve this goal, it is necessary to construct the fundamental solution of system (9) and obtain Green's formulas, then it will be possible to prove the above-mentioned theorems.

3. Fundamental Solution

Let τ_1^2, τ_2^2, $\dots$, τ_7^2 be the roots of the algebraic equation $\Lambda_1(-\xi) = 0$ (with respect to ξ), where

$$\Lambda_1(\Delta) = \frac{1}{\mu_0'\alpha_0' k_0} \det \mathbf{B}(\Delta) = \prod_{j=1}^{7} (\Delta + \lambda_j^2),$$

$\mu_0' = \lambda' + 2\mu'$, $\alpha_0' = \det\boldsymbol{\alpha}'$, $k_0 = \det\mathbf{k}$, $\mathbf{B}(\Delta) = (B_{lj}(\Delta))_{7\times 7}$ is a matrix differential operator with the following elements:

$$B_{11}(\Delta) = \mu_0'\Delta + \rho\omega^2, \quad B_{1;j+1}(\Delta) = -\gamma_j'\Delta, \quad B_{1;j+4}(\Delta) = i\omega\beta_j\Delta,$$
$$B_{l+1;1}(\Delta) = b_l', \quad B_{l+1;j+1}(\Delta) = \alpha_{lj}'\Delta + c_{lj}', \quad B_{l+1;j+4}(\Delta) = i\omega m_{lj},$$
$$B_{l+4;1}(\Delta) = -\beta_l, \quad B_{l+4;j+1}(\Delta) = m_{lj}, \quad B_{l+4;j+4}(\Delta) = k_{lj}\Delta + a_{lj}'.$$

We assume that the values $\tau_1^2,\ \tau_2^2, \dots, \tau_8^2$ are distinct and $\mathrm{Im}\tau_j > 0$ $(j = 1, 2, \dots, 8)$, where $\tau_8^2 = \dfrac{\rho\omega^2}{\mu'}$.

Furthermore, we introduce the following notation:

(i)

$$\mathbf{M}(\mathbf{D_x}) = (M_{lj}(\mathbf{D_x}))_{9\times 9}\,, \qquad M_{lj}(\mathbf{D_x}) = \frac{1}{\mu'}\Lambda_1(\Delta)\,\delta_{lj} + n_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j},$$
$$M_{lr}(\mathbf{D_x}) = n_{1;r-2}(\Delta)\frac{\partial}{\partial x_l}, \qquad M_{rl}(\mathbf{D_x}) = n_{r-2;1}(\Delta)\frac{\partial}{\partial x_l}, \tag{13}$$
$$M_{rs}(\mathbf{D_x}) = n_{r-2;s-2}(\Delta), \qquad l, j = 1, 2, 3, \qquad r, s = 4, 5, \dots, 9,$$

where

$$n_{j1}(\Delta) = -\frac{1}{\mu'\mu_0'\alpha_0' k_0}\left[(\lambda'+\mu')B_{j1}^*(\Delta) - \gamma_l' B_{j;l+1}^*(\Delta) + \beta_l' B_{j;l+4}^*(\Delta)\right],$$
$$n_{jr}(\Delta) = \frac{1}{\mu_0'\alpha_0' k_0} B_{jr}^*(\Delta), \qquad j=1,2,\dots,7, \qquad r=2,3,\dots,7$$

and B_{lj}^* is the cofactor of the element B_{lj} of matrix $\mathbf{B}$.

(ii)

$$\mathbf{\Psi}(\mathbf{x}) = \left(\Psi_{lr}(\mathbf{x})\right)_{9\times 9}, \qquad \Psi_{11}(\mathbf{x}) = \Psi_{22}(\mathbf{x}) = \Psi_{33}(\mathbf{x}) = \sum_{j=1}^{8} \eta_{2j}\gamma^{(j)}(\mathbf{x}),$$
$$\Psi_{44}(\mathbf{x}) = \Psi_{55}(\mathbf{x}) = \cdots = \Psi_{99}(\mathbf{x}) = \sum_{j=1}^{7} \eta_{1j}\gamma^{(j)}(\mathbf{x}), \tag{14}$$
$$\Psi_{lr}(\mathbf{x}) = 0, \qquad l \neq r, \qquad l,r = 1,2,\dots,9,$$

where

$$\gamma^{(j)}(\mathbf{x}) = -\frac{e^{i\tau_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|} \tag{15}$$

and

$$\eta_{1r} = \prod_{l=1,\, l\neq r}^{7} (\tau_l^2 - \tau_r^2)^{-1}, \quad \eta_{2j} = \prod_{l=1,\, l\neq j}^{8} (\tau_l^2 - \tau_j^2)^{-1}, \quad r=1,2,\dots,7, \quad j=1,2,\dots,8.$$

It is not difficult to prove the following relation:

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{M}(\mathbf{D}_\mathbf{x}) = \mathbf{\Lambda}(\Delta), \tag{16}$$

where

$$\mathbf{\Lambda}(\Delta) = \left(\Lambda_{lj}(\Delta)\right)_{9\times 9}, \qquad \Lambda_{11}(\Delta) = \Lambda_{22}(\Delta) = \Lambda_{33}(\Delta) = \Lambda_2(\Delta),$$
$$\Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) = \cdots = \Lambda_{99}(\Delta) = \Lambda_1(\Delta), \qquad \Lambda_{lj}(\Delta) = 0,$$
$$l \neq j, \qquad l,j = 1,2,\dots,9.$$

Obviously, $\mathbf{\Psi}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{\Lambda}(\Delta)$, i.e.,

$$\mathbf{\Lambda}(\Delta)\mathbf{\Psi}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{17}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{9\times 9}$ is the unit matrix and $\mathbf{x} \in \mathbb{R}^3$.

