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SOME FIXED POINT RESULTS OF ENRICHED CONTRACTIONS BY KRASNOSELSKIJ ITERATIVE METHOD IN PARTIALLY ORDERED BANACH SPACES

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Abstract. In this paper, we prove some fixed point results for enriched contraction and enriched Kannan contraction in partially ordered Banach spaces. Also, some examples are given to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

S. Banach [6] introduced a fixed point result, which is well known as the Banach contraction principle. Since then, several authors proved many fixed point results for the Banach contraction principle (refer to [3, 12, 16, 17, 23, 25, 26, 28–31]). V. Berinde [7] introduced the technique of enrichment of nonexpansive mappings in Hilbert spaces and proved some fixed points results for enriched nonexpansive mappings. In 2020, Berinde and Pacurar proved some fixed point results for enriched contraction mapping [8], enriched Kannan mapping [9] and enriched Chatterjea mapping [10] in the setting of a Banach space. They used the Krasnoselskij iteration for approximate the fixed points of enriched mappings. In this work, we prove some fixed point results for enriched contractions and enriced Kannan contractions in partially ordered Banach spaces. A self-mapping T on X is called a Picard operator (abbreviated P.O.) if $\operatorname{Fix}(T) = \{p\}$ and $\lim\limits_{n\to+\infty} T^n = p$ for any $x \in X$ [24]. We need the Krasnoselskij iterative method, for which we prove fixed point results in the class of enriched contractions. Indeed, fixed points of such mappings can be approximated by a suitable Krasnoselskij iteration. Let X be a linear space and C be a convex subset of X and $T : C \to C$. For any $\lambda \in (0, 1)$, define $T_\lambda x = (1 - \lambda)x + \lambda Tx$ for all $x \in C$. For $\lambda = 0$, we get $T_0 = I_X$, the identity mapping on X. In this case, T_λ is called an averaged mapping. Berinde and Pacurar in [8] introduced the following definition.

Definition 1 ([8])**.** Let $(X, ||.||)$ be a linear normed space. A mapping $T : X \to X$ is called an enriched contraction if there exist $k \in (0, +\infty)$ and $h \in [0, k + 1)$ such that for all $x, y \in X$

$$||k(x - y) + Tx - Ty|| \leq h||x - y||.$$

Then T is called a (k, h)-enriched contraction. Obviously, if T is a contraction mapping with contraction constant h, then T is a $(0, h)$-enriched contraction with $k = 0$.

Berinde and Pacurar [8] proved that any enriched contraction has a unique fixed point.

Theorem 1. *Let $(X, ||.||)$ be a Banach space and $T : X \to X$ be a (k, h)-enriched contraction. Then*
1) $\operatorname{Fix}\{T\} = \{p\}$.
2) *There exists $\lambda \in (0, 1]$ such that the iterative sequence $\{x_n\}_{n=0}^{+\infty}$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n n \geq 0,$$

converges to p, for any $x_0 \in X$.
3) *For all $n = 0, 1, 2, \ldots$ and $i = 1, 2, 3, \ldots$, we have*

$$||x_{n+i-1} - p|| \leq \frac{\delta^i}{1 - \delta}||x_n - x_{n-1}||,$$

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where $\delta = \frac{h}{1+k}$.

Berinde and Pacurar [9] introduced a class of contractive mappings, called enriched Kannan contraction and proved some fixed point results for such contractions in Banach spaces.

Definition 2. Let $(X, ||.||)$ be a linear normed space. A mapping $T : X \to X$ is called an enriched Kannan contraction if there exist $k \in [0, +\infty)$ and $h \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$||k(x-y) + Tx - Ty|| \leq h(||x - Tx|| + ||y - Ty||).$$

Then T is called a (k, h)-enriched Kannan mapping. Obviously, if T is a Kannan contraction with constant h, then T is a $(0, h)$-enriched Kannan mapping with $k = 0$.

Theorem 2. *Let $(X, ||.||)$ be a Banach space and also $T : X \to X$ be a (k, h)-enriched Kannan contraction. Then we have*
1) $\mathrm{Fix}\{T\} = p$.
2) *There exists $\lambda \in (0, 1]$ such that the iterative sequence $\{x_n\}_{n=0}^{+\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \ \ n \geq 0,$$

converges to p, for any $x_0 \in X$.
3) *For all $n = 0, 1, 2, \ldots$ and $i = 1, 2, 3, \ldots$, we have*

$$||x_{n+i-1} - p|| \leq \frac{\delta^i}{1-\delta} ||x_n - x_{n-1}||,$$

where $\delta = \frac{h}{1-h}$.

