Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 2, 309–314

EMBEDDINGS AND REGULARITY OF POTENTIALS IN GRAND VARIABLE EXPONENT FUNCTION SPACES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this note, grand variable exponent Hajłasz–Morrey spaces are introduced and embeddings from these spaces to Hölder spaces are established under the log-Hölder continuity condition on exponents. The boundedness of the fractional integral operator from a grand variable exponent Morrey space to a grand variable parameter Hölder space is also proved. In general, the function spaces are defined on quasi-metric measure spaces, however, the results are new even for the Euclidean spaces.

Our aim is to introduce grand variable exponent Hajłasz–Morrey spaces ($GVEHMS$ briefly) $(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ and to explore embeddings from these spaces to Hölder spaces with a variable parameter $H^{\lambda(\cdot)}(X)$, under the log-Hölder continuity condition on the exponents. The boundedness of the fractional integral operator defined on an open set Ω in $\mathbb{R}^d$ with Ahlfors upper $n-$ regular condition Borel measure μ on Ω from grand variable exponent Morrey spaces ($GVEMS$ briefly) to grand variable exponent Hölder spaces ($GVEHS$ briefly) is also established. This work may be considered as a continuation of the investigation carried out in [7]. In particular, in that paper, the embeddings from grand variable exponent Hajłasz–Sobolev space ($GVEHSS$ briefly) to $GVEHS$ were established. It should be emphasised that the study of such problems in the variable exponent setting has been initiated in [2].

Investigation of function spaces with a variable exponent is a very efficient area of research nowadays. A variable exponent Lebesgue space ($VELS$ briefly) $L^{p(\cdot)}$ is a special case of that introduced by W. Orlicz in the 1930s and subsequently generalized by I. Musielak and W. Orlicz. For the mapping properties of operators of Harmonic Analysis in $VELS$ we refer to the monographs [4, 5, 16] and for the classical Sobolev spaces and embeddings in these spaces we refer, e.g, to the monograph [18].

The grand Lebesgue spaces were introduced in the 1990s by T. Iwaniec and C. Sbordone [11]. Later on, quite a number of problems of Harmonic Analysis and the theory of non-linear differential equations were studied in these spaces (see, e.g., the monograph [17] and references cited therein). The grand variable exponent Lebesgue spaces were introduced in [14] (see also [6] for more precise spaces).

Morrey spaces describe local regularity more exactly than Lebesgue spaces. As a result, Morrey spaces may be widely used not only in Harmonic Analysis, but also in the theory of PDEs. $GVEMS$ $M^{p(\cdot)}_{q(\cdot),\theta}$ were introduced in [15], where the authors studied the mapping properties of operators of Harmonic Analysis in these spaces (for a constant exponent these spaces appeared first in [19]). A variant of $GVEMS$ defined on quasi-metric measure spaces with a non-doubling measure is introduced in [20] and Sobolev's inequality is obtained for appropriate fractional integrals. However, the authors do not provide the appropriate norm estimates that lead to the desired result.

1. Preliminaries

Let (X, d, μ) be a quasi-metric measure space, i.e., X is a topological space endowed with a locally finite complete measure μ and quasi-metric $d : X \times X \mapsto \mathbb{R}_+$ satisfying the following conditions:

2020 *Mathematics Subject Classification.* 26A33, 42B20, 42B25, 42B35.

Key words and phrases. Grand variable exponent Hajłasz–Morrey spaces; Hölder spaces; Spaces of homogeneous type; Embedding; Maximal operator; Sobolev embeddings.

(i) $d(x,y)=0$ if and only if $x=y$;
(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(iii) there exists a constant $\kappa\geq 1$ such that for all $x,y,z\in X$,

$$d(x,y)\leq \kappa[d(x,z)+d(z,y)];$$

(iv) for every neighbourhood V of a point $x\in X$ there exists $r>0$ such that the ball $B(x,r)=\{y\in X: d(x,y)<r\}$ with center x and radius r is contained in V.

Let

$$d_X:=\operatorname{diam}(X):=\sup\{d(x,y): x,y\in X\}$$

be the diameter of X. It is also assumed that all balls $B(x,r):=\{y\in X: d(x,y)<r\}$ in X are measurable, $\mu\{x\}=0$ for all $x\in X$, μ is a finite measure (i.e., $\mu(X)<\infty$) and the class of continuous functions with compact supports is dense in the space of integrable functions on X.