Let us introduce the matrix $\mathbf{\Gamma}(\mathbf{x})$ by

$$\mathbf{\Gamma}(\mathbf{x}) = \mathbf{M}(\mathbf{D}_\mathbf{x})\mathbf{\Psi}(\mathbf{x}), \tag{18}$$

where $\mathbf{M}(\mathbf{D}_\mathbf{x})$ and $\mathbf{\Psi}(\mathbf{x})$ are defined by (13) and (14), respectively. In view of relations (16) and (17), we can write $\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}$. Hence the following theorem is valid.

Theorem 1. *The matrix* $\mathbf{\Gamma}(\mathbf{x}) = \left(\Gamma_{lj}(\mathbf{x})\right)_{7\times 7}$ *defined by* (18) *is the fundamental solution of system* (9).

Now we introduce the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x}) = \left(\Gamma_{lj}^{(0)}(\mathbf{x})\right)_{9\times 9}$ with the following elements

$$\Gamma_{lj}^{(0)}(\mathbf{x}) = -\frac{\lambda'+3\mu'}{8\pi\mu_0'\mu'}\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{\lambda'+\mu'}{8\pi\mu_0'\mu'}\frac{x_l x_j}{|\mathbf{x}|^3}, \qquad \Gamma_{l+3;j+3}^{(0)}(\mathbf{x}) = \frac{\alpha_{lj}'^*}{\alpha_0'}\gamma^{(0)}(\mathbf{x}),$$
$$\Gamma_{l+6;j+6}^{(0)}(\mathbf{x}) = \frac{k_{lj}^*}{k_0}\gamma^{(0)}(\mathbf{x}), \qquad \Gamma_{lr}^{(0)}(\mathbf{x}) = \Gamma_{rl}^{(0)}(\mathbf{x}) = \Gamma_{l+3;j+6}^{(0)}(\mathbf{x}) = \Gamma_{l+6;j+3}^{(0)}(\mathbf{x}) = 0,$$
$$\gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi\,|\mathbf{x}|}, \qquad l,j=1,2,3, \qquad r=4,5,\dots,9,$$

where $\alpha_{lj}'^*$ and k_{lj}^* are the cofactors of the elements α_{lj}' and k_{lj} of matrices $\boldsymbol{\alpha}'$ and $\mathbf{k}$, respectively.

Theorem 1 leads to the following basic properties of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$.

Theorem 2.
(i) *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of the homogeneous equation* $\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x})= \mathbf{0}$ *at every point* $\mathbf{x}\in\mathbb{R}^3$ *except the origin of* $\mathbb{R}^3$.
(ii) *The relations*

$$\Gamma_{lj}(\mathbf{x})=O(|\mathbf{x}|^{-1}),\qquad \Gamma_{l+3;j+3}(\mathbf{x})=O(|\mathbf{x}|^{-1}),\qquad \Gamma_{l+6;j+6}(\mathbf{x})=O(|\mathbf{x}|^{-1}),$$
$$\Gamma_{lr}(\mathbf{x})=O(1),\qquad \Gamma_{rl}(\mathbf{x})=O(1),\qquad \Gamma_{l+3;j+6}(\mathbf{x})=O(1),$$
$$\Gamma_{l+6;j+3}(\mathbf{x})=O(1),\qquad l,j=1,2,3,\qquad r=4,5,\dots,9$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.
(iii) *The relations*

$$\Gamma_{lj}(\mathbf{x})-\Gamma_{lj}^{(0)}(\mathbf{x})=\mathrm{const}+O(|\mathbf{x}|),\qquad l,j=1,2,\dots,9 \tag{19}$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

Thus the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed explicitly by means of eight elementary functions $\gamma^{(j)}(j=1,2,\dots,8)$ (see (15)). Moreover, on the basis of Theorem 2, the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin of $\mathbb{R}^3$.

4. Boundary value problems and Green's identities

In this section, the basic internal and external BVPs of the theory under consideration are formulated and Green's identities are obtained.

Let Ω^+ be a finite domain in $\mathbb{R}^3$ and is surrounded with a smooth closed surface S, $\overline{\Omega^+}=\Omega^+\cup S$, $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega^+}$, $\overline{\Omega^-}=\Omega^-\cup S$. In what follows, the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$ is denoted by $\mathbf{n}(\mathbf{z})$. We denote the scalar product of two vectors $\mathbf{U}=(U_1,U_2,\dots,U_9)$ and $\mathbf{V}=(V_1,V_2,\dots,V_9)$ by $\mathbf{U}\cdot\mathbf{V}=\sum_{j=1}^{9}U_j\bar{V}_j$, where $\bar{V}_j$ is the complex conjugate of V_j.

A vector function $\mathbf{U}=(U_1,U_2,\dots,U_9)$ is called *regular* in Ω^- (or in Ω^+) if
(i)

$$U_j\in C^2(\Omega^-)\cap C^1(\overline{\Omega^-})\qquad (\text{or }\ U_j\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})),$$

and (ii)

$$U_j(\mathbf{x})=O(|\mathbf{x}|^{-1}),\qquad U_{j,l}(\mathbf{x})=o(|\mathbf{x}|^{-1})\qquad\text{for}\qquad |\mathbf{x}|\gg 1, \tag{20}$$

where $j=1,2,\dots,9$, $l=1,2,3$.

