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ON CERTAIN APPLICATIONS OF QUASI-POWER INCREASING SEQUENCES

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Abstract. Recently, in [7], we proved a main theorem dealing with the $\varphi- \mid C,\alpha \mid_k$ summability factors of infinite series. In the present paper, we have generalized this theorem for the $\varphi-\mid C,\alpha*\beta \mid_k$ summability methods by using a quasi-f-power increasing sequence instead of an almost increasing sequence. Some new results are also obtained.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exist a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [2]). A positive sequence $X=(X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K=K(X,f)\geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n\geq m\geq 1$, where $f=\{f_n(\sigma,\delta)\}=\{n^\sigma(\log n)^\delta,$ $\delta\geq\ 0,\ 0<\sigma<1\}$ (see [14]). If we take $\delta=0$, then we get a quasi-σ-power increasing sequence and every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse need not be true as can be seen by taking $X_n=n^{-\sigma}$ (see [12]). For any sequence (λ_n), we write that $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$. Let $\sum a_n$ be an infinite series. We denote by $t_n^{\alpha*\beta}$ the nth convolution Cesàro mean of order $(\alpha*\beta)$, with $\alpha+\beta>-1$, of the sequence (na_n), i.e., (see [8])

$$t_n^{\alpha*\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{1}$$

where

$$A_n^{\alpha+\beta}\simeq\frac{n^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)},\quad A_{-n}^{\alpha+\beta}=0\quad\text{for}\quad n>0.$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi-\mid C,\alpha*\beta\mid_k$, $k\geq 1$, if (see [4])

$$\sum_{n=1}^{\infty}\frac{1}{n^k}\mid\varphi_n t_n^{\alpha*\beta}\mid^k<\infty.$$

If we set $\beta=0$, then the $\varphi-\mid C,\alpha*\beta\mid_k$ summability reduces to the $\varphi-\mid C,\alpha\mid_k$ summability (see [1]). If we take $\varphi_n=n^{1-\frac{1}{k}}$, then the $\varphi-\mid C,\alpha*\beta\mid_k$ summability is the same as the $\mid C,\alpha*\beta\mid_k$ summability (see [9]). Also if we take $\varphi_n=n^{1-\frac{1}{k}}$ and $\beta=0$, then we have the $\mid C,\alpha\mid_k$ summability (see [10]). Furthermore, if we take $\varphi_n=n^{\delta+1-\frac{1}{k}}$ and $\beta=0$, then the $\varphi-\mid C,\alpha*\beta\mid_k$ summability reduces to $\mid C,\alpha;\delta\mid_k$ summability (see [11]). Finally, if we take $\varphi_n=n^{\delta+1-\frac{1}{k}}$, then the $\varphi-\mid C,\alpha*\beta\mid_k$ summability reduces to the $\mid C,\alpha*\beta;\delta\mid_k$ summability (see [5]).

2. Known Result

The following main theorem dealing with absolute Cesàro summability factors of infinite series is known.

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Theorem A ([7])**.** *Let* (φ_n) *be a sequence of complex numbers and* (w_n^α) *be a sequence defined by* (*see* [13])

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1\leq v\leq n} |t_v^\alpha|, & 0<\alpha<1. \end{cases}$$

Let (κ_n) be a positive sequence and (X_n) be an almost increasing sequence. Suppose also that there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing. If the conditions

$$\sum_{n=1}^{\infty} n \left|\Delta^2\lambda_n\right| X_n < \infty, \tag{2}$$

$$|\lambda_n| X_n = O(1) \quad \text{as} \quad n \to \infty, \tag{3}$$

$$\kappa_n = O(1) \quad \text{as} \quad n \to \infty, \tag{4}$$

$$n\Delta\kappa_n = O(1) \quad \text{as} \quad n \to \infty, \tag{5}$$

$$\sum_{v=1}^{n} \frac{(\mid \varphi_v \mid w_v^\alpha)^k}{v^k {X_v}^{k-1}} = O(X_n) \quad \text{as} \quad n \to \infty$$

hold, then the series $\sum a_n\lambda_n\kappa_n$ is summable $\varphi - \mid C,\alpha \mid_k$, where $0<\alpha\leq 1$, $\epsilon + (\alpha-1)\mathtt{k} > 0$ and $\mathtt{k} \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem A for the $\varphi - \mid C, \alpha * \beta \mid_k$ summability method by taking a wider class of increasing sequences. Now, we prove the following main theorem.

