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ON GEOMETRIC PROPERTIES OF HENSTOCK–ORLICZ SPACES

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Abstract. In this paper, we extend the theory of Henstock–Orlicz spaces with respect to vector measure. We study the integral representation of operators. Lastly, we study the uniform convexity, reflexivity and the Radon–Nikodym property of the Henstock–Orlicz spaces $\mathscr{H}^{\theta}(\mu_\infty)$.

1. Introduction and Preliminaries

In the early 19^{th} century, Lebesgue's Theory of integration has taken a center stage in concrete problem of analysis. It was seen as early as in 1915 with the publications of de-la Vallèe Poussin, although it is the Banach space researches of 1920's that formally gave birth to what are later called the Orlicz spaces, first proposed by Z. W. Birnbaum and W. Orlicz. Later on, this space was further developed by Orlicz himself. Their monograph [18], Kransoselskii and Rutickii devoted entirely to Orlicz spaces. We refer to [7, 9, 21, 24, 28, 29] for detailed discussion of Orlicz space. Brooks and Dinculeanu have developed a theory of vector integration for a bounded family of measures (see [5]). In [22], Roy and Chakraborty developed a theory of Orlicz spaces for the case of Banach space valued functions with respect to a $\sigma-$bounded family of measures. In [23], Roy and Chakraborty developed integral representation as an application of their previous work [22]. In [16, 17], Kaminska discussed the criteria for uniform convexity of Orlicz spaces in the case of a non-atomic measure as well as in the case of a purely atomic measure. It is known that if f is bounded with a compact support, then the following conditions are equivalent:

(a) f is Henstock–Kurzweil integrable,
(b) f is Lebesgue integrable,
(c) f is Lebesgue measurable.

In general, every Henstock–Kurzweil integrable function is measurable, and f is Lebesgue integrable if and only if both f and $|f|$ are Henstock–Kurzweil integrable. This means that the Henstock–Kurzweil integral can be thought as a "non-absolutely convergent version of Lebesgue integral". The detailed on the Henstock–Kurzweil integral can be found in [11, 13, 14, 19, 26, 27, 30]. The Henstock–Orlicz spaces are developed in [12]. In this paper, we develop a theory of Henstock–Orlicz spaces for the vector valued functions with respect to a $\sigma-$bounded family of measures. Detales on vector measures can be found in [4–6, 15]. Throughout this paper, Σ denotes a $\sigma-$algebra of subsets of an abstract set $T \neq \emptyset$. $\mathscr{P}(T)$ is the class of all subsets of T, $\Sigma \subset \mathscr{P}(T)$ is a $\sigma-$algebra, X is a Banach space and X' is its topological dual. For each $A \in \Sigma$, the characteristic function of A

$$ch_A(t) = \begin{cases} 1 & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \in T \setminus A. \end{cases}$$

Recall, a vector measure is a $\sigma-$additive set function $\mu_\infty : \Sigma \to X$, the $\sigma-$additivity of μ_∞ is equivalent to the $\sigma-$additivity of the scalar-valued set functions $x'\mu_\infty : A \to x'(\mu_\infty(A))$ on Σ for every $x' \in X'$.

Definition 1.1 ([4])**.** The variation $|\mu_\infty|$ of μ_∞ is defined by

$$|\mu_\infty|(A) = \sup\left\{ \sum_{i=1}^{r} ||\mu_\infty(A_i)|| \ : A_i \in \Sigma,\ i = 1, 2, \ldots, r; A_i \cap A_j = \emptyset \ \text{for} \ i \neq j;\ \bigcup_{i=1}^{r} A_i \subset A \right\}.$$

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The semi-variation $||\mu_\infty||$ of μ_∞ is given by

$$||\mu_\infty||(A) = \sup_{x'\in X', ||x'||\leq 1} |x'\mu_\infty|(A).$$

A function $f : T \to \mathbb{R}$ is said to be μ_∞-measurable if

$$f^{-1}(B) \cap \{t \in T : f(t) \neq 0\} \in \Sigma$$

for each Borel subset $B \subset \mathbb{R}$.

Definition 1.2 ([4])**.** We say that a Σ-measurable function $f : T \to \mathbb{R}$ is Kluvánek–Lewis–Henstock–Kurzweil μ_∞-integrable, shortly (HKL) μ_∞-integrable, if the following properties hold:

$$f \quad \text{is} \quad |x'\mu_\infty| - \text{Henstock–Kurzweil integrable for each} \quad x' \in X',$$

and for every $A \in \Sigma$, there is $x_A^{(HK)} \in X$ with

$$x'(x_A^{(HK)}) = (HK)\int_A f\, d|x'\mu_\infty| \quad \text{for all} \quad x' \in X',$$

where the symbol (HK) denotes the usual Henstock–Kurzweil integral of a real-valued function with respect to an (extended) real-valued measure.

Definition 1.3 ([22])**.** A function $f : T \to X$ is $M-$measurable if there is a sequence of simple functions from $\mathscr{S}_X(\Sigma)$ converging to f M-a.e.

M denotes a $\sigma-$bounded family of positive measures defined on Σ. This means that for each $E \in \Sigma$, there exist pairwise disjoint collections $\{E_i\}_{i=1}^{\infty}$, $E_i \in \Sigma$ such that $E = \bigcup_{i=1}^{\infty} E_i$ and $\mathscr{S}_X(E)$ is a set of all X valued simple functions. If f is $M-$measurable, then f is μ_∞-measurable for each $\mu_\infty \in M$, we define $M(f) = \sup_{\mu_\infty \in M} H\int_T ||f||d|x'\mu_\infty|$. In this paper, we consider all functions which are $M-$measurable. Let $\mathscr{M}_X(M)$ denote the space of all $M-$ measurable functions $f : T \to X$ for which $M(f) < \infty$, then $\mathscr{M}_X(M)$ is complete with respect to the semi-norm $M(.)$. If $M(f) < \infty$, then f is integrable in $\mathscr{M}_X(E)$.