Therewith, we use the matrix differential operator $\mathbf{R}(\mathbf{D_x},\mathbf{n})=(R_{lj}(\mathbf{D_x},\mathbf{n}))_{9\times 9}$, where

$$R_{lj}(\mathbf{D_x},\mathbf{n})=\mu'\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+\mu' n_j\frac{\partial}{\partial x_l}+\lambda' n_l\frac{\partial}{\partial x_j},\quad R_{l;j+3}(\mathbf{D_x},\mathbf{n})=b_j' n_l,$$
$$R_{l;j+6}(\mathbf{D_x},\mathbf{n})=-\beta_j n_l,\quad R_{l+3;j+3}(\mathbf{D_x},\mathbf{n})=\alpha_{lj}'\frac{\partial}{\partial\mathbf{n}},\quad R_{l+6;j+6}(\mathbf{D_x},\mathbf{n})=k_{lj}\frac{\partial}{\partial\mathbf{n}},$$
$$R_{rl}(\mathbf{D_x},\mathbf{n})=R_{l+3;j+6}(\mathbf{D_x},\mathbf{n})=R_{l+6;j+3}(\mathbf{D_x},\mathbf{n})=0,\quad l,j=1,2,3,\quad r=4,5,\dots,9$$

and $\frac{\partial}{\partial\mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

Now, we formulate the basic internal and external BVPs of steady vibrations in the coupled linear theory of viscoelastic materials with a triple porosity as follows:

Find a regular (classical) solution $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\mathbf{p})$ to system (11) for $\mathbf{x}\in\Omega^{\pm}$ satisfying the boundary condition

$$\lim_{\Omega^{\pm}\ni\mathbf{x}\to\ \mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^{\pm}=\mathbf{f}(\mathbf{z})$$

in *Problem* $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^{\pm}\ni\mathbf{x}\to\ \mathbf{z}\in S}\mathbf{R}(\mathbf{D_x},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x})\equiv\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^{\pm}=\mathbf{f}(\mathbf{z})$$

in *Problem* $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are the known nine-component smooth vector functions, $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

Let $\mathbf{U}' = (\mathbf{u}', \boldsymbol{\varphi}', \mathbf{p}')$ be a nine-component smooth vector function, $\mathbf{u}' = (u'_1, u'_2, u'_3)$, $\boldsymbol{\varphi}' = (\varphi'_1, \varphi'_2, \varphi'_3)$, $\mathbf{p}' = (p'_1, p'_2, p'_3)$. We introduce the notation

$$
\begin{aligned}
G^{(0)}(\mathbf{u},\mathbf{u}') &= \frac{1}{3}(3\lambda' + 2\mu')\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\overline{\mathbf{u}'} + \frac{\mu'}{2}\sum_{l,j=1;\,l\neq j}^{3}\Big(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\Big)\Big(\frac{\partial \overline{u'_j}}{\partial x_l} + \frac{\partial \overline{u'_l}}{\partial x_j}\Big)\\
&\quad + \frac{\mu'}{3}\sum_{l,j=1}^{3}\Big(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\Big)\Big(\frac{\partial \overline{u'_l}}{\partial x_l} - \frac{\partial \overline{u'_j}}{\partial x_j}\Big),\\
G^{(1)}(\mathbf{U},\mathbf{u}') &= G^{(0)}(\mathbf{u},\mathbf{u}') - \rho\omega^2\mathbf{u}\cdot\mathbf{u}' + (b'_l\varphi_l - \beta_l p_l)\,\mathrm{div}\,\overline{\mathbf{u}'},\\
G^{(2)}(\mathbf{U},\boldsymbol{\varphi}') &= \alpha'_{lj}\nabla\varphi_j\cdot\nabla\varphi'_l + \big(\gamma'_l\mathrm{div}\mathbf{u} - c'_{lj}\varphi_j - m_{lj}p_j\big)\,\overline{\varphi'_l},\\
G^{(3)}(\mathbf{U},\mathbf{p}') &= k_{lj}\nabla p_j\cdot\nabla p'_l - \big(\beta'_l\mathrm{div}\mathbf{u} + m'_{lj}\varphi_j + a_{lj}p_j\big)\,\overline{p'_l},\\
G(\mathbf{U},\mathbf{U}') &= G^{(1)}(\mathbf{U},\mathbf{u}') + G^{(2)}(\mathbf{U},\boldsymbol{\varphi}') + G^{(3)}(\mathbf{U},\mathbf{p}').
\end{aligned}
\tag{21}
$$

It is not very difficult to prove the following result.

Lemma 1. *If* $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ *and* $\mathbf{U}' = (\mathbf{u}', \boldsymbol{\varphi}', \mathbf{p}')$ *are regular vector in* $\Omega^{\pm}$*, then*

$$
\begin{aligned}
\int_{\Omega^{\pm}}\Big[\mathbf{A}^{(1)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + G^{(1)}(\mathbf{U},\mathbf{u}')\Big]\,d\mathbf{x} &= \pm\int_S \mathbf{R}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,\\
\int_{\Omega^{\pm}}\Big[\mathbf{A}^{(2)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\boldsymbol{\varphi}'(\mathbf{x}) + G^{(2)}(\mathbf{U},\boldsymbol{\varphi}')\Big]\,d\mathbf{x} &= \pm\int_S \boldsymbol{\alpha}'\frac{\partial\boldsymbol{\varphi}(\mathbf{z})}{\partial\mathbf{n}}\cdot\boldsymbol{\varphi}'(\mathbf{z})\,d_{\mathbf{z}}S,\\
\int_{\Omega^{\pm}}\Big[\mathbf{A}^{(3)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{p}'(\mathbf{x}) + G^{(3)}(\mathbf{U},\mathbf{p}')\Big]\,d\mathbf{x} &= \pm\int_S \mathbf{k}\frac{\partial\mathbf{p}(\mathbf{z})}{\partial\mathbf{n}}\cdot\mathbf{p}'(\mathbf{z})\,d_{\mathbf{z}}S,
\end{aligned}
\tag{22}
$$