Recently, Berinde and Pacurar [10] used the technique of enrichment of contractive type mappings to the class of Chatterjea mappings and proved the following fixed point theorem.

Definition 3. Let $(X, ||.||)$ be a linear normed space. A mapping $T : X \to X$ is called an enriched Chatterjea mapping if there exist $k \in [0, +\infty)$ and $h \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$||k(x-y) + Tx - Ty|| \leq h(||(k+1)(x-y) + yTy|| + ||(k+1)(y-x) + x - Tx||).$$

Then T is called a (k, h)-enriched Chatterjea mapping. Obviously, if T is a Chatterjea contraction with constant h, then T is a $(0, h)$-enriched Chatterjea mapping with $k = 0$.

Theorem 3. *Let $(X, ||.||)$ be a Banach space and also $T : X \to X$ be a (k, h)-enriched Chatterjea mapping. Then we have*
1) $\mathrm{Fix}\{T\} = \{p\}$.
2) *There exists $\lambda \in (0, 1]$ such that the iterative sequence $\{x_n\}_{n=0}^{+\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, n \geq 0,$$

converges to p, for any $x_0 \in X$.
3) *For all $n = 0, 1, 2, \ldots$ and $i = 1, 2, 3, \ldots$, we have*

$$||x_{n+i-1} - p|| \leq \frac{\delta^i}{1-\delta} ||x_n - x_{n-1}||,$$

where $\delta = \frac{h}{1-h}$.

Let X be an ordered normed space, i.e., a vector space over the real one be equipped with a partial order $\preccurlyeq$ and a norm $||.||$. For every $\alpha \geq 0$ and $x, y \in X$ with $x \preccurlyeq y$ one has that $x + z \preccurlyeq y + z$ and $\alpha x \preccurlyeq \alpha y$. Two elements $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A self-mapping T on X is called non-decreasing if $Tx \preccurlyeq Ty$ whenever $x \preccurlyeq y$ for all $x, y \in X$. In [21], Ran and Reurings introduced the fixed point theory on partially ordered sets. The following results is an extension of Banach contraction principle in an ordered metric space.

Theorem 4 ([21])**.** *Let $(X, \preccurlyeq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing map and the following conditions hold:*
(i) *there exists $k \in [0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$ with $x \preccurlyeq y$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$;*
(iii) *f is continuous, or*
(iv) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*
Then f has a fixed point p.

Thereafter, several authors obtained many fixed point results in an ordered metric space (see [1, 2, 4, 11, 15, 18, 20, 27, 32] and references therein).

2. Main Results

In this section, we prove some fixed point results for enriched contractions in partially ordered Banach spaces. For this purpose, we use the Krasnoselskij iteration for approximate the fixed points of enriched mappings.

Definition 4. Let $(X, ||.||)$ be a partially ordered norm space. A mapping $T : X \to X$ is called an enriched contraction if there exist $k \in [0, +\infty)$ and $\alpha \in [0, k + 1)$ such that

$$||k(x - y) + Tx - Ty|| \leq \alpha||x - y||, \tag{1}$$

for all $x \preccurlyeq y$. Then T is called a $(k, \alpha)_p$-enriched mapping.

Example 1. Suppose $X = \mathbb{R}$ is endowed with the usual norm and order $\leq$. Define $T : X \to X$ by $Tx = -3x$, for all $x \in \mathbb{R}$. For $x = 1$ and $y = 0$, we obtain

$$3 = ||T1 - T0|| \geq \alpha||1 - 0|| = \alpha,$$

where $0 < \alpha < 1$. So, T is not a α-contraction mapping for any $\alpha < 1$. Now, we show that T is an enriched contraction. Using condition (1), for all $x \leq y$, we have

$$|(k - 3)(x - y)| \leq \alpha|x - y|,$$

where $k \geq 0$ and $\alpha \in [0, k + 1)$. We can easily check that T is a $(3.8, 0.9)_p$-enriched contraction for any $x, y \in X$.