Theorem. *Let* (φ_n) *be a sequence of complex numbers and let* $(\omega_n^{\alpha*\beta})$ *be a sequence defined by* (*see* [3])

$$\omega_n^{\alpha*\beta} = \begin{cases} \left|t_n^{\alpha*\beta}\right|, & \alpha = 1, \ \ \beta > -1 \\ \max\limits_{1\leq v\leq n} \left|t_v^{\alpha*\beta}\right|, & 0<\alpha<1, \ \ \beta > -1. \end{cases}$$

Let (κ_n) be a positive sequence and let (X_n) be a quasi-f-power increasing sequence. Suppose also that there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing. If the conditions (2)–(5) and

$$\sum_{v=1}^{n} \frac{(\mid \varphi_v \mid w_v^{\alpha*\beta})^k}{v^k {X_v}^{k-1}} = O(X_n) \quad \text{as} \quad n \to \infty$$

hold, then the series $\sum a_n\lambda_n\kappa_n$ is summable $\varphi - \mid C, \alpha*\beta \mid_k$, where $0<\alpha\leq 1$, $\epsilon + (\alpha+\beta-1)\mathtt{k} > 0$, and $\mathtt{k} \geq 1$.

4. Lemmas

We need the following lemmas for the proof of theorem.

Lemma 1 ([3])**.** *If* $0<\alpha\leq 1$, $\beta > -1$, *and* $1 \leq v \leq n$, *then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^\beta a_p\right| \leq \max_{1\leq m\leq v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^\beta a_p\right|.$$

Lemma 2 ([6])**.** *Under the conditions on* (X_n) *and* (λ_n) *as taken in the statement of the theorem, the following conditions*

$$nX_n \left|\Delta\lambda_n\right| = O(1) \quad \textit{as} \quad n \to \infty,$$

$$\sum_{n=1}^{\infty} X_n \left|\Delta\lambda_n\right| < \infty$$

hold.

5. Proof of the Theorem

Let $(T_n^{\alpha*\beta})$ be the nth $(C,\alpha*\beta)$ mean, with $0<\alpha\leq 1$, of the sequence $(na_n\lambda_n\kappa_n)$. Then by (1), we have

$$T_n^{\alpha*\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^{\beta} v a_v \lambda_v \kappa_v.$$