Definition 1.4 ([23])**.** Let $\chi : [0,\infty] \to [0,\infty]$ be a non-decreasing left-continuous function such that $\chi(0) = 0$. The inverse function ξ of χ is defined by $\xi(0) = 0$ and $\xi(v) = \sup\{u : \ \chi(u) < v\}$. If $\lim_{u\to\infty} \chi(u) = A < \infty$, then $\xi(v) = \infty$ for all $v > A$. The functions defined below

$$\theta(u) = \int_0^u \chi(t)dt \quad \text{and} \quad \phi(v) = \int_0^v \xi(t)dt$$

are called conjugate Young's functions and they satisfy Young's inequality

$$uv \leq \theta(u) + \phi(v), \ u, v \geq 0.$$

2. Henstock–Orlicz Space and Vector Measure

If $f \in \mathscr{H}^\theta(|x'\mu_\infty|)$, then there is a $k > 0$ such that $\theta(\frac{f}{k}) \in H(|x'\mu_\infty|) = L^1(|x'\mu_\infty|)$, where $H(|x'\mu_\infty|)$ and $L^1(|x'\mu_\infty|)$ are the Henstock integrable space and the Lebesgue integrable space, respectively. The space $\mathscr{H}^\theta(|x'\mu_\infty|)$ endowed with the Lexemburg norm

$$||f||_{\mathscr{H}^\theta} = \inf\left\{k > 0 \ : \ H\int_T \theta\left(\frac{|f|}{k}\right) d|x'\mu_\infty| \leq 1\right\} \tag{2.1}$$

is a Banach space.

Definition 2.1. For an $M-$measurable function $f : T \to X$, let us define

$$||f||_{\theta,M} = \sup_{\mu_\infty\in M}\ \sup_{g\in\mathscr{S}(\Sigma)} \left\{H\int_T ||fg||d|x'\mu_\infty| \ : M_\phi(g) \leq 1\right\}. \tag{2.2}$$

Clearly, (2.1) and (2.2) are equivalent.

Definition 2.2 ([23]). If X and Y are two Banach spaces and $\mu_\infty : \Sigma \to L(X,Y)$ is a countable additive measure, then $(\theta, M)-$variation of μ_∞ is defined by

$$|\mu_\infty|_{\theta,M}(E) = \sup \sum_{i=1}^{n} ||\mu_\infty(E_i)x_i|| \text{ for each } \quad E \in \Sigma.$$

Here, the supremum is taken over all simple functions $f = \sum_{i=1}^{n} x_i ch(E_i)$ belonging to $\mathscr{S}_X(\Sigma)$ with pairwise disjoint $E_i \subset E$ and $M_\theta(f) \leq 1$. The (θ, M) semi-variation of μ_∞, for $E \in \Sigma$ is written as

$$||\mu_\infty||_{\theta,M}(E) = \sup \Big\| \sum_{i=1}^{n} \mu_\infty(E_i)x_i \Big\|,$$

where the supremum is taken over all simple functions $f = \sum_{i=1}^{n} x_i ch(E_i)$ belonging to $\mathscr{S}_X(\Sigma)$ with pairwise disjoint $E_i \subset E$ and $M_\theta(f) \leq 1$.

Definition 2.3 ([23]). We call $\theta \in \Delta_2$ if $\theta(2t) \leq k\theta(t),\ k > 0$.

Theorem 2.1. *$\mathscr{H}^\theta(M,X)$ is complete with respect to $||.||_{\theta,M}$.*

Proof. Suppose $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathscr{H}^\theta(M,X), ||.||_{\theta,M})$, then $||f_n - f_m||_{\theta,M} \leq \frac{1}{2}$ for $m > n$. In general, $n_k,\ k \in \mathbb{N}$ such that $||f_n - f_m||_{\theta,M} < \frac{1}{2^k}$ for $m > n$. Since M is a $\sigma-$bounded, we can write $T = \bigcup_{i=1}^{\infty} E_i,\ E_i \cap E_j = \emptyset$ for $i \neq j$ and $M(E_i) < \infty$ for each $i = 1, 2, \ldots$. Let $t > 0$ be such that $M(E_i)\phi(t) \leq 1$. If $g(x) = t$ on E_i and $g(x) = 0$ otherwise, then we have $M_\phi(g) \leq 1$ and, consequently, $M(f_n - f_m) < \epsilon$ for $n, m \geq M_0$ so, $M(f_n - f_m) < \frac{\epsilon}{t}$ for $n, m \geq M_0$. This implies that $\{f_n\}$ is a Cauchy sequence in $\mathscr{S}_X(M)$. As (f_n) is Cauchy sequence in $\mathscr{S}_X(M)$, there exist a positive integer M_1 and a function $f_{M_1} \in \mathscr{S}_X(M)$ such that $M(f_n - f_{M_1}) < \frac{1}{2}$ for $n \geq M_1$. Similarly, on $E_2,\ M(f_n - f_{M_2}) < \frac{1}{2^2}$ for $n \geq M_2$ and so on. The series $M(f_{M_1}) + M(f_{M_2} - f_{M_1}) + \cdots$ is convergent. The fact that $\mathscr{S}_X(M)$ is complete implies that $(f_{M_1}) + (f_{M_2} - f_{M_1}) + \cdots$ converges a.e. to a function, say f in $\mathscr{S}_X(M)$, so, $M(f_n - f)(g) < \epsilon$ whenever $n \geq M_0$ and $M_\phi(g) \leq 1$. Thus $f_n - f \in \mathscr{H}^\theta(M,X)$ and we have $f \in \mathscr{H}^\theta(M,X)$. So, $(\mathscr{H}^\theta(M,X), ||.||_{\theta,M})$ is complete. □

Theorem 2.2. *Let $f_n \in \mathscr{H}^\theta(M,X)$. For $n = 1, 2, \ldots,$ then followings conditions are equivalent:*

(1) *$f \in \mathscr{H}^\theta(M,X)$ and $||f_n - f||_{\theta,M} \to 0$ as $n \to \infty$.*
(2) *$||f_n - f_m||_{\theta,M} \to 0$ as $m, n \to \infty$ and hence there exists a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ $M-$a.e.*

Proof. Proofs are similar to [22, Theorem 3.3]. □

Definition 2.4. We denote $H(M,X)$ is the collection of all functions $f : T \to X$ which are $M-$measurable and for which $\theta(||f||) \in H(M,\mathbb{R})$, where $H(M,\mathbb{R})$ are the Henstock–Kurzweil integrable function spaces.