Theorem 5. *Let $(X, ||.||)$ be a partially ordered Banach space and $T : X \to X$ be a $(k, \alpha)_p$-enriched mapping. Suppose that the following hypotheses hold:*
(i) *There exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$;*
(ii) *$S = \frac{1}{k+1}(kI_X + T)$ is a nondecreasing mapping;*
(iii) *T is continuous; or*
(iv) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*
Then we have
(1) *T has a fixed point p.*
(2) *There exists $\lambda \in (0, 1]$ such that the iterative sequence $\{x_n\}$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \tag{2}$$

converges to p.

Proof. Define the averaged mapping T_λ by

$$T_\lambda x = (1 - \lambda)x + \lambda Tx, \tag{3}$$

for all $x \in X$, where $\lambda \in (0, 1]$. It can be easily seen that

$$\mathrm{Fix}\,(T_\lambda) = \mathrm{Fix}\,(T).$$

Case 1. Assume that $k > 0$ and set $\lambda = \frac{1}{k+1} < 1$. Then $k = \frac{1}{\lambda} - 1$, and we have

$$\frac{1}{k + 1}(kI_X + T) = \lambda\Big(\Big(\frac{1}{\lambda} - 1\Big)I_X + T\Big) = ((1 - \lambda)I_X + \lambda T) = T_\lambda,$$

and $T_\lambda x = (1-\lambda)x + \lambda Tx$ for all $x \in X$. From condition (ii), $T_\lambda : X \to X$ is a nondecreasing function. Using the contraction condition (1), we get

$$\Big\|\Big(\frac{1}{\lambda} - 1\Big)(x-y) + Tx - Ty\Big\| \le \alpha||x-y||, \quad \text{for } x \preccurlyeq y.$$

Then we have

$$||T_\lambda x - T_\lambda y|| \le ||x-y||, \quad \text{for all} \quad x \preccurlyeq y, \tag{4}$$

where $\delta = \lambda\alpha$. Then T_λ is a contraction mapping. Since $x_0 \preccurlyeq Tx_0$, we have $x_0 \preccurlyeq T_\lambda x_0$. On the other hand, T_λ is a nondecreasing mapping, then we obtain

$$x_0 \preccurlyeq T_\lambda x_0 \preccurlyeq T_\lambda^2 x_0 \preccurlyeq T_\lambda^3 x_0 \preccurlyeq \cdots \preccurlyeq T_\lambda^n x_0 \preccurlyeq \cdots .$$

Set $x_{n+1} = T_\lambda x_n$ for all $n = 0, 1, 2, 3, \ldots$. Then the elements x_{n+1} and x_n are comparable, that is $xn \preccurlyeq x_{n+1}$ for all $n \ge 0$. Substituting $x = x_n$ and $y = x_{n-1}$ into (4), we obtain

$$||x_{n+1} - x_n|| \le \delta||x_n - x_{n-1}||, \tag{5}$$

for all $n \in N$. Using (5) for $m \in \mathbb{N}$ and $n \ge 0$, we have

$$||x_{n+m} - x_n|| \le \delta^n \frac{1-\delta^m}{1-\delta}||x_1 - x_0|| \tag{6}$$

and

$$||x_{n+m} - x_n|| \le \delta \frac{1-\delta^m}{1-\delta}||x_n - x_{n-1}||. \tag{7}$$

From (6), $\{x_n\}$ is a Cauchy sequence in a partially ordered Banach space $(X, ||.||)$ so, there exists $p \in X$ such that

$$\lim_{n\to+\infty} x_n = p. \tag{8}$$

Now, we show that p is a fixed point of T_λ. First, suppose T is continuous, then T_λ is continuous so, we obtain

$$p = \lim_{n\to+\infty} x_{n+1} = \lim T_\lambda x_n = T_\lambda p.$$

Consequently, $p \in \operatorname{Fix}(T_\lambda) = \operatorname{Fix}(T)$ and so, T has a fixed point p. By condition (iv) and using (8), we get $x_n \preccurlyeq p$ for all $n \in \mathbb{N}$. Using (4), we obtain

$$||x_{n+1} - T_\lambda p|| \le \delta||x_n - p||.$$

Taking the limit as $n \to +\infty$ in the above inequalities and using (8), we obtain

$$||p - T_\lambda p|| = 0,$$

that is, $T_\lambda p = p$ so, $Tp = p$. Now, conclusion (2) follows immediately from (8).