Now, applying Abel's transformation first and then using Lemma 1, we obtain

$$\begin{aligned}
T_n^{\alpha*\beta} =& \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\Delta(\lambda_v\kappa_n)\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p + \frac{\lambda_n\kappa_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\\
=& \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}\lambda_v\Delta\kappa_v + \kappa_{v+1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p + \frac{\lambda_n\kappa_n}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v.\\
|T_n^{\alpha*\beta}| \leq& \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\lambda_v\Delta\kappa_v|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\right| + \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}|\kappa_{v+1}\Delta\lambda_v|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}A_p^{\beta}pa_p\right|\\
&+ \frac{|\lambda_n\kappa_n|}{A_n^{\alpha+\beta}}\left|\sum_{v=1}^{v}A_{n-v}^{\alpha-1}A_v^{\beta}va_v\right|\\
\leq& \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha*\beta}|\lambda_v||\Delta\kappa_v| + \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha*\beta}|\kappa_{v+1}||\Delta\lambda_v| + |\lambda_n||\kappa_n|w_n^{\alpha*\beta}\\
=& T_{n,1}^{\alpha*\beta} + T_{n,2}^{\alpha*\beta} + T_{n,3}^{\alpha*\beta}.
\end{aligned}$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^k}|\ \varphi_n T_{n,r}^{\alpha*\beta}\ |^k < \infty, \quad \text{for} \quad r=1,2,3.$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n^k}|\ \varphi_n T_{n,1}^{\alpha*\beta}\ |^k \leq& \sum_{n=2}^{m+1} n^{-k}(A_n^{\alpha+\beta})^{-k}|\ \varphi_n\ |^k\left\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha*\beta}|\Delta\kappa_v||\lambda_v|\right\}^k\\
=& O(1)\sum_{n=2}^{m+1}\frac{|\ \varphi_n\ |^k}{n^{k+(\alpha+\beta)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\Delta\kappa_v|^k|\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1}\\
=& O(1)\sum_{n=2}^{m+1}\frac{|\ \varphi_n\ |^k}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\lambda_v|^k\frac{1}{v^k}\\
=& O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k v^{-k}|\lambda_v|^k\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}\ |\ \varphi_n\ |^k}{n^{1+\epsilon+(\alpha+\beta-1)k}}\\
=& O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k v^{-k}|\lambda_v|^k v^{\epsilon-k}\ |\ \varphi_v\ |^k\sum_{n=v+1}^{m+1}\frac{1}{n^{1+\epsilon+(\alpha+\beta-1)k}}\\
=& O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k v^{-k}|\lambda_v|^k v^{\epsilon-k}\ |\ \varphi_v\ |^k\int_{v}^{\infty}\frac{dx}{x^{1+\epsilon+(\alpha+\beta-1)k}}\\
=& O(1)\sum_{v=1}^{m}(w_v^{\alpha*\beta})^k|\lambda_v||\lambda_v|^{k-1}\frac{|\ \varphi_v\ |^k}{v^k} = O(1)\sum_{v=1}^{m}|\lambda_v|\frac{(w_v^{\alpha*\beta}\ |\ \varphi_v\ |)^k}{v^k X_v^{k-1}}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{(w_r^{\alpha*\beta}\mid\varphi_r\mid)^k}{r^kX_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\frac{(w_v^{\alpha*\beta}\mid\varphi_v\mid)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m=O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the theorem and Lemma 2. Again, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\frac{1}{n^k}\mid\varphi_nT_{n,2}^{\alpha*\beta}\mid^k &\le \sum_{n=2}^{m+1}n^{-k}(A_n^{\alpha+\beta})^{-k}\mid\varphi_n\mid^k\left\{\sum_{v=1}^{n-1}A_v^{\alpha+\beta}w_v^{\alpha*\beta}|\kappa_{v+1}||\Delta\lambda_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{\mid\varphi_n\mid^k}{n^{k+(\alpha+\beta)k}}\left\{\sum_{v=1}^{n}v^{\alpha+\beta}(w_v^{\alpha*\beta})|\Delta\lambda_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{\mid\varphi_n\mid^k}{n^{k+(\alpha+\beta)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\Delta\lambda_v|^k\left\{\sum_{v=1}^{n-1}1\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\frac{\mid\varphi_n\mid^k}{n^{1+(\alpha+\beta)k}}\sum_{v=1}^{n-1}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\Delta\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\Delta\lambda_v||\Delta\lambda_v|^{k-1}\sum_{n=v+1}^{m+1}\frac{\mid\varphi_n\mid^k}{n^{1+(\alpha+\beta)k}}\\
&= O(1)\sum_{v=1}^{m}v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\Delta\lambda_v||\Delta\lambda_v|^{k-1}\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}\mid\varphi_n\mid^k}{n^{1+\epsilon+(\alpha+\beta-1)k}}\\
&= O(1)\sum_{v=1}^{m}\frac{v^{(\alpha+\beta)k}(w_v^{\alpha*\beta})^k|\Delta\lambda_v|}{v^{k-1}X_v^{k-1}}v^{\epsilon-k}\mid\varphi_v\mid^k\int_v^\infty\frac{dx}{x^{1+\epsilon+(\alpha+\beta-1)k}}\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{(w_v^{\alpha*\beta}|\varphi_v\mid)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v}\frac{(w_r^{\alpha*\beta}|\varphi_r\mid)^k}{r^kX_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{(w_v^{\alpha*\beta}|\varphi_v\mid)^k}{v^kX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|X_v+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the theorem and Lemma 2. Finally, as in $T_{n,1}^{\alpha*\beta}$, we have that

$$
\sum_{n=1}^{m}\frac{1}{n^k}|T_{n,3}^{\alpha*\beta}\varphi_n|^k=\sum_{n=1}^{m}\frac{1}{n^k}|\lambda_n\ \kappa_n\ w_n^{\alpha*\beta}\varphi_n|^k=O(1)\sum_{n=1}^{m}|\lambda_n|\frac{(w_n^{\alpha*\beta}|\varphi_n\mid)^k}{n^kX_n^{k-1}}=O(1)\quad\text{as}\quad m\to\infty,
$$

by the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

6. Particular Cases

1. If we take $\beta=0$ and (X_n) as an almost increasing sequence, then we have Theorem A.
2. Also, if we take $\varphi_n=n^{\delta+1-\frac{1}{k}}$ and $\beta=0$, then we obtain a new result dealing with the $\mid C,\alpha;\delta\mid_k$ summability factors of infinite series.
3. Finally, if we set $\varphi_n=n^{\delta+1-\frac{1}{k}}$, then we have a new result concerning the $\mid C,\alpha*\beta;\delta\mid_k$ summability factors of infinite series.

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