Theorem 2.3. *If $f \in \mathscr{H}^\theta(M,X)$, there exists a constant $N > 0$ such that $Nf \in H(M,X)$. Moreover, if $f \in \mathscr{H}^\theta(M,X)$ and $f(x) \neq 0$, then*

$$\sup_{\mu_\infty \in M} H\int_T \theta\Big(\frac{||f||}{||f||_{\theta,M}}\Big) d|x'\mu_\infty| \leq 1.$$

Proof. Let $f \in \mathscr{H}^\theta(M,X)$ and $g \in \mathscr{S}(\Sigma)$, we need to prove

$$H\int_T \frac{||fg||}{||f||_{\theta,M}} d|x'\mu_\infty| \leq \max\{1, \mu_{\infty_\phi}(g)\}$$

for each $\mu_\infty \in M$. If $\alpha > 1$ and $t > 0$, then $\theta(\alpha t) \geq \alpha\phi(t)$. So, let $f \in \mathscr{H}^\theta(M, X)$ and $g \in \mathscr{S}(\Sigma)$, then

$$H\int_T ||fg||d|x'\mu_\infty| \leq ||f||_{\theta,M} \ \ ,provided\ \ \mu_{\infty_\phi} \leq 1. \tag{2.3}$$

If $1 < \mu_{\infty_\phi}(g) < \infty$, then

$$\begin{aligned}\phi(|g|) &= \phi\Big(\mu_{\infty_\phi}\frac{|g|}{\mu_{\infty_\phi}(g)}\Big)\\ &\geq \mu_{\infty_\phi}(g)\phi\Big(\frac{|g|}{\mu_{\infty_\phi}(g)}\Big).\end{aligned}$$

That is, $\frac{\phi(|g|)}{\mu_{\infty_\phi}(g)} \geq \phi\left(\frac{|g|}{\mu_{\infty_\phi}(g)}\right)$. So,

$$\begin{aligned}H\int_T \phi\Big(\frac{|g|}{\mu_{\infty_\phi}(g)}\Big)d|x'\mu_\infty| &\leq H\int_T \frac{\phi(|g|)}{\mu_{\infty_\phi}(g)}d|x'\mu_\infty|\\ &\leq 1.\end{aligned}$$

Thus $H\int_T \frac{||fg||}{\mu_{\infty_\phi}(g)}d|x'\mu_\infty| \leq ||f||_{\theta,M}$. This means

$$H\int_T ||fg||d|x'\mu_\infty| \leq ||f||_{\theta,M}\mu_{\infty_\phi}(g). \tag{2.4}$$

From (2.3) and (2.4), we get

$$\begin{aligned}&H\int_T ||fg||d|x'\mu_\infty| \leq ||f||_{\theta,M}\max\{1, \mu_{\infty_{\phi}(g)}, \}\\ &\text{i.e., } \ H\int_T \frac{||fg||}{||f||_{\theta,M}}d|x'\mu_\infty| \leq \max\{1, \mu_{\infty_\phi}(g)\}.\end{aligned}$$

Next, suppose that $f \in \mathscr{H}^\theta(M, X)$ and $||f||_{\theta,M} > 0$, from here we get two cases.

Case 1. If f is positive bounded and has a support, say E, of finite $M-$measurable. Let $g = \frac{f}{||f||_{\theta,M}}$. Since g is bounded on E, $\phi(\chi(||g||))$ is bounded so, $\sup\limits_{\mu_\infty\in M} H\int_T \phi(\chi(||g||))d|x'\mu_\infty|$ exists. Now, allowing the Young's inequality to an equality, we obtain

$$\begin{aligned}&\sup_{\mu_\infty\in M} H\int_T \theta(||g||)d|x'\mu_\infty|\\ &= \sup_{\mu_\infty\in M}\Big\{H\int_T \theta(||g||)d|x'\mu_\infty| + H\int_T \phi(\chi(||g||))d|x'\mu_\infty| - H\int_T \phi(\chi(||g||))d|x'\mu_\infty|\Big\}\\ &= \sup_{\mu_\infty\in M}\Big\{H\int_T ||g\chi(||g||)||d|x'\mu_\infty| - H\int_T \phi(\chi(||g||))d|x'\mu_\infty|\Big\}.\end{aligned}$$

As $\chi(||g||)$ is a positive bounded M-measurable function, there exists a non-decreasing sequence of positive simple functions $\{g_n\}$ such that $g_n \to \chi(||g||)$ $M-$a.e. and $g_n \leq \chi(||g||)$ and $||gg_n|| \leq ||g\chi(||g||)||$. So, [5, Equation 7 of page 353],

$$\sup_{\mu_\infty\in M} H\int_T ||g\chi(||g||)||d|x'\mu_\infty| \leq \sup_{\mu_\infty\in M}\lim_n H\int_T ||gg_n||d|x'\mu_\infty|.$$