We say that the measure μ of the quasi-metric measure space (X,d,μ) is Ahlfors upper $\alpha-$ regular ($\alpha>0$) (or satisfies the growth condition) if there is a positive constant C such that for all $x\in X$ and $R>0$,

$$\mu\big(B(x,R)\big)\leq CR^{\alpha}. \tag{1}$$

A quasi-metric measure space with growth condition is called a space of non-homogeneous type (in which the doubling condition given below might not be satisfied).

A measure μ is said to satisfy a doubling condition ($\mu\in DC(X)$) if there is a constant $D_\mu>0$ such that

$$\mu B(x,2r)\leq D_\mu \mu B(x,r), \tag{2}$$

for every $x\in X$ and all $r>0$. The best possible constant in (2) is called the doubling constant for μ which will be denoted again by D_μ.

Further, it can be checked (see also [10, Lemma 14.6]) that if $\mu\in DC(X)$, there is a positive constant C such that whenever $0<r\leq\rho<d_X$, $x\in X$ and $y\in B(x,r)$,

$$\frac{\mu B(x,\rho)}{\mu B(y,r)}\leq C\Big(\frac{\rho}{r}\Big)^N,$$

where

$$N=\log_2 D_\mu \tag{3}$$

and D_μ is the doubling constant. Consequently, taking $\rho=d_X$ in the latter estimate, we find that there is a positive constant C_N such that

$$\mu(B(x,r))\geq C_N r^N. \tag{4}$$

whenever $x\in X$ and $0<r<d_X$, where N is defined by (3).

Throughout this note, by N will be meant the constant given by (3).

A quasi-metric measure space (X,d,μ) with doubling measure μ is called a space of homogeneous type (SHT).

Examples of an SHT are: (a) a domain Ω in $\mathbb{R}^d$ satisfying the condition: there is a positive constant $C>0$ such that $|\Omega\cap B(x,r)|\geq Cr^n$, where $|E|$ is the Lebesgue measure induced on Ω; here, $N=n$; (b) regular curves, i.e., rectifiable curves Γ in $\mathbb{C}$ satisfying the condition: $\nu\big(\Gamma\cap D(x,r)\big)\leq Cr$, where $D(x,r)$ is the disc with center x and radius $r>0$ and ν is the arc-length measure on Γ (in this case, $N=1$); (c) a nilpotent Lie group G with an appropriate distance and Haar measure is an SHT. The Heisenberg group is a special case of such a group.

For the basic properties and other examples of an SHT we refer, e.g., to [3].

We denote by $P_0(X)$ (resp., $P(X)$) the family of all real-valued μ- measurable functions $p(\cdot)$ on X such that

$$0<p_-\leq p_+<\infty, \quad (\text{resp., } 1<p_-\leq p_+<\infty),$$

where

$$p_-:=p_-(X):=\inf_X p(x), \qquad p_+:=p_+(X):=\sup_X p(x).$$

It is clear that $P_0(X)\subset P(X)$.

We say that a pair of exponents $\big(p(\cdot), q(\cdot)\big) \in \widetilde{P}(X)$ if for all x in X,

$$1 < q_- \le q(x) \le p(x) \le p_+ < \infty.$$

We say that a function $p(\cdot) \in P_0(X)$ belongs to the class $\mathcal{P}^{\log}(X)$ (or $p(\cdot)$ satisfies the log-Hölder continuity condition) if there is a positive constant ℓ such that for all $x, y \in X$ with $0 < d(x,y) < 1/2$,

$$|p(x) - p(y)| \le \frac{\ell}{-\ln(d(x,y))}. \tag{5}$$

The class of exponents $\mathcal{P}^{\log}(X)$ plays an important role in the theory of mapping properties of integral operators in $L^{p(\cdot)}$ spaces. For example, maximal, fractional and singular integral operators are bounded in $L^{p(\cdot)}$ under the condition $p(\cdot) \in \mathcal{P}^{\log}$ (see, e.g., the monographs [4,5,16] and references cited therein).