where

$$
\begin{gathered}
\mathbf{A}^{(l)}(\mathbf{D_x}) = \Big(A^{(1)}_{jr}(\mathbf{D_x})\Big)_{3\times 9}, \quad A^{(1)}_{jr}(\mathbf{D_x}) = A_{jr}(\mathbf{D_x}), \quad A^{(2)}_{jr}(\mathbf{D_x}) = A_{j+3;r}(\mathbf{D_x}),\\
A^{(3)}_{jr}(\mathbf{D_x}) = A_{j+6;r}(\mathbf{D_x}), \quad \mathbf{R}^{(1)}(\mathbf{D_x},\mathbf{n}) = \Big(R^{(1)}_{jr}(\mathbf{D_x},\mathbf{n})\Big)_{3\times 9}, \quad R^{(1)}_{jr}(\mathbf{D_x},\mathbf{n}) = R_{jr}(\mathbf{D_x},\mathbf{n}),\\
l,j = 1,2,3, \quad r = 1,2,\dots,9.
\end{gathered}
$$

Combining the relations of (22), we can obtain the following

Theorem 3. *If* $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ *and* $\mathbf{U}' = (\mathbf{u}', \boldsymbol{\varphi}', \mathbf{p}')$ *are the regular vectors in* $\Omega^{\pm}$*, then*

$$
\int_{\Omega^{\pm}}\big[\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + G(\mathbf{U},\mathbf{U}')\big]\,d\mathbf{x} = \pm\int_S \mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S,
\tag{23}
$$

where $G(\mathbf{U},\mathbf{U}')$ *is defined by* (4)*.*

Now, we introduce the following matrix differential operators $\tilde{\mathbf{A}}(\mathbf{D_x})$ and $\tilde{\mathbf{R}}(\mathbf{D_x},\mathbf{n})$, where $\tilde{\mathbf{A}}(\mathbf{D_x}) = \mathbf{A}^{\top}(-\mathbf{D_x})$ (the superscript $\top$ denotes transposition) and

$$
\begin{gathered}
\tilde{\mathbf{R}}(\mathbf{D_x},\mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D_x},\mathbf{n}))_{9\times 9}, \qquad \tilde{R}_{lj}(\mathbf{D_x},\mathbf{n}) = R_{lj}(\mathbf{D_x},\mathbf{n}),\\
\tilde{R}_{l;j+3}(\mathbf{D_x},\mathbf{n}) = \gamma'_j n_l, \qquad \tilde{R}_{l;j+6}(\mathbf{D_x},\mathbf{n}) = -\beta'_j n_l, \qquad \tilde{R}_{sr}(\mathbf{D_x},\mathbf{n}) = R_{sr}(\mathbf{D_x},\mathbf{n}),\\
l,j = 1,2,3, \qquad r = 1,2,\dots,9, \qquad s = 4,5,\dots,9.
\end{gathered}
\tag{24}
$$

By a direct calculation, we get the following results.

Theorem 4. *Let* $\tilde{\mathbf{U}}_j$ *be the* j*-th column of the matrix* $\tilde{\mathbf{U}} = (\tilde{U}_{lj})_{9\times 9}$. *If* $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ *and* $\tilde{\mathbf{U}}_j$ $(j = 1, 2, \dots, 9)$ *are regular vectors in* $\Omega^{\pm}$, *then*

$$\int\limits_{\Omega^{\pm}} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^{\top}\mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^{\top}\mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \Big\}\, d\mathbf{y}$$
$$= \pm \int\limits_{S} \Big\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^{\top}\mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^{\top}\mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}}S. \tag{25}$$

Let $\tilde{\boldsymbol{\Gamma}}(\mathbf{x})$ be the fundamental matrix of the operator $\tilde{\mathbf{A}}(\mathbf{D_x})$. Clearly, the matrix $\tilde{\boldsymbol{\Gamma}}(\mathbf{x})$ satisfies the following condition:

$$\tilde{\boldsymbol{\Gamma}}(\mathbf{x}) = \boldsymbol{\Gamma}^{\top}(-\mathbf{x}), \tag{26}$$

where $\boldsymbol{\Gamma}(\mathbf{x})$ is the fundamental solution of system (9).

Applying (25) with $\tilde{\mathbf{U}}(\mathbf{y}) = \tilde{\boldsymbol{\Gamma}}(\mathbf{y} - \mathbf{x})$ and using equation (26), we obtain the following results.

Theorem 5. *If* $\mathbf{U}$ *is a regular vector in* $\Omega^{\pm}$, *then*

$$\mathbf{U}(\mathbf{x}) = \pm \int\limits_{S} \Big\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})\boldsymbol{\Gamma}^{\top}(\mathbf{x} - \mathbf{z})]^{\top}\mathbf{U}(\mathbf{z}) - \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{z})\, \mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}}S$$
$$+ \int\limits_{\Omega^{\pm}} \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) d\mathbf{y} \qquad \textit{for} \quad \mathbf{x} \in \Omega^{\pm}. \tag{27}$$

Formulas (23), (25) and (27) are, respectively, Green's first, second and third identities in the considered theory for the domain $\Omega^{\pm}$.