Hence

$$\sup_{\mu_\infty \in M} H\int_T \theta(||g||)d|x'\mu_\infty| \leq \sup_{\mu_\infty \in M} \max\{1-\mu_{\infty_\phi}(\chi||g||,0)$$
$$\leq 1.$$

Case 2: Suppose $f \in \mathscr{H}^\theta(M,X)$ and $||f||_{\theta,M} > 0$. Since M is $\sigma-$bounded, we can choose increasing sequences of sets $\{E_n\}$ of finite $M-$measurable such that $T = \bigcup_{n=1}^{\infty} E_n$. Using (2.2) for $M(||f_n|| \to M||f||)$ for each $n = 1,2,\dots,$ we have

$$\sup_{\mu_\infty \in M} H\int_T \theta\Big(\frac{||f_n||}{||f_n||_{\theta,M}}\Big)d|x'\mu_\infty| \leq 1. \qquad \square$$

Proposition 2.1. *Suppose $\theta \in \Delta_2$, then $\mathscr{H}^\theta(M,X) = H(M,X)$, also if $f_n \in \mathscr{H}^\theta(M,X)$ with $\sup_{\mu_\infty \in M} H\int_T \theta(||f_n||)d|x'\mu_\infty| \to 0$, then $||f_n||_{\theta,M} \to 0$.*

Proof. Let $f : T \to X$ be an $M-$measurable function and $g \in \mathscr{S}(\Sigma)$. Then by Young's inequality,

$$H\int_T ||fg||d|x'\mu_\infty| \leq H\int_T \theta(||f||d|x'\mu_\infty|.$$

So, if $f \in H(M,X)$, then $f \in \mathscr{H}^\theta(M,X)$. Let $f \in \mathscr{H}^\theta(M,X)$ and $||f||_{\theta,M} \neq 0$. Now, from Theorem 2.3, $\frac{f}{||f||_{\theta,M}} \in H(M,X)$. Now, as $f \in \mathscr{H}^\theta(M,X)$ implies $||f||_{\theta,M} < \infty$, that is, $||f||_{\theta,M} \leq 2^m;\ m > 0$. Therefore

$$\theta(||f||) = \theta\Big(\frac{||f||}{||f||_{\theta,M}}\Big)||f||_{\theta,M}$$
$$\leq 2^m\theta\Big(\frac{||f||}{||f||_{\theta,M}}\Big).$$

Thus $\theta(||f||) \in H(M,X)$, so, $f \in H(M,X))$. Hence $\mathscr{H}^\theta(M,X) = H(M,X)$. $\square$

Theorem 2.4. *$\mathscr{S}_X(\Sigma)$ is dense in $\mathscr{H}^\theta(M,X)$.*

Proof. Choose arbitrarily $\epsilon > 0$ and $f \in \mathscr{H}^\theta(M,X)$, that is, $g \in H(M,X)$ with

$$||g-f||_{\mathscr{H}^\theta(M,X)} \leq \frac{\epsilon}{N+1}.$$

Moreover, in correspondence with ϵ and g, we find a $\Sigma-$simple function s, with

$$||s-g||_{H(M,X)} \leq \frac{\epsilon}{N+1}.$$

As $||.||_{\mathscr{H}^\theta(M,X)} \leq M||.||_{H(M,X)}$ hence we obtain

$$\begin{aligned}||s-f||_{\mathscr{H}^\theta(M,X)} &\leq ||s-f||_{\mathscr{H}^\theta(M,X)} + ||g-f||_{\mathscr{H}^\theta(M,X)}\\ &\leq N(||s-g||_{H(M,X)}) + ||g-f||_{\mathscr{H}^\theta(M,X)}\\ &\leq \frac{N\epsilon}{N+1} + \frac{\epsilon}{N+1}\\ &= \epsilon.\end{aligned}$$

So, $\mathscr{S}_X(\Sigma)$ is dense in $\mathscr{H}^\theta(M,X)$. $\square$

Proposition 2.2. *Suppose $\mu_\infty : \Sigma \to L(X,Y)$ is countable additive with $||\mu_{\infty_{X,Y}}||(T) < \infty$, then $\mathscr{H}^\theta(M,X) \subset H(\mu_{\infty_{X,Y}},X)$ and $\mu_{\infty_{X,Y}}(f) \leq ||\mu_{\infty_{\theta,M}}||(T)||f||_{\theta,M}$ for $f \in \mathscr{H}^\theta(M,X)$.*

Proof. Let $f \in \mathscr{H}^\theta(M,X)$. Then by Theorem 2.4, there is a sequence $\{f_n\}$ of simple functions converging to f in $\mathscr{H}^\theta(M,X)$. Let $y_1 \in Y'$ be such that $|y_1| \le 1$. If $f = \sum\limits_{i=1}^{n} x_i ch(E_i) \in \mathscr{S}_X(E)$, then

$$H\int_T ||f||d|\mu_{\infty_i} = \sum_{i=1}^{n} ||x_i|||\mu_{\infty_{y_1}}|(E_i).$$

Suppose $\epsilon > 0$ is arbitrary and for each i, let $\{\mathscr{F}_{ij}\}$ be a finite family of disjoint sets in Σ contained in E_i such that

$$|\mu_{\infty_{y_i}}|(E_i) < \sum_{ij} ||\mu_{\infty_{y_1}}(\mathscr{F}_{ij})|| + \frac{\epsilon}{n||x_i||}.$$

Then

$$H\int_T ||f||d|\mu_{\infty_{y_1}}| < \sum_{ij} ||\mu_{\infty_{y_1}}(\mathscr{F}_{ij})||||x_i|| + \epsilon.$$