Let $p(\cdot) \in P(X)$. The variable exponent Lebesgue space ($VELS$), denoted by $L^{p(\cdot)}(X)$ (or by $L^{p(x)}(X)$), is the class of all μ- measurable functions f on X for which

$$S_p(f) := \int\limits_X |f(x)|^{p(x)} d\mu(x) < \infty.$$

The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf \Big\{ \lambda > 0 : S_p\big(f/\lambda\big) \le 1 \Big\}.$$

1.1. **Variable Exponent Morrey Spaces ($VEMS$).** A large number of various results regarding the mapping properties of integral operators in non-standard spaces have been obtained during the last decade. One of such a non-standard function space is the variable exponent Morrey space which was introduced in [1] in the Euclidean spaces and in [12] for quasi-metric measure spaces.

Let $p(\cdot)$ and $q(\cdot)$ be variable exponents defined on an SHT such that $\big(p(\cdot), q(\cdot)\big) \in \widetilde{P}(X)$. A measurable locally integrable function f on X belongs to the class $M^{p(\cdot)}_{q(\cdot)}(X)$ (see [12]) if

$$\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)} := \sup_{x \in X, 0 < r < d_X} (\mu B(x,r))^{\frac{1}{p(x)} - \frac{1}{q(x)}} \|f\|_{L^{q(\cdot)}(B(x,r))} < \infty.$$

If $p(\cdot) = q(\cdot)$, then $M^{p(\cdot)}_{q(\cdot)}(X) = L^{p(\cdot)}(X)$ is $VELS$.

The following relation:

$$M^{p(\cdot)}_{r(\cdot)}(X) \hookrightarrow M^{p(\cdot)}_{q(\cdot)}(X), \quad 1 < q(\cdot) \le r(\cdot) \le p(\cdot) \le p_+ < \infty; \quad r(\cdot), q(\cdot) \in \mathcal{P}^{\log}(X)$$

holds for $VEMSs$ (see [15]).

1.2. **Grand Variable Exponent Morrey Spaces.** Let $q(\cdot) \in P(X)$ be a variable exponent on X. We say that a function $\psi(\cdot)$ defined on $(0, q_- - 1)$ belongs to the class $\mathcal{A}_{q(\cdot)}$ if it is increasing on $(0,\delta)$ for some small positive δ, and $\lim\limits_{x \to 0+} \psi(x) = 0$.

Let (X, d, μ) be a quasi-metric measure space. Suppose that $\big(p(\cdot), q(\cdot)\big) \in \widetilde{P}(X)$, $\varphi(\cdot) \in \mathcal{A}_{q(\cdot)}$. We define the grand variable exponent Morrey space as follows (see [15]): we say that $f \in M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ if

$$\|f\|_{M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} := \sup \Big\{ \varphi(\eta)^{\frac{1}{q_- - \eta}} \|f\|_{M^{p(\cdot)}_{q(\cdot)-\eta}(X)} : 0 < \eta < q_- - 1 \Big\} < \infty.$$

Here, η is a constant.

Further, the embeddings

$$M^{p(\cdot)}_{q(\cdot)}(X) \hookrightarrow M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X) \hookrightarrow M^{p(\cdot)}_{q(\cdot)-\eta}(X), \quad 0 < \eta < p_- - 1$$

hold.

The following statement was proved in [15] for $\varphi(t) = t^{\theta}$, and the proof for $\varphi(\cdot) \in \mathcal{A}_{q(\cdot)}$ is the same:

Proposition 1.1. *Let (X,d,μ) be an SHT. Suppose that $\big(p(\cdot),q(\cdot)\big) \in \widetilde{P}(X)$ and $\varphi(\cdot) \in \mathcal{A}_{q(\cdot)}$. Then $M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ is a Banach space. Further, the following equality:*

$$\lim_{\eta\to 0} \varphi(\eta)^{\frac{1}{q_- - \eta}} \|f\|_{M^{p(\cdot)}_{q(\cdot)-\eta}(X)} = 0$$

holds for all $f \in \Big[M^{p(\cdot)}_{q(\cdot)}(X)\Big]_{M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)}$, where $\Big[M^{p(\cdot)}_{q(\cdot)}(X)\Big]_{M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)}$ denotes the closure of $M^{p(\cdot)}_{q(\cdot)}(X)$ in $M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$.

1.3. **Grand Variable Exponent Hajłasz–Morrey and Hölder spaces.** Now, we define the Variable Exponent Hajłasz–Sobolev and Hölder spaces. Let p_c be a constant such that $1 < p_c < \infty$. We use the ideas of P. Hajłasz [9], where the space $(HS)^{p_c}(X)$ (Hajłasz–Sobolev space with a constant exponent) was introduced as a generalization of the classical Sobolev spaces W^{1,p_c} to the general setting of quasi-metric measure spaces.