5. Uniqueness Theorems

We are now in a position to prove the uniqueness theorems for classical solutions of the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{+}_{\mathbf{F},\mathbf{f}}$. We have the following results.

Theorem 6. *The internal BVP* $(K)^{+}_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Proof. Let's say the problem $(K)^{+}_{\mathbf{F},\mathbf{f}}$ $(K = I, II)$ has two regular solutions. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(K)^{+}_{\mathbf{0},\mathbf{0}}$. Hence $\mathbf{U}$ is a regular solution of the homogeneous system of equations

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^{+} \tag{28}$$

satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^{+} = \mathbf{0}, \tag{29}$$

for $K = I$ and

$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^{+} = \mathbf{0}, \qquad \mathbf{z} \in S \tag{30}$$

for $K = II$.

On the basis of equations (28)–(30), from (22), we can deduce that

$$\int\limits_{\Omega^{+}} G^{(1)}(\mathbf{U}, \mathbf{u}) d\mathbf{x} = 0, \quad \int\limits_{\Omega^{+}} G^{(2)}(\mathbf{U}, \boldsymbol{\varphi}) d\mathbf{x} = 0, \quad \int\limits_{\Omega^{+}} G^{(3)}(\mathbf{U}, \mathbf{p}) d\mathbf{x} = 0. \tag{31}$$

With the help of relations (4), we obtain

$$\begin{gathered} G^{(0)}(\mathbf{u}, \mathbf{u}) = \frac{1}{3}(3\lambda' + 2\mu')\, |\mathrm{div}\, \mathbf{u}|^2 + \frac{\mu'}{2} \sum_{l,j=1;\, l\neq j}^{3} \left| \frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j} \right|^2 + \frac{\mu'}{3} \sum_{l,j=1}^{3} \left| \frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j} \right|^2, \\ G^{(1)}(\mathbf{U}, \mathbf{u}) = G^{(0)}(\mathbf{u}, \mathbf{u}) - \rho\omega^2 |\mathbf{u}|^2 + (b'_l \varphi_l - \beta_l p_l)\mathrm{div}\overline{\mathbf{u}}, \\ G^{(2)}(\mathbf{U}, \boldsymbol{\varphi}) = \alpha'_{lj} \nabla\varphi_j \cdot \nabla\varphi_l + \left(\gamma'_l \mathrm{div}\mathbf{u} - c'_{lj}\varphi_j - m_{lj} p_j\right) \overline{\varphi_l}, \\ G^{(3)}(\mathbf{U}, \mathbf{p}) = k_{lj} \nabla p_j \cdot \nabla p_l - \left(\beta'_l \mathrm{div}\mathbf{u} + m'_{lj}\varphi_j + a_{lj} p_j\right) \overline{p_l}. \end{gathered} \tag{32}$$

By virtue of the relations of (5), we can easily verify that

$$-\frac{1}{\omega}\mathrm{Im}G^{(0)}(\mathbf{u},\mathbf{u})=\frac{1}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2+\frac{\mu^*}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right|^2+\frac{\mu^*}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2,$$

$$-\frac{1}{\omega}\mathrm{Im}G^{(1)}(\mathbf{U},\mathbf{u})=-\frac{1}{\omega}\mathrm{Im}G^{(0)}(\mathbf{u},\mathbf{u})+\frac{1}{\omega}b_l\mathrm{Im}(\mathrm{div}\mathbf{u}\overline{\varphi_l})+b_l^*\mathrm{Re}(\mathrm{div}\mathbf{u}\overline{\varphi_l})-\frac{1}{\omega}\beta_l\mathrm{Im}(\mathrm{div}\mathbf{u}\overline{p_l}),$$

$$-\frac{1}{\omega}\mathrm{Im}G^{(2)}(\mathbf{U},\boldsymbol{\varphi})=\alpha_{lj}^*\nabla\varphi_j\cdot\nabla\varphi_l+\gamma_l^*\mathrm{Re}\,(\mathrm{div}\mathbf{u}\overline{\varphi_l})-\frac{1}{\omega}b_l\mathrm{Im}(\mathrm{div}\mathbf{u}\overline{\varphi_l})-\zeta_{lj}^*\varphi_j\overline{\varphi_l}-\frac{1}{\omega}m_{lj}\mathrm{Im}(\varphi_j\overline{p_l}),$$

$$\begin{aligned}\mathrm{Re}G^{(3)}(\mathbf{U},\mathbf{p})=&\,k_{lj}\nabla p_j\cdot\nabla p_l+\omega\beta_l\mathrm{Im}\,(\mathrm{div}\mathbf{u}\overline{p_l})+\omega m_{lj}\mathrm{Im}(\varphi_j\overline{p_l})\\&+\gamma_1|p_1-p_2|^2+\gamma_2|p_1-p_3|^2+\gamma_3|p_2-p_3|^2.\end{aligned}$$