For each $\mathscr{F}_{ij}$, we can choose an x_{ij} with $||x_{ij}|| = 1$ such that

$$||\mu_{\infty_{y_1}}(\mathscr{F}_{ij})|| < |\mu_{\infty_{y_1}}(\mathscr{F}_{ij}x_{ij})| + \frac{\epsilon}{k},$$

where k is a constant, and by Theorem 2.3,

$$\begin{aligned} H\int_T ||f||d|\mu_{\infty_{y_1}}| &\le \sum_{ij} |\mu_{\infty_{y_1}}(\mathscr{F}_{ij})x_{ij}||x_{ij}||| + \epsilon \\ &= \sum_{i,j} \Big|\mu_{\infty_{y_1}}(\mathscr{F}_{ij})x_{ij}\frac{||x_i||}{||f||_{\theta,M}}\Big| ||f||_{\theta,M} + \epsilon \\ &\le |\mu_{\infty_{y_1}}|_{\theta,M}(T)||f||_{\theta,M} + \epsilon. \end{aligned}$$

So,

$$H\int_T ||f||d|\mu_{\infty_{y_1}}| \le |\mu_{\infty_{y_1}}|_{\theta,M}(T)||f||_{\theta,M}.$$

Therefore

$$\mu_{\infty_{X,Y}}(f) \le ||\mu_\infty||_{\theta,M}(T)||f||_{\theta,M}.$$

From the above, we have

$$\mu_{\infty_{X,Y}}(f_n - f_m) \le ||\mu_\infty||_{\theta,M}(T)||f_n - f_m||_{\theta,M}.$$

This means that $\{f_n\}$ is the Cauchy sequence in $H(\mu_{\infty_{X,Y}}, X)$. Thus $f \in H(\mu_{\infty_{X,Y}}, X)$ as $f_n \to f$ is $\mu_{\infty_{X,Y}}$-a.e. and $\mu_{\infty_{X,Y}}(f_n - f) \to 0$ as $n \to \infty$. So, $\mu_{\infty_{X,Y}}(f) \le ||\mu_{\infty_{\theta,M}}||(T)||f||_{\theta,M}$ for $f \in \mathscr{H}^\theta(M,X)$ as this is true for each n. □

We now prove that $(\mathscr{H}^\theta, ||.||_{\mathscr{H}^\theta})$ is a solid Banach lattice with a weak order unit. Recall, a partially ordered Banach space Z, which is also a vector lattice, is a Banach lattice if $||x|| \le ||y||$ for every $x, y \in Z$ with $|x| \le |y|$. A weak order unit of Z is a positive element $e \in Z$ such that if $x \in Z$ and $x \Lambda e = 0$, then $x = 0$. Let Z be a Banach lattice and $A \ne \emptyset \subset B \subset Z$. We say that A is solid in B if for each x, y with $x \in B,\ y \in A$ and $|x| \le |y|$, it is $x \in A$.

Theorem 2.5. *$(\mathscr{H}^\theta, ||.||_{\mathscr{H}^\theta})$ is a solid Banach lattice with a weak order unit.*

Proof. It is clear that $(\mathscr{H}^\theta, ||.||_{\mathscr{H}^\theta})$ is a solid normed lattice with respect to the usual order.
By the Rybakov theorem (see also [8, Theorem IX.2.2]), there is $x_1' \in X'$ with $||x_1'|| \le 1$ such that $\lambda = x_1'\mu_\infty$ is a control measure of μ_∞. If $f, g \in \mathscr{H}^\theta(|x'\mu_\infty|),\ |f| \le |g|$ λ-a.e, $k \in \mathbb{N}$ and $x' \in X'$ with $||x'|| \le 1$, then

$$\inf\Big\{k > 0 :\ \theta\Big(\frac{|f|}{k}\Big)d|x'\mu_\infty| \le 1\Big\} \le \inf\Big\{k > 0 :\ \theta\Big(\frac{|g|}{k}\Big)d|x'\mu_\infty| \le 1\Big\}.$$

That is, $||f||_{\mathscr{H}^\theta} \le ||g||_{\mathscr{H}^\theta}$. So, $(\mathscr{H}^\theta(|x'\mu_\infty|), ||.||_{\mathscr{H}^\theta})$ is a Banach lattice. Moreover, $\mathscr{H}^\theta(|x'\mu_\infty|)$ is easily a weak order unit. □

Theorem 2.6. *$(\mathscr{H}^\theta, ||.||_{\mathscr{H}^\theta})$ is a Köthe function spaces.*

3. Integral Representations

We discuss the integral representations of operators from $\mathscr{H}^\theta(M, X)$ into Y. We assume that M is relatively weakly compact in $Ca(\Sigma)$, then we can find a control measure $\lambda : \Sigma \to \mathbb{R}^+$ for M such that $\lambda \leq M$ and $M << \lambda$.

Proposition 3.1. *Let $f : T \to C$ be a $\lambda-$measurable function such that $fg \in H(\lambda)$ for each $g \in \mathscr{H}^\theta(M)$, then $\sup\{H\int_T |fg|d\lambda :\ g \in \mathscr{H}^\theta(M),\ ||g||_{\theta,M} \leq 1\} < \infty$.*

Proof. If possible, let $\sup\{H\int_T |fg|d\lambda :\ g \in \mathscr{H}^\theta(M),\ ||g||_{\theta,M} \leq 1\} = \infty$, then for each n, there exists a sequence $\{g_n\} \subset \mathscr{H}^\theta(M)$ with $||g_n||_{\theta,M} \leq 1$ such that $H\int_T |fg_n|d\lambda \geq n.2^n$. If $g_0(r) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}|g_n(r)|$, then

$$\begin{aligned} ||g_0||_{\theta,M} &\leq \frac{1}{2^n}||g_n||_{\theta,M} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 1. \end{aligned}$$

So, $g_0 \in \mathscr{H}^\theta(M)$ and hence $fg_0 \in H(\lambda)$, as

$$H\int_T |fg_0|d\lambda \geq \frac{1}{2^n} H\int_T |fg_n|d\lambda \geq n \ \text{ for each } \ n.$$

This implies that fg_0 does not belong to $H(\lambda)$. Hence

$$\sup\left\{H\int_T |fg|d\lambda : g \in \mathscr{H}^\theta(M),\ ||g||_{\theta,M} \leq 1\right\} < \infty.$$ □

We now discuss the existence of a countable additive measure for a continuous linear operator from $\mathscr{H}^\theta(M, X)$ into Y.