Case 2. Let $k = 0$. Then in this case, $\lambda = 1$ and hence we obtain $T = T_1$. Thus Kasnoselskij iteration (2) reduces to the Picard sequence

$$x_{n+1} = Tx_n. \qquad \square$$

Corollary 1. *Putting $k = 0$, Theorem 5 reduces to Theorem 4.*

Now, the uniqueness of the fixed point in Theorems 6 can be obtained by adding the following hypothesis [19]:

$$\text{for all } x, y \in X, \text{ there exists } z \in X, \text{ which is comparable to } x \text{ and } y. \tag{9}$$

Theorem 6. *Adding condition (9) to the hypotheses of Theorem 5, we obtain the uniqueness of the fixed point of T.*

Proof. Suppose there exist $u, v \in X$ such that $T_\lambda u = Tu = u$ and $T_\lambda v = Tv = v$.

Case 1. Let u be comparable to v. From (4), we obtain

$$\begin{aligned} ||u-v|| &= ||T_\lambda u - T_\lambda v|| \\ &\le \delta ||u-v||, \end{aligned}$$

a contradiction. Then $||u-v|| = 0$ and this implies $u = v$.

Case 2. Now, suppose u is not comparable to v. By condition (9), there exists $x \in X$ such that x is comparable to u and v. Since T_λ is a nondecreasing maping, $T_\lambda^n x$ is comparable to $T_\lambda^n u$ and $T_\lambda^n v$ for all $n = 0, 1, 2, 3, \ldots$. Using (4), we have

$$\begin{aligned} ||u-v|| &\le ||T_\lambda^n x - T_\lambda^n u|| + ||T_\lambda^n x - T_\lambda^n v|| \\ &\le \delta^n(||x-u|| + ||x-v||). \end{aligned}$$

Letting $n \to +\infty$ in the above inequalities, we obtain $||u-v|| = 0$, that is, $u = v$ and T has a unique fixed point. □

Example 2. Suppose $X = \mathbb{R}^2$ is endowed with the norm $||.||_1$, which is defined as follows:

$$||(x_1, x_2)||_1 = |x_1| + |x_2|, \quad x_1, x_2 \in \mathbb{R}.$$

Also, we define a partial order on R^2 as follows:

$$(a, b) \preccurlyeq (c, d) \text{ if only if } a \le c, b \le d, \quad a, b, c, d \in \mathbb{R}.$$

Then $(X, ||.||_1)$ is a partially ordered norm space. Define $T : X \to x$ by $T(a, b) = (\frac{5-3a}{2}, \frac{7-4b}{3})$ for all $(a, b) \in \mathbb{R}^2$. For $x = (0, 1)$ and $y = (1, 1)$, we obtain

$$\frac{3}{2} = ||T(1,1) - T(0,1)||_1 \ge \alpha ||(1,1) - (0,1)||_1 = \alpha,$$

where $0 < \alpha < 1$. So, T is not a α-contraction mapping for any $\alpha < 1$. Now, we show that T is an enriched contraction. Using condition (1), for all $(x_1, y_1) \le (x_2, y_2)$, we have

$$\left|k - \frac{3}{2}\right| |x_2 - x_1| + \left|k - \frac{4}{3}\right| |y_2 - y_1| \le \alpha(|x_2 - x_1| + |y_2 - y_1|),$$

where $k \ge 0$ and $\alpha \in [0, k+1)$. We can easily check that T is a $(2, 0.9)_p$-enriched contraction. On the other hand, $SY = \frac{1}{k+1}(kY + TY)$ is a nondecreasing mapping for $k = 2$ and for any $Y \in R^2$. Indeed,

$$S(x, y) = \frac{1}{3}\Big(\frac{x+5}{2}, \frac{2y+7}{3}\Big),$$

for all $(x, y) \in \mathbb{R}^2$. Also, $(0, 1) \le T(0, 1) = (\frac{5}{2}, 1)$. Then all the conditions of Theorem 5 and Theorem 6 are satisfied and T has a unique fixed point $(1, 1)$.