Now, on the basis of these relations and condition (12), it follows that

$$\begin{aligned}&-\frac{1}{\omega}\left[\mathrm{Im}G^{(1)}(\mathbf{U},\mathbf{u})+\mathrm{Im}G^{(2)}(\mathbf{U},\boldsymbol{\varphi})\right]+\frac{1}{\omega^2}\mathrm{Re}G^{(3)}(\mathbf{U},\mathbf{p})\\&=\frac{1}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2+(b_l^*+\gamma_l^*)\mathrm{Re}(\mathrm{div}\mathbf{u}\overline{\varphi_l})+\zeta_{lj}^*\varphi_j\overline{\varphi_l}\\&+\frac{1}{\omega^2}\left[\gamma_1|p_1-p_2|^2+\gamma_2|p_1-p_3|^2+\gamma_3|p_2-p_3|^2\right]+\alpha_{lj}^*\nabla\varphi_j\cdot\nabla\varphi_l\\&+\frac{1}{\omega^2}k_{lj}\nabla p_j\cdot\nabla p_l+\frac{\mu^*}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right|^2+\frac{\mu^*}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2\geq 0\end{aligned}$$

and from (31), we obtain

$$\begin{gathered}\frac{1}{3}(3\lambda^*+2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2+(b_l^*+\gamma_l^*)\mathrm{Re}(\mathrm{div}\mathbf{u}\overline{\varphi_l})+\zeta_{lj}^*\varphi_j\overline{\varphi_l}=0,\quad \alpha_{lj}^*\nabla\varphi_j\cdot\nabla\varphi_l=0,\\ k_{lj}\nabla p_j\cdot\nabla p_l=0,\qquad \sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\right|^2=0,\qquad \sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right|^2=0.\end{gathered}\tag{33}$$

Then, using again inequalities (12), from (33), we get

$$\begin{gathered}\mathrm{div}\,\mathbf{u}(\mathbf{x})=0,\qquad \frac{\partial u_j(\mathbf{x})}{\partial x_l}+\frac{\partial u_l(\mathbf{x})}{\partial x_j}=0,\qquad \frac{\partial u_l(\mathbf{x})}{\partial x_l}-\frac{\partial u_j(\mathbf{x})}{\partial x_j}=0,\\ \nabla p_1(\mathbf{x})=\nabla p_2(\mathbf{x})=\mathbf{0},\qquad l,j=1,2,3\end{gathered}\tag{34}$$

and

$$\boldsymbol{\varphi}(\mathbf{x})=\mathbf{0}\tag{35}$$

for $\mathbf{x}\in\Omega^+$.

On the other hand, based on equations (4) and (31), we have the following relation $G^{(1)}(\mathbf{U},\mathbf{u})=\rho\omega^2|\mathbf{u}(\mathbf{x})|^2=0$ and obtain

$$\mathbf{u}(\mathbf{x})=\mathbf{0}\qquad\text{for}\quad \mathbf{x}\in\Omega^+.\tag{36}$$

Quite similarly, from (4) and (31), it follows that

$$G^{(3)}(\mathbf{U},\mathbf{p})=i\omega a_{lj}p_jp_l+\gamma_1|p_1-p_2|^2+\gamma_2|p_1-p_3|^2+\gamma_3|p_2-p_3|^2=0$$

thus we get

$$\mathbf{p}(\mathbf{x})=\mathbf{0}\qquad\text{for}\quad \mathbf{x}\in\Omega^+.\tag{37}$$

Finally, by relations (35)–(37), we obtain the desired result $\mathbf{U}(\mathbf{x})=\mathbf{0}$ for $\mathbf{x}\in\Omega^+$. □

Using condition (20), the next theorem can be proved similarly.

Theorem 7. *The external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K=I,II$.

Hence each of the basic BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ in the class of regular vectors has the unique classical solution.

6. Potentials and singular integral operators

We introduce the following notation:

(i) $\mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}) = \int\limits_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the single-layer potential,

(ii) $\mathbf{P}^{(2)}(\mathbf{x},\mathbf{g}) = \int\limits_S [\widetilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top \mathbf{g}(\mathbf{y})d_{\mathbf{y}}S$ is the double-layer potential,

(iii) $\mathbf{P}^{(3)}(\mathbf{x},\mathbf{h},\Omega^\pm) = \int\limits_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{h}(\mathbf{y})d\mathbf{y}$ is the volume potential,

where $\mathbf{g}$ and $\mathbf{h}$ are the nine-component vector functions, $\mathbf{\Gamma}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D}_{\mathbf{x}})$ defined by (18) and $\widetilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}},\mathbf{n}(\mathbf{y}))$ is given by (24).

In the next four theorems, using the results of Theorems 4 and 5, we establish the basic properties of these potentials.

Theorem 8. *If* $S\in C^{2,\nu},\ \mathbf{g}\in C^{1,\nu'}(S),\ 0<\nu'<\nu\le 1,$ *then:*

(a) $\mathbf{P}^{(1)}(\cdot,\mathbf{g})\in C^{0,\nu'}(\mathbb{R}^3)\cap C^{2,\nu'}\left(\overline{\Omega^\pm}\right)\cap C^\infty(\Omega^\pm)$;

(b) $\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{P}^{(1)}(\mathbf{x},\mathbf{g})=\mathbf{0}$;

(c) $\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g})$ *is a singular integral;*

(d)
$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}), \tag{38}$$
where $\mathbf{x}\in\Omega^\pm$ *and* $\mathbf{z}\in S$.

Theorem 9. *If* $S\in C^{2,\nu},\ \mathbf{g}\in C^{1,\nu'}(S),\ 0<\nu'<\nu\le 1,$ *then:*

(a) $\mathbf{P}^{(2)}(\cdot,\mathbf{g})\in C^{1,\nu'}\left(\overline{\Omega^\pm}\right)\cap C^\infty(\Omega^\pm)$;

(b) $\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{P}^{(2)}(\mathbf{x},\mathbf{g})=\mathbf{0}$,

(c) $\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral,*

(d)
$$\{\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g})\}^\pm = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}), \tag{39}$$

(e) $\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g})\}^+ = \{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g})\}^-$, *where* $\mathbf{x}\in\Omega^\pm$ *and* $\mathbf{z}\in S$.