Theorem 3.1. *Suppose $\Delta : \mathscr{H}^\theta(M, X) \to Y$ is a continuous linear operator, then there exists a countable additive measure $\mu_\infty : \Sigma \to L(X, Y)$ such that*

$$\Delta f = H\int_T f d\mu_\infty \quad \text{for } \ f \in \mathscr{H}^\theta(M, X).$$

Proof. Suppose $\Delta : \mathscr{H}^\theta(M, X) \to Y$ is a continuous linear operator. Let us define $\mu_\infty : \Sigma \to L(X, Y)$ by $\mu_\infty(E)x = \Delta(xch(E))$. The linearity of $\mu_\infty(E)$, for each $E \in \Sigma$ is very obvious. Also,

$$\begin{aligned} ||\mu_\infty(E)x|| &= ||\Delta(xch(E))|| \\ &\leq ||\Delta||||xch(E)||_{\theta,M} \\ &\leq ||\Delta||||ch(E)||_{\theta,M}||x||. \end{aligned}$$

Therefore Δ is continuous linear operator. Now, we need to show that μ_∞ is a countable additive measure. Since

$$\begin{aligned} M_\theta(ch(E)) &= \sup_{\mu_\infty \in M} H\int_T \theta(ch(E))d\mu_\infty \\ &= \theta(1)M(E). \end{aligned}$$

If $E_n \to \emptyset$, then $M_\theta(ch(E_n)) \to 0$. Using Proposition 2.1, $||ch(E_n)||_{\theta,M} \to 0$. Since μ_∞ is finitely additive, the above limit shows that μ_∞ is countable additive and $\Delta f = H\int_T f d\mu_\infty$ for $f \in \mathscr{S}_X(\Sigma)$. Now, using Theorem 2.4, $\Delta f = H\int_T f d\mu_\infty$ for $f \in \mathscr{H}^\theta(M, X)$. □

Theorem 3.2. *If* $\Delta : \mathscr{H}^\theta(M, X) \to Y$ *defined by* $\Delta f = H\int_T f d\mu_\infty$ *for* $f \in \mathscr{H}^\theta(M, X)$ *with* $||\mu_\infty||_{\theta,M}(T) < \infty$, *then* $||\Delta|| \le \mu_{\infty_{\theta,M}}(T) \le 2||\Delta||$.

We now show the converse of Theorem 3.2 holds for a countable additive measure.

Theorem 3.3. *If* μ_∞ *is a countable additive measure and* $\Delta : \mathscr{H}^\theta(M, X) \to Y$ *defined by* $\Delta f = H\int_T f d\mu_\infty$ *for* $f \in \mathscr{H}^\theta(M, X)$ *with* $||\mu_\infty||_{\theta,M}(T) < \infty$, *then* Δ *is a continuous linear operator.*

Proof. Suppose $\mu_\infty : \Sigma \to L(X, Y)$ is a countable additive measure such that $||\mu_\infty||_{\theta,M}(T) < \infty$, then the mapping $\Delta : \mathscr{H}^\theta(M, X) \to Y$ defined by $\Delta f = H\int_T f d\mu_\infty$ for all $f \in \mathscr{H}^\theta(M, X)$ is linear. Also,

$$\begin{aligned}
||\Delta f|| &= ||H\int_T f d\mu_\infty|| \\
&= \sup_{||y_1||\le 1} \left| H\int_T d|y_1\mu_\infty| \right| \\
&\le \sup_{||y_1||\le 1} H\int_T ||f|| d|y_1\mu_\infty| \\
&= \mu_{\infty_{X,Y}}(f) \\
&\le ||\mu_\infty||_{\theta,M}(T)||f||_{\theta,M} \;\; \text{using the Proposition 2.2.}
\end{aligned}$$

Hence $\Delta : \mathscr{H}^\theta(M, X) \to Y$ is continuous. □

4. Geometrical Properties

In this section, we discuss about the uniformly convexity, reflexivity of $\mathscr{H}^\theta(M, X)$ and the Radon–Nikodym property of the Henstock–Orlicz spaces $\mathscr{H}^\theta(\mu_\infty)$. Before we proceed to discussing the uniform convexity and reflexivity of $\mathscr{H}^\theta$, we state about Modular spaces. The modular spaces can be defined as $B_{\theta,|x'\mu_\infty|}(f) = H\int_T \theta(f)d|x'\mu_\infty|$. For a complementary Young's functions θ and ϕ, we define

$$\mu_{\infty_\phi}(g) = H\int_T \phi(|g|)d\mu_\infty, \; g \in S(\Sigma),$$
$$M_\phi(g) = \sup_{\mu_\infty \in M} \mu_{\infty_\phi}(g).$$

Definition 4.1 ([20])**.** A Banach space $(X, ||.||)$ is uniformly convex if for each $\epsilon > 0$, there exists $p(\epsilon) \in (0, 1)$ such that $||x|| = ||y|| = 1, \; ||x - y|| \ge \epsilon$ implies $||\frac{x+y}{2}|| \le 1 - p(\epsilon)$.
Similarly, the modular $B_{\theta,|x'\mu_\infty|}$ is uniformly convex if for each $\epsilon > 0$, there exists $p(\epsilon) \in (0, 1)$ such that $B_{\theta,|x'\mu_\infty|}(x) = B_{\theta,|x'\mu_\infty|}(y) = 1, \; B_{\theta,|x'\mu_\infty|}(x - y) \ge \epsilon$ implies $B_{\theta,|x'\mu_\infty|}(\frac{x+y}{2}) \le 1 - p(\epsilon)$.