Let $\big(p(\cdot),q(\cdot)\big) \in \widetilde{P}(X)$ and let $\varphi(\cdot) \in \mathcal{A}_{q(\cdot)}$. We say that a function $f \in M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ (resp., $f \in M^{p(\cdot)}_{q(\cdot)}(X)$) belongs to the grand Hajłasz–Morrey space $(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ (resp., Hajłasz–Morrey space $(HM)^{p(\cdot)}_{q(\cdot)}(X)$) if there exists a non-negative function $\mathrm{g} \in M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\mathrm{X})$ (resp., $\mathrm{g} \in M^{p(\cdot)}_{q(\cdot)}(\mathrm{X})$) such that the inequality

$$|f(x) - f(y)| \le d(x,y)[g(x) + g(y)]$$

holds $\mu-$ a.e. in X.

In this case, the function g is called a generalized gradient of f. The space $(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ (resp., $(HM)^{p(\cdot)}_{q(\cdot)}(X)$) is a Banach space with respect to the norm

$$\|f\|_{(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} = \|f\|_{M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} + \inf \|g\|_{M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)}$$
$$\Big(\text{ resp., } \|f\|_{(HM)^{p(\cdot)}_{q(\cdot)}(X)} = \|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)} + \inf \|g\|_{M^{p(\cdot)}_{q(\cdot)}(X)}\Big),$$

where the infimum is taken over all generalized gradients g of f.

The following relation holds between the norms $\|\cdot\|_{(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)}$ and $\|\cdot\|_{(HM)^{p(\cdot)}_{q(\cdot)}(X)}$:

$$\|\cdot\|_{(HM)^{p(\cdot)}_{q(\cdot)}(X)} \hookrightarrow \|\cdot\|_{(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)} \hookrightarrow \|\cdot\|_{(HM)^{p(\cdot)}_{q(\cdot)-\varepsilon}(X)}, \quad 0 < \varepsilon < q_- - 1.$$

For $\varphi(x) := x^\theta$, we denote the space $M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)$ by $M^{p(\cdot)}_{q(\cdot),\theta}(X)$.

Taking formally $\theta = 0$, $(HM)^{p(\cdot)}_{q(\cdot),\theta}(X)$ is a variable exponent Hajłasz–Sobolev space denoted by $HS^{p(\cdot)}(X)$ and introduced in [2].

If $p(\cdot) \equiv p_c = \text{const}$, $\theta = 0$, then we have the space $HS^{p_c}(X)$ which was introduced and studied by P. Hajłasz [9] as a generalization of the classical Sobolev spaces W^{1,p_c} to the general setting of quasi-metric measure spaces.

Suppose that $p_- > N$. We say that a bounded function f belongs to the variable exponent Hölder space ($VEHS$ briefly) $H^{p(\cdot)}(X)$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \le C d(x,y)^{\max\{1-N/p(x),1-N/p(y)\}},$$

for every $x, y \in X$ (see [2] for this definition).

The norms in these spaces are defined as follows:

$$\|f\|_{H^{p(\cdot)}(X)} = \|f\|_{L^\infty(X)} + \big[f\big]_{H^{p(\cdot)}(X)}$$

where

$$\big[f\big]_{H^{p(\cdot)}(X)} := \sup_{\substack{x,y\in X\\ 0<d(x,y)\le 1}} \frac{|f(x) - f(y)|}{d(x,y)^{\max\{1-N/p(x),1-N/p(y)\}}}.$$

Let Ω be an open set in $\mathbb{R}^d$. Suppose that γ and η are the constants such that $1<\gamma<\eta<\infty$. We say that $f\in \widetilde{H}^{p(\cdot)}_{\gamma,\eta}(\Omega)$ if

$$[f]_{\widetilde{H}^{p(\cdot)}_{\gamma,\eta}(\Omega)} := \sup_{x,x+h\in\Omega,\ 0<|h|\le 1} \frac{|f(x+h)-f(x)|}{h^{\gamma-\eta/p(x)}}<\infty.$$

2. Main Results

Now, we formulate the main results of this note.

