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A RADEMACHER SERIES CONVERGENT TO EACH REAL-VALUED FUNCTION CONTINUOUS OVER $(0,1)$ ON CERTAIN DENSE SUBSETS OF $(0,1)$

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In the paper, the theorems implying the existence of a Rademacher series, convergent to each real-valued function, piecewise continuous over $(0,1)$ on certain dense subsets of $(0,1)$, are announced.

The set of all Rademacher series with the above-mentioned property is fully described. Among the elements of this set are both almost everywhere convergent and almost everywhere divergent Rademacher series.

Let k be a non-negative integer and $r_k(t)$ be a Rademacher function defined over $[0,1]$. Namely, for every non-negative integer k, the following equalities hold:

$$r_k(t) = (-1)^i, \quad \text{where} \quad t \in \left(\frac{i}{2^{k+1}}, \frac{i+1}{2^{k+1}}\right) \quad \text{and} \quad i = 0,1,2,\dots,2^{k+1}-1,$$

$$r_k\left(\frac{i}{2^{k+1}}\right) = 0, \quad \text{where} \quad i = 0,1,2,\dots,2^{k+1}-1.$$

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers.

The following theorems, due to Rademacher and Kolmogorov, are well-known:

Theorem A (Rademacher [5]). *If $\sum\limits_{k=0}^{\infty} a_k^2 < \infty$, then a Rademacher series*

$$\sum_{k=0}^{\infty} a_k r_k(t) \tag{1}$$

converges almost everywhere over $[0,1]$.

Theorem B (Kolmogorov [3]). *If $\sum\limits_{k=0}^{\infty} a_k^2 = \infty$, then the series* (1) *diverges almost everywhere over* $[0,1]$.

Everywhere below, $S_n(t)$ stands for the n-th partial sum of the series (1) at a point t, that is,

$$S_n(t) = \sum_{k=0}^{n} a_k r_k(t).$$

The following result is also known (see [2]).

Theorem C (Kaczmarz and Steinhaus). *If the series* (1) *is such that*

$$a_k \to 0, \quad \textit{when} \quad k \to \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} |a_k| = \infty, \tag{2}$$

then for any constants A and B such that $-\infty \le A \le B \le +\infty$, there exists a subset of $[0,1]$ *of cardinality continuum such that for every $t \in E$, the following equalities hold:*

$$\varliminf_{n\to\infty} S_n(t) = A \quad \textit{and} \quad \varlimsup_{n\to\infty} S_n(t) = B.$$

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It directly follows from this theorem that if the series (1) is such that the conditions (2) are satisfied, then for an arbitrary constant γ, there exists a subset E of $[0,1]$ of cardinality continuum such that the series (1) converges to γ at every point of E.

Similar results are established by Beyer [1] and Muromskii [4]. Also, Muromskii [4] proved that there exists a function $f(t)$, continuous over $[0,1]$ such that no Rademacher series converges to $f(t)$ over the set with the positive Lebesgue linear measure.

Below, we formulate two theorems: Theorem 1 and Theorem 2. Theorem 1 is a generalization, in a certain sense, of Theorem C, while Theorem 2 is a corollary of Theorem 1, which shows that any series (1) such that the coefficients of this series satisfy the conditions (2) is a universal series in the sense of the representation of an arbitrary function, continuous over $[0,1]$ on the corresponding dense subset of $[0,1]$ with the cardinality continuum.

Let us formulate some notation and definitions we need below. $C(a,b)$ denotes the set of all continuous functions over the interval (a,b), and μE denotes the Lebesgue linear measure of a subset E of $[0,1]$.

Definition 1. We say that a function $f(t)$ is piecewise continuous over the interval $(0,1)$ if there exists an open set $G=\bigcup\limits_n (a_n,b_n)$ such that $G\subset(0,1)$, $\mu G=1$ and $(a_i,b_i)\bigcap(a_j,b_j)=\emptyset$ if $i\neq j$ and $f(t)=f_n(t)$, when $t\in(a_n,b_n)$ and $f_n(t)\in C(a_n,b_n)$ for every natural n.

Definition 2. We say that a series (1) is a universal one in the sense of the representation of any function, piecewise continuous over the interval $(0,1)$ on a dense subset of $(0,1)$ with the cardinality continuum and we call such a series of CD type universal series, if for any function $f(t)$, piecewise continuous over the interval $(0,1)$, there exists a dense subset E of $(0,1)$ with the cardinality continuum such that

$$\sum_{k=0}^{\infty} a_k r_k(t) = f(t), \quad \text{for every} \quad t\in E.$$

The following statements hold.

Theorem 1. a) *Let a Rademacher series*

$$\sum_{k=0}^{\infty} a_k r_k(t)$$

be such that

$$a_k\to 0, \quad \textit{when} \quad k\to\infty \ \ \textit{and} \ \sum_{k=0}^{\infty}|a_k|=\infty,$$

then for any function $f(t)$*, continuous over the interval* $(a,b)\subset[0,1]$*, there exists a dense subset* E *of* (a,b) *with the cardinality continuum such that*

$$\sum_{k=0}^{\infty} a_k r_k(t) = f(t), \quad \textit{for every} \quad t\in E;$$

b) *If* $\lim\limits_{k\to\infty} a_k \neq 0$ *or* $\sum\limits_{k=0}^{\infty}|a_k|<\infty$*, then there exists a real number* γ *such that the series* (1) *converges to* γ *at no point of* $[0,1]$.

The following statement is a corollary of Theorem 1:

Theorem 2. *It is necessary and sufficient for a series* (1) *to be of* CD *type universal series that*

$$a_k\to 0, \quad \textit{when} \quad k\to\infty \ \ \textit{and} \ \sum_{k=0}^{\infty}|a_k|=\infty.$$

Note that Theorem A, Theorem B and Theorem 2 directly imply the existence of both almost everywhere convergent and almost everywhere divergent CD type universal Rademacher series.

Remark. There also exists the following definition of Rademacher functions:

$$r_k(t) = (-1)^i, \quad \text{if } k \text{ is a non-negative integer number}, \quad t \in \left[\frac{i}{2^{k+1}}, \frac{i+1}{2^{k+1}}\right)$$

and $i = 0, 1, 2, \dots, 2^{k+1} - 1$.

Note that in the case of the latter definition of Rademacher functions, all of the above-presented theorems remain valid.

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