Definition 5. Let $(X, ||.||)$ be a partially ordered normed space. A mapping $T : X \to X$ is called an enriched Kannan mapping if there exist $k \in [0, +\infty)$ and $\alpha \in [0, \frac{1}{2})$ such that

$$||k(x-y) + Tx - Ty|| \le \alpha(||x - Tx|| + ||y - Ty||), \tag{10}$$

for all $x \preccurlyeq y$. Then T is called a $(k, \alpha)_p$-enriched Kannan mapping.

Theorem 7. *Let $(X, ||.||)$ be a partially ordered Banach space and $T : X \to X$ be a $(k, \alpha)_p$-enriched Kannan mapping. Suppose that the following hypotheses hold:*

i) *There exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$;*

ii) *$S = \frac{1}{k+1}(kI_X + T)$ is a nondecreasing mapping;*

iii) *T is continuous; or*

iv) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*

Then we have

1) *T has a fixed point p.*

2) *There exists $\lambda \in (0, 1]$ such that the iterative sequence $\{x_n\}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n \tag{11}$$

converges to p.

Proof. Define the averaged mapping T_λ by

$$T_\lambda x = (1-\lambda)x + \lambda Tx, \tag{12}$$

for all $x \in X$, where $\lambda \in (0,1]$. It can be easily seen that

$$\mathrm{Fix}\,(T_\lambda) = \mathrm{Fix}\,(T).$$

Case 1. Assume that $k > 0$ and set $\lambda = \frac{1}{k+1} < 1$, Then $k = \frac{1}{\lambda} - 1$, and we get

$$\frac{1}{k+1}(kI_X + T) = \lambda\Big(\Big(\frac{1}{\lambda} - 1\Big)I_X + T\Big) = ((1-\lambda)I_X + \lambda T) = T_\lambda$$

and $T_\lambda x = (1-\lambda)x + \lambda Tx$ for all $x \in X$. From condition (ii), $T_\lambda : X \to X$ is a nondecreasing function. Using (10), we get

$$\Big\|\Big(\frac{1}{\lambda} - 1\Big)(x-y) + Tx - Ty\Big\| \le \alpha(||x - Tx|| + ||y - Ty||), \quad \text{for all } \ x \preccurlyeq y.$$

Then we have

$$||T_\lambda x - T_\lambda y|| \le \alpha(||x - Tx|| + ||y - Ty||), \quad \text{for all } x \preccurlyeq y. \tag{13}$$

Since $x_0 \preccurlyeq Tx_0$, we have $x_0 \preccurlyeq T_\lambda x_0$. Also, T_λ is a nondecreasing mapping, and then we obtain

$$x_0 \preccurlyeq T_\lambda x_0 \preccurlyeq T_\lambda^2 x_0 \preccurlyeq \cdots \preccurlyeq T_\lambda^n x_0 \preccurlyeq \cdots .$$

Set $x_{n+1} = T_\lambda x_n$ for all $n = 0, 1, 2, \ldots$. Then the elements x_{n+1} and x_n are comparable, that is $x_n \preccurlyeq x_{n+1}$ for all $n \ge 0$. Substituting $x = x_n$ and $y = x_{n-1}$ in (13), we obtain

$$||x_{n+1} - x_n|| \le \alpha(||x_n - x_{n+1}|| + ||x_{n-1} - x_n||),$$

which implies

$$||x_{n+1} - x_n|| \le \delta||x_n - x_{n-1}||, \tag{14}$$

for all $n \in N$, where $\delta = \frac{\alpha}{1-\alpha}$. Using (14), for $m \in N$ and $n \ge 0$, we have

$$||x_{n+m} - x_n|| \le \delta^n \frac{1-\delta^m}{1-\delta}||x_1 - x_0|| \tag{15}$$

and

$$||x_{n+m} - x_n|| \le \delta \frac{1-\delta^m}{1-\delta}||x_n - x_{n-1}||. \tag{16}$$

From (15), $\{x_n\}$ is a Cauchy sequence in the partially ordered Banach space $(X, ||.||)$ so, there exists $p \in X$ such that

$$\lim_{n\to+\infty} x_n = p. \tag{17}$$

Now, we show that p is a fixed point of T_λ. First, suppose T is continuous, then T_λ is continuous so, we obtain

$$p = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} T_\lambda x_n = T_\lambda p.$$