Proposition 4.1. *Let* θ *satisfy the* Δ_2 *condition with* μ_∞ *is atomless and* $\mu_\infty(T) < \infty$, *then* $\mathscr{H}^\theta(M, X)$ *is locally uniformly convex if and only if pseudomodular* $B_{\theta,|x'\mu_\infty|}(M, X)$ *is locally uniformly convex.*

Proof. Proof is similar to the technique of [17, Lemma 3]. □

Theorem 4.1. $\mathscr{H}^\theta(M, X)$ *is uniformly convex if and only if the modular* $B_{\theta,|x'\mu_\infty|}(M, X)$ *is uniformly convex.*

Proof. We can prove it by using Proposition 4.1 with similar technique [16, Lemma 1]. □

Theorem 4.2. *Assume* M *is non-atomic, then* $\mathscr{H}^\theta(M, X)$ *is reflexive.*

Proof. Uniform convex implies reflexivity, so, this is obvious. □

Recalling a Banach space X is said to have Radon–Nikodym property, shortly RNP, if given a finite measure spaces (T,Σ,μ) and a vector measure $\mu_\infty:\Sigma\to X$ of finite variation and absolutely continuous with respect to μ_∞, there exists a Bochner integrable function $g:\Omega\to X$ such that $\mu_\infty(E)=\int_E g d\mu$ for any $E\in\Sigma$. The fact that the Bochner integral is McShane integral (see also [10]), every McShane integral is a Henstock integral (see [14, page 158]). With this known fact, we can define the Radon–Nikodym property as follows for our setting:

Definition 4.2. $\mathscr{H}^\theta(\Omega)$ is said to have Radon–Nikodym property, shortly RNP, if given a finite measure spaces (T,Σ,μ) and a vector measure $\mu_\infty:\Sigma\to\mathscr{H}^\theta$ of finite variation and absolutely continuous with respect to μ_∞, there exists a Henstock integrable function $g:T\to\mathscr{H}^\theta$ such that $\mu_\infty(E)=H\int_E g d\mu$ for any $E\in\Sigma$.

Definition 4.3. We say that $f\in X$ has absolute norms if for every decreasing sequence $\{G_n\}$ of a subset of Ω satisfying $\mu(G_n)\to 0$, we have $||fch(G_n)||\to 0$.

Theorem 4.3. *$(\mathscr{H}^\theta, H_\theta)$ has an absolutely continuous norm.*

Proof. Let $E_n=\{x\in\Omega:\ n\le\theta(x)<n+1\},\ n\in\mathbb{N}$. Then there exists a sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $\mu_\infty(E_{n_k})>0$. Let $c_k>0$ be such that $H\int_{E_{n_k}} c_k^{\theta(x)} d|x'\mu_\infty|=1,\ k\in\mathbb{N}$.

We have $f(x)=\sum\limits_{k=1}^{\infty} c_k ch(E_{n_k})(x),\ x\in\Omega$ and $E_j=\bigcup\limits_{k=j}^{\infty} E_{n_k}$. Then $E_n\to\emptyset$.

Now,

$$\begin{aligned}||f||&=\inf\left\{\lambda>0:\ H\int\limits_{E_{n_k}}\theta\left(\frac{|f|}{\lambda}\right)d|x'\mu_\infty|\le 1\right\}\\&\le\inf\left\{\lambda>1:\ \sum_{k=1}^{\infty}\left(\frac{1}{\lambda}\right)^{n_k}\le 1\right\}\\&\le 2.\end{aligned}$$

So, $f\in\mathscr{H}^\theta(\mu_\infty)$. Hence

$$||fch(E_n)||=\inf\left\{\lambda>0\ :H\int\limits_{E_{n_l}}\theta\left(\frac{c_l}{\lambda}\right)d|x'\mu_\infty|\le 1\right\}.$$

If $\mu_\infty(E_n)\to 0$ implies $||fch(E_n)||\to 0$, so, $\mathscr{H}^\theta(\mu_\infty)$ has an absolutely continuous norm. □

Theorem 4.4. *Let $T=\bigcup\limits_{n=1}^{\infty} A_n$ be a union of measurable sets. If the subspace $\mathscr{H}^\theta_n=\{f\in\mathscr{H}^\theta:\ f=0\ \text{on}\ T\setminus A_n\}$ of $\mathscr{H}^\theta$, then $\mathscr{H}^\theta$ has RNP.*

Proof. Let (T,Σ,μ) be a finite measure space and let $\mu_\infty:\Sigma\to\mathscr{H}^\theta$ be a vector measure of finite variation which is absolutely continuous with respect to μ. We define projections $\mathscr{P}_n:\mathscr{H}^\theta\to\mathscr{H}^\theta_n$ by $\mathscr{P}_n(f)=fch(A_n)$ and $\mu_{\infty_n}=\mathscr{P}_n(\mu_\infty)$, then each μ_{∞_n} is an $\mathscr{H}^\theta_n-$valued vector measure of finite variation which is absolutely continuous with respect to μ. With the fact that each space $\mathscr{H}^\theta_n$ has the RNP, there exists a Henstock integrable function $h_n:T\to\mathscr{H}^\theta_n$ satisfying $\mu_{\infty_n}(E)=\int_T h_n d\mu$ for each $E\in\Sigma$. Now, for $E\in\Sigma$ and $n\in\mathbb{N}$, we have

$$H\int\limits_E\left\|\sum_{k=1}^{n}h_k\right\|_{H_\theta}d\mu\le|\mu_\infty|(E),\tag{4.1}$$

and so, for $\mu-$almost all $\alpha \in \Omega$, there exists $h(\alpha) \in \mathscr{H}^\theta$ such that $h(\alpha) = \sum_{k=1}^{\infty} h_k(\alpha)$. Using (4.1) and Fatou's lemma, h is also Bochner integrable and so Henstock integrable. Finally,

$$\begin{aligned}\mu_\infty(E) &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu_{\infty_k}(E) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} H \int_E h_n d\mu \\ &= H \int_E g d\mu.\end{aligned}$$