Theorem 2.1. *Let (X,d,μ) be an SHT. Let $1\le N<q_-\le q(x)\le p(x)\le p_+<\infty$ and let $\varphi(\cdot)\in\mathcal{A}_{q(\cdot)}$. Suppose that $p(\cdot),q(\cdot)\in\mathcal{P}^{\log}(X)$. Then*

$$(HM)^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(X)\hookrightarrow H^{p(\cdot)}(X).$$

To formulate the next statement, we need the following class of variable exponents. We say that $p(\cdot)\in\mathcal{P}(X)$ if there is a positive constant C such that

$$\mu(\mathrm{B(x,R)})^{\mathrm{p_-(B(x,R))-p_+(B(x,R))}}\le \mathrm{C},$$

for all $x\in X$ and small positive R.

It is known (see, e.g., [16]) that if (X,d,μ) is an SHT, then $\mathcal{P}^{\log}(X)=\mathcal{P}(X)$.

Let Ω be an open set in $\mathbb{R}^d$ and let μ be a Borel measure on Ω. The next statement shows the regularity of fractional integrals

$$J^\gamma_\Omega f(x)=\int_\Omega \frac{f(y)}{|x-y|^{n-\gamma}}d\mu(y),\qquad 0<\gamma<n,\quad x\in\Omega,$$

for $f\in M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$, where the measure μ on Ω satisfies the following condition: there are positive constants c_0 and n such that for all $x\in\Omega$ and $R>0$,

$$\mu\big(D(x,R)\big)\le c_0R^n,\qquad D(x,R):=B(x,R)\cap\Omega. \tag{6}$$

Sometimes, in this case, they say that μ is upper Ahlfors $n-$ regular.

Theorem 2.2. *Let μ be a finite Borel measure on Ω satisfying condition* (6). *Suppose that $p(\cdot)$ and $q(\cdot)$ are variable exponents on Ω such that $1<q_-\le q(x)\le p(x)\le p_+<\infty$, $x\in\Omega$. Assume that $\varphi(\cdot)\in\mathcal{A}_{q(\cdot)}$. Suppose that γ and ε are thepositive constants such that $\frac{n}{\gamma}<p_-\le p_+<\frac{n}{\gamma-\varepsilon}$. Let $q(\cdot)\in\mathcal{P}(\Omega)$ and let $p(\cdot)\in\mathcal{P}^{log}(\Omega)$. Then the operator J^γ_Ω is bounded from $M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$ to $\widetilde{H}^{p(\cdot)}_{\gamma,n}(\Omega)$, i.e., there is a positive constant c_0 such that for all $f\in M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$,*

$$[J^\gamma_\Omega f]_{\widetilde{H}^{p(\cdot)}_{\gamma,n}(\Omega)}\le c_0\|f\|_{M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)}. \tag{7}$$

For Theorem 2.2 in different settings see [13, 22].

We recall that the doubling condition for a Borel measure μ on an open set $\Omega\subset\mathbb{R}^d$ reads as follows: there is a constant $C_{dc}>0$ such that for all $x\in\Omega$ and $R>0$,

$$\mu\big(D(x,2R)\big)\le C_{dc}\mu\big(D(x,R)\big).$$

Corollary 2.1. *Let μ be a finite doubling Borel measure on Ω satisfying condition* (6). *Suppose that $p(\cdot)$ and $q(\cdot)$ are the variable exponents on Ω such that $1<q_-\le q(x)\le p(x)\le p_+<\infty$, $x\in\Omega$, and $q(\cdot),p(\cdot)\in\mathcal{P}^{log}(\Omega)$. Suppose that $\varphi(\cdot)\in\mathcal{A}_{q(\cdot)}$. Assume that γ and ε are the positive constants such that $\gamma>\varepsilon$ and $\frac{n}{\gamma}<p_-\le p_+<\frac{n}{\gamma-\varepsilon}$. Then the operator J^γ_Ω is bounded from $M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$ to $\widetilde{H}^{p(\cdot)}_{\gamma,n}(\Omega)$, i.e., there is a positive constant c_0 such that for all $f\in M^{p(\cdot)}_{q(\cdot),\varphi(\cdot)}(\Omega)$,* (7) *holds.*

Acknowledgement

The second and third authors were supported by Batumi Shota Rustaveli State University Grant. Contract numbers: No. 01-50/108, 1.02.2023 (D. Makharadze) and No. 01-50/109, 1.02.2023 (A. Meskhi).

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(Received 15.01.2023)

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