Theorem 10. *If* $S\in C^{1,\nu},\ \mathbf{h}\in C^{0,\nu'}(\Omega^+),\ 0<\nu'<\nu\le 1,$ *then:*

(a) $\mathbf{P}^{(3)}(\cdot,\mathbf{h},\Omega^+)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^+)\cap C^{2,\nu'}(\overline{\Omega_0^+})$,

(b) $\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{P}^{(3)}(\mathbf{x},\mathbf{h},\Omega^+)=\mathbf{h}(\mathbf{x})$, *where* $\mathbf{x}\in\Omega^+$, Ω_0^+ *is a finite domain in* $\mathbb{R}^3$ *and* $\Omega_0^+\subset\Omega^+$.

Theorem 11. *If* $S\in C^{1,\nu}$, $\operatorname{supp}\mathbf{h}=\Omega\subset\Omega^-$, $\mathbf{h}\in C^{0,\nu'}(\Omega^-)$, $0<\nu'<\nu\le 1,$ *then:*

(a) $\mathbf{P}^{(3)}(\cdot,\mathbf{h},\Omega^-)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^-)\cap C^{2,p'}\left(\overline{\Omega_0^-}\right)$,

(b) $\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{P}^{(3)}(\mathbf{x},\mathbf{h},\Omega^-)=\mathbf{h}(\mathbf{x})$, *where* $\mathbf{x}\in\Omega^-$, Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^-}\subset\Omega^-$.

Let us consider the following integral operators
$$\begin{gathered}
\mathscr{N}^{(1)}\mathbf{g}(\mathbf{z})\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\qquad \mathscr{N}^{(2)}\mathbf{g}(\mathbf{z})\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathscr{N}^{(3)}\mathbf{g}(\mathbf{z})\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\qquad \mathscr{N}^{(4)}\mathbf{g}(\mathbf{z})\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathscr{N}_\varsigma\mathbf{g}(\mathbf{z})\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\varsigma\,\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g})
\end{gathered} \tag{40}$$

for $\mathbf{z} \in S$, where ς is a complex parameter. On the basis of Theorems 8 and 9, $\mathscr{N}^{(j)}$ $(j = 1, 2, 3, 4)$ and $\mathscr{N}_\varsigma$ are singular integral operators.

Let $\boldsymbol{\theta}^{(j)} = (\theta_{lm}^{(j)})_{9\times 9}$ be the symbol (symbolic matrix) of the singular integral operator $\mathscr{N}^{(j)}$ $(j = 1, 2, 3, 4)$ (for the basic definitions in the theory of singular integral equations see, e.g., [19]). By virtue of Theorem 5 and relations (12), (33) and (40), for $\det\boldsymbol{\theta}^{(j)}$, we obtain

$$\det\boldsymbol{\theta}^{(1)} = -\det\boldsymbol{\theta}^{(2)} = -\det\boldsymbol{\theta}^{(3)} = \det\boldsymbol{\theta}^{(4)} = -\frac{(\lambda' + \mu')(\lambda' + 3\mu')}{512(\lambda' + 2\mu')^2}. \tag{41}$$

In view of inequalities of (12), from (41), we get $\det\boldsymbol{\theta}^{(j)} \neq 0$ which proves that the singular integral operator $\mathscr{N}^{(j)}$ is of the normal type, where $j = 1, 2, 3, 4$.

Furthermore, let $\boldsymbol{\theta}_\varsigma$ and $\operatorname{ind}\mathscr{N}_\varsigma$ be, respectively, the symbol and the index of the operator $\mathscr{N}_\varsigma$. It can be easily shown that

$$\det\boldsymbol{\theta}_\varsigma = -\frac{(\lambda' + \varsigma\mu')(\lambda' + 3\varsigma\mu')}{512(\lambda' + 2\mu')^2}$$

and $\det\boldsymbol{\theta}_\varsigma$ vanishes only at two points $\varsigma_1 = -\dfrac{\lambda'}{\mu'}$, $\varsigma_2 = -\dfrac{\lambda'}{3\mu'}$ of the complex plane. By virtue of (41) and owing to $\det\boldsymbol{\sigma}_1 = \det\boldsymbol{\sigma}^{(1)}$, we get $\varsigma_l \neq 1$ $(l = 1, 2)$ and $\operatorname{ind}\mathscr{N}^{(1)} = \operatorname{ind}\mathscr{N}_1 = 0$.

The relation $\operatorname{ind}\mathscr{N}^{(2)} = 0$ is proved in a quite similar manner. Clearly, the operators $\mathscr{N}^{(3)}$ and $\mathscr{N}^{(4)}$ are the adjoint operators for $\mathscr{N}^{(2)}$ and $\mathscr{N}^{(1)}$, respectively. Hence

$$\operatorname{ind}\mathscr{N}^{(3)} = -\operatorname{ind}\mathscr{N}^{(2)} = 0, \qquad \operatorname{ind}\mathscr{N}^{(4)} = -\operatorname{ind}\mathscr{N}^{(1)} = 0.$$

Thus the singular integral operator $\mathscr{N}^{(j)}$ $(j = 1, 2, 3, 4)$ is of the normal type with an index equal to zero and, consequently, Fredholm's theorems are valid for $\mathscr{N}^{(j)}$.