Consequently, $p \in \mathrm{Fix}\,(T_\lambda) = \mathrm{Fix}\,(T)$ and so, T has a fixed point p. By condition (iv), $\{x_n\}$ is a nondecreasing sequence in X and also $\lim\limits_{n\to+\infty} x_n = p$, then $x_n \preccurlyeq p$ for any $n \ge 1$. Using (13), we obtain

$$||x_{n+1} - T_\lambda p|| \le \alpha(||x_n - T_\lambda x_n|| + ||p - T_\lambda p||).$$

Taking the limit as $n \to +\infty$ in the above inequalities and using (17), we obtain

$$||p - T_\lambda p|| \le \alpha||p - T_\lambda p||,$$

that is, $T_\lambda p = p$ so, Tp = p. Now, conclusion (2) follows immediately from (17).

Case 2. Let $k = 0$. Then in this case, $\lambda = 1$ and hence, we obtain $T = T_1$. Thus Krasnoselskij iteration (11) reduces to the Picard sequence

$$x_{n+1} = Tx_n. \qquad \square$$

Now, the uniqueness of the fixed point in Theorems 7 can be obtained by adding the following hypothesis [13]:

$$\text{for all } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x, y \text{ and } z \preccurlyeq Tz. \tag{18}$$

Theorem 8. *Adding condition* (18) *to the hypotheses of Theorem* 7*, we obtain the uniqueness of the fixed point of* T*.*

Proof. Suppose there exist $u, v \in X$ such that $T_\lambda u = Tu = u$ and $T_\lambda v = Tv = v$.

Case 1. Let u be comparable to v. From (13), we obtain

$$\begin{aligned} ||u - v|| &= ||T_\lambda u - T_\lambda v|| \\ &\leq \delta(||u - T_\lambda u|| + ||v - T_\lambda v||) = 0. \end{aligned}$$

Then $||u - v|| = 0$ and this implies $u = v$.

Case 2. Now, suppose u is not comparable to v. By condition (18), there exists $x \in X$ such that x is comparable to u, v and $x \preccurlyeq Tx$. Since T_λ is a nondecreasing maping, $T_\lambda^n x$ is comparable to $T_\lambda^n u = u$ and $T_\lambda^n v = v$ for all $n = 0, 1, 2, \ldots$. Using (13), we have

$$\begin{aligned} ||u - v|| &\leq ||T_\lambda^n x - T_\lambda^n u|| + ||T_\lambda^n x - T_\lambda^n v|| \\ &\leq \alpha(||T_\lambda^{n-1} x - T_\lambda^n x|| + ||T_\lambda^{n-1} u - T_\lambda^n u||) + \alpha(||T_\lambda^{n-1} x - T_\lambda^n x|| + ||T_\lambda^{n-1} v - T_\lambda^n v||). \end{aligned}$$

Since $x \preccurlyeq Tx$, we get $lim_{n \to +\infty} T_\lambda^n x = p$, where p is a fixed point of T which implies $||T_\lambda^{n-1} x - T_\lambda^n x|| \to 0$ as $n \to +\infty$. Letting $n \to +\infty$ in the above inequalities, we obtain $||u - v|| = 0$, that is, $u = v$. Then T has a unique fixed point. □

Example 3. Let $X = \mathbb{R}$ be endowed with the usual norm and ordering $\leq$. Define $T : X \to X$ as $Tx = \frac{1-3x}{2}$ for all $x \in X$. Then T is not a Kannan contraction because for $x = \frac{1}{5}$ and $y = 1$, we have

$$\left\| T(\frac{1}{5}) - T(1) \right\| = \frac{6}{5} \geq \alpha\Big(\left\| \frac{1}{5} - T(\frac{1}{5}) \right\| + ||1 - T(1)|| \Big) = 2\alpha,$$

where $2\alpha < 1$. But T is a $(2, \frac{1}{5})_p$-enriched Kannan contraction and $0 \leq T(0)$. On the other hand, $Sx = \frac{1}{k+1}(kx + Tx)$ is a nondecreasing mapping for $k = 2$. Indeed,

$$Sx = \frac{1}{3}\Big(2x + \frac{1 - 3x}{2}\Big) = \Big(\frac{x + 1}{6}\Big),$$

for all $x \in X$. Then all the conditions of Theorem 7 and Theorem 8 are satisfied and T has a unique fixed point $x = \frac{1}{5}$.

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