Hence the proof is complete. □

5. Conclusion

In this article, we discuss about Henstock–Orlicz space with vector measure. In Geometrical property, we discuss about Uniform convexity, Reflexivity and, finally, about Radon Nikodym Property. Our one purpose of this article was to discuss about RNP without Δ_2 property, but we unable to prove our assumed result "$\mathscr{H}^\theta(M, X)$ has RNP if and only if X has RNP". In our research, to prove the above result, we need Henstock differentiation. Interested Researcher can think about Henstock-differentiation with the technique of Bochner differentiation.

References

1. M. A. Al-Gwaiz, *Theory of Distributions.* Monographs and Textbooks in Pure and Applied Mathematics, 159. Marcel Dekker, Inc., New York, 1992.
2. A. Alexiewicz, Linear functionals on Denjoy-integrable functions. *Colloq. Math.* **1** (1948), 289–293.
3. C. Bennett, R. Sharpley, *Interpolation of Operators.* Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.
4. A. Boccuto, B. Hazarika, H. Kalita, Kuelbs–Steadman Spaces for Banach Space-Valued Measures, *Mathematics*, **8** (2020), no. 6, 1005.
5. J. K. Brooks, N. Dinculeanu, Lebesgue-type spaces for vector integration, linear operators, weak completeness and weak compactness. *J. Math. Anal. Appl.* **54** (1976), no. 2, 348–389.
6. J. Diestel, J. J. Uhl, *Vector Measures.* With a foreword by B. J. Pettis. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.
7. T. K. Donaldson, N. S. Trudinger, Orlicz-Sobolev spaces and imbedding theorems. *J. Functional Analysis* **8** (1971), 52–75.
8. J. J. Duistermaat, *J. A. C Kolk Distribution: Theory and Application*, Berlin Heidelberg, Newyork, Springer, 2006.
9. M. Et, Y. Altin, B. Choudhary, B.C. Tripathy, On some classes of sequences defined by sequences of Orlicz functions. *Math. Inequal. Appl.* **9** (2006), no. 2, 335–342.
10. D. L. Forkert, *The Banach space-valued integrals of Riemann, Mcshane, Henstock–Kurzweil and Bochner*, Bachelor Thesis, Vienna University of Technology, November 22, 2012.
11. R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock.* Graduate Studies in Mathematics, 4. American Mathematical Society, Providence, RI, 1994.
12. B. Hazarika, H. Kalita, Henstock-Orlicz space and its dense space. *Asian-Eur. J. Math.* **14** (2021), no. 7, Paper no. 2150114, 17 pp.
13. R. Henstock, The equivalence of generalized forms of the Ward, variational, Denjoy-Stieltjes, and Perron-Stieltjes integrals. *Proc. London Math. Soc. (3)* **10** (1960), 281–303.
14. R. Henstock, *The General Theory of Integration.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.
15. H. Kalita, B. Hazarika, Countable additivity of Henstock-Dunford integrable functions and Orlicz space. *Anal. Math. Phys.* **11** (2021), no. 2, Paper no. 96, 13 pp.
16. A. Kaminska, On uniform convexity of Orlicz spaces. *Nederl. Akad. Wetensch. Indag. Math.* **44** (1982), no. 1, 27–36.
17. A. Kaminska, The criteria for local uniform rotundity of Orlicz spaces. *Studia Math.* **79** (1984), no. 3, 201–215.
18. M. A. Krasonsel'skii, Ya. B. Rutickii, *Convex Functions and Orlicz Spaces.* Translated from the first Russian edition by Leo F. Boron P. Noordhoff Ltd., Groningen 1961.
19. J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. (Russian) *Czechoslovak Math. J.* **7** (1957), no. 82, 418–449.
20. W. A. J. Luxemburg, *Banach Function Spaces.* Thesis. Technische Hogeschool te Delft, Delft, 1955.

21. M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*. Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.
22. S. K. Roy, N. D. Chakraborty, Orlicz spaces for a family of measures. *Anal. Math.* **12** (1986), no. 3, 229–235.
23. S. K. Roy, N. D. Chakraborty, On Orlicz spaces for a family of measures. *J. Math. Anal. Appl.* **145** (1990), no. 2, 485–503.
24. Ben-Zion A. Rubshtein, G. Ya. Grabarnik, M. A. Muratov, Y. S. Pashkova, *Foundations of Symmetric Spaces of Measurable Functions*. Lorentz, Marcinkiewicz and Orlicz spaces. Developments in Mathematics, 45. Springer, Cham, 2016.
25. N. Subramanian, R. Nallaswamy, N. Saivaraju, Characterization of entire sequences via double Orlicz space. *Int. J. Math. Math. Sci.* 2007, Art. ID 59681, 10 pp.
26. C. Swartz, *Introduction to Gauge Integrals*. World Scientific Publishing Co., Singapore, 2001.
27. B. S. Thomson, *Theory of Integral*. Classical Real Analysis. Com., 2008.
28. T. V. Thung, Some collections of functions Dense in an Orlicz Space. *Acta Math. Vietnamica.* **25**(2)(2000), no. 2, 195–208.
29. B. C. Tripathy, R. Goswami, Vector valued multiple sequence spaces defined by Orlicz function. *Bol. Soc. Parana. Mat.* **33** (2015), no. 1, 69–81.
30. L. T. Yeong, *Henstock-Kurzweil Integration on Euclidean Spaces*. Series in real analysis, vol. 12, World Scientific, 2011.

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