7. Existence Theorems

In this section, the existence theorems for regular (classical) solutions of the BVPs of steady vibrations $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$ are proved by using the potential method. Obviously, on the basis of Theorems 10 and 11, the volume potential $\mathbf{P}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^{\pm})$ is a partial regular solution of the nonhomogeneous equation (11). Consequently, we further consider problem $(K)^{\pm}_{\mathbf{0},\mathbf{f}}$ for $K = I, II$.

Problem $(I)^{+}_{\mathbf{0},\mathbf{f}}$**.** We seek a regular solution to problem $(I)^{+}_{\mathbf{0},\mathbf{f}}$ in the form of a double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) \qquad \text{for} \qquad \mathbf{x} \in \Omega^+, \tag{42}$$

where $\mathbf{g}$ is the required nine-component vector function. By Theorem 9, the vector function $\mathbf{U}$ is a solution of the homogeneous equation

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\,\mathbf{U}(\mathbf{x}, \mathbf{g}) = \mathbf{0} \tag{43}$$

for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition and identity (39), from (42), to determine the unknown vector $\mathbf{g}$, we have a singular integral equation

$$\mathscr{N}^{(1)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \qquad \mathbf{z} \in S \tag{44}$$

for which Fredholm's theorems are valid. We prove that (44) is always solvable for an arbitrary vector $\mathbf{f}$. Let us consider the adjoint homogeneous equation

$$\mathscr{N}^{(4)}\mathbf{h}_0(\mathbf{z}) = \mathbf{0} \qquad \text{for} \quad \mathbf{z} \in S, \tag{45}$$

where $\mathbf{h}_0$ is the required nine-component vector function. Towards this end, it suffices to show that the integral equation (45) has only the trivial solution.

Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (45). On the basis of Theorem 8 and equation (45), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of problem $(II)^{-}_{\mathbf{0},\mathbf{0}}$. In view of Theorem 7, problem $(II)^{-}_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, i.e.,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{46}$$

Furthermore, using Theorem 8 and (46), we can write $\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0}$ for $\mathbf{z} \in S$, i.e., the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Now, in view of Theorem 6, problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{47}$$

By virtue of (47), (48) and identity (39), we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ \equiv \mathbf{0} \qquad \text{for} \qquad \mathbf{z} \in S.$$

Thus the homogeneous equation (45) has only the trivial solution and therefore (44) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following theorem.

Theorem 12. *If $S \in C^{2,p}$, $\mathbf{f} \in C^{1,p'}(S)$, $0 < p' < p \leq 1$, then a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by the double-layer potential* (43)*, where $\mathbf{g}$ is a solution of the singular integral equation* (44) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$. Now, we are looking for a regular solution to problem $(II)^-_{\mathbf{0},\mathbf{f}}$ in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{48}$$

where $\mathbf{h}$ is the required nine-component vector function. Clearly, by Theorem 8, the vector function $\mathbf{U}$ is a solution of (43) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition and using (38), from (48), to determine the unknown vector $\mathbf{h}$, we obtain a singular integral equation

$$\mathscr{N}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{49}$$

It has been proved above that the corresponding homogeneous equation (45) has only the trivial solution. Hence on the basis of Fredholm's theorems, it follows that (49) is always solvable. We have thereby proved the following consequence.

Theorem 13. *If $S \in C^{2,p}$, $\mathbf{f} \in C^{0,p'}(S)$, $0 < p' < p \leq 1$, then a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by the single-layer potential* (48)*, where $\mathbf{h}$ is a solution of the singular integral equation* (49) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Quite similarly, we can prove the following results.

Theorem 14. *If $S \in C^{2,p}$, $\mathbf{f} \in C^{0,p'}(S)$, $0 < p' < p \leq 1$, then a regular solution of problem $(II)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by the single-layer potential $\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g})$ for $\mathbf{x} \in \Omega^+$, where $\mathbf{g}$ is a solution of the singular integral equation $\mathscr{N}^{(2)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z})$ for $\mathbf{z} \in S$ which is always solvable for an arbitrary vector $\mathbf{f}$.*

Theorem 15. *If $S \in C^{2,p}$, $\mathbf{f} \in C^{1,p'}(S)$, $0 < p' < p \leq 1$, then a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by the double-layer potential $\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(2)}(\mathbf{x}, \mathbf{h})$ for $\mathbf{x} \in \Omega^-$, where $\mathbf{h}$ is a solution of the singular integral equation $\mathscr{N}^{(3)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z})$ for $\mathbf{z} \in S$ which is always solvable for an arbitrary vector $\mathbf{f}$.*

8. Concluding Remarks

1. In this paper the coupled linear model of viscoelasticity for Kelvin–Voigt materials with a triple porosity is presented. The governing systems of equations of motion and steady vibrations are established. Further, the basic internal and external BVPs of steady vibrations of this theory are investigated by using the potential method and the theory of singular integral equations. Indeed,

(i) The fundamental solution of the system of steady vibration equations is constructed and its basic properties are established;
(ii) Green's identities are obtained for the bounded and unbounded domains in the above-considered theory;
(iii) The surface and volume potentials are introduced and their properties are given;
(iv) Some useful properties of the required singular integral operators are studied;
(v) Finally, the existence and uniqueness theorems for classical solutions of the mentioned above BVPs of steady vibrations are proved;

2. On the basis of the results obtained in this paper, it becomes possible to present the basic equations of the coupled linear theory of thermoviscoelasticity for materials with a triple porosity and to investigate the BVPs of steady vibrations of this theory.

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Faculty of Exact and Natural Sciences, Tbilisi State University, I. Chavchavadze Ave., 3, Tbilisi 0179, Georgia

Email address: maia.svanadze@gmail.com