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MULTISET MIXED TOPOLOGICAL SPACE

BINOD CHANDRA TRIPATHY[1] AND RAKHAL DAS[2*]

Abstract. In this paper, an attempt has been made to extend the concept of multiset topological spaces in the context of multisets (mset). The multiset quasi-coincidence between sets have been introduced. The notion of multiset mixed topological space relative to quasi-coincidence has also been introduced. Some properties of multiset quasi-coincidence and multiset mixed topological space have been investigated.

1. Introduction

The domain of crisp set theory or Cantor's set is not sufficient for representing the solution of all kind of mathematical problems. Since, using an element once at a time, sometimes the exact picture cannot be explicitly expressed. For example, the fundamental theorem of algebra. From the practical point of view, there are many situations where repetition of elements plays a vital role. This led the introduction of the theory of multisets, which was first studied by Blizard (see [1]) in 1989. Thus a multiset is a collection of elements in which certain elements may occur more than once and a number of times an element occurs is called its multiplicity. The different types of collections of msets such as power msets, power whole msets and power full msets are submsets of the mset space and operations under such collections are defined.

The number of elements in a multiset may be infinite, but the multiplicity of any element is considered as a finite quantity. That is, the mset space $[X]^w$ is the collection of msets whose elements are from X such that no element in the mset occurs more than w times. The application of a multiset is not only limited to philosophy, logic, linguistics, and physics, but a good number of them has been observed in mathematics and computer science, which led to the formulation of a comprehensive theory of multisets. Multiset like sets, are collections of elements that elements can be repeated. This concept is the generalization of scrip set theory, which can be applied to different branches of mathematics allowing to get better result from an ordinary set.

The notion of M-topological space and the concept of open msets are introduced. More precisely, an M-topology is defined as a set of msets as points. Furthermore, the notions of basis, subbasis, closed sets, closure and interior in topological spaces have been extended to the M-topological spaces and many related theorems have been established. The paper concludes with the definition of continuous mset functions and related properties, in particular, the comparison of discrete topology and discrete M-topology are established. The concept of a multiset topological space was studied by Girish and John (see [2]). The concept of a mixed multiset topological space and quasi-coincidence with a multiset point to a set is studied by Shravan and Tripathy (see [5]).

2. Definitions and Preliminaries

In this section, we present some basic definitions, notations and results which will be used in this paper (one may refer to Shravan and Tripathy (see [3]).

Definition 1. A domain X is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are from X such that no element occurs more

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*Corresponding author.

than w times. Throughout this paper, we denote a multiset drawn from the multiset space $[X]^w$ by M.

Definition 2. An mset M drawn from the set X is represented by a count function M or $C_M : X \to N$, where N represents the set of nonnegative integers. Here, $C(x)$ is the number of occurrences of the element x in the mset M drawn from the set $X = \{x_1, x_2, \ldots, x_n\}$ as $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, \ldots, n$ in the mset M. The elements which are not included in the mset M have zero count.

Remark 1. We introduce some new notions on a multiset topological space on the basis of the count function. When $C_M(x) = 1$ for all $x \in X$, the notion of multisets becomes structurally equivalent to Cantor's set. So, whatever results and definitions we establish when restricted to this condition, they must be equivalent to some results in the classical set theory. Consider two msets M and N drawn from a set X. The following are the operations defined on the msets that will be used in this article. For details of binary operations on multisets such as addition, union, intersection, subtraction, compliment, one may refer to Girish and John (see [2]). Further, the notation of support set, empty mset, partial sub mset, power mset etc., one may refer to Shravan and Tripathy (see [3]).

The following are the basic operations under collection of msets. Let $[X]^w$ be an mset space with $C_Z(x)$ as the multiplicities of $x \in X$, and let $\{M_1, M_2, \ldots\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under arbitrary collections of msets.

1. The Union is defined by $\bigcup_{i\in I} M_i = \{C_M(x)/x : C_M(x) = \max\{C_{M_i}(x) : i \in I\}$, for all $x \in X\}$.
2. The intersection is defined by $\bigcap_{i\in I} M_i = \{C_M(x)/x : C_M(x) = \min\{C_{M_{i(x)}} : i \in I\}$, for all $x \in X\}$.
3. The mset complement is defined by $M^c = Z \ominus M = \{C^c_M(x)/x : C^c_M(x) = C_Z(x) - C_M(x)$, for all $x \in X\}$. For the more details of the above operation one may refer to Girish and John (see [2]), Jakaria (see [12]), Syropoulus (see [6]), Venkateswaram (see [7]), Shravan and Tripathy (see [3]).

Definition 3. Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$.

Definition 4. An mset M is said to be an empty set if for all $x \in X, C_M(x) = 0$.

Definition 5. A submset N of M is a whole submset of M with each element in N having full multiplicity as in M i.e., $C_N(x) = C_M(x)$ for every $x \in N$.

Definition 6. A submset N of M is a partial whole submset of M with at least one element in N having full multiplicity as in M, i.e., $C_N(x) = C_N(x)$ for some x in N.

Definition 7. A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in M, i.e., $C_N(x) \leq C_M(x)$ for every $x \in N$. For details of different types of multisets and their definition and operation we may refer to Jakaria (see [12]), Syropoulus (see [6]), Venkateswaram (see [7]), Shravan and Tripathy (see [3]). Girish and John (see [2]) introduced the notion of a multiset topology in considering multisets in place of Cantor set, called as M-topology.

Mathematically, a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \subseteq [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Multiset topology is abbreviated as M-Topology. The elements of τ are called open msets. The complement of an open mset in an M-Topological space is said to be a closed mset.

Definition 8. Given a submset A of M-topological space M in $[X]^w$, the interior of A is denoted by $\mathrm{Int}(A)$ and is defined as the mset union of all open msets contained in A, i.e., $C_{\mathrm{Int}(A)}(x) = \max\{C_G(x) : G \subseteq A\}$.

Definition 9. Given a submset A of an M-topological space M in$[X]^w$, the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by $C_{l(A)}$, i.e., $C_{Cl(A)}(x) = \min\{C_K(x) : A \subseteq K\}$.

Definition 10. Let (M, τ) be an M-topological space and N be a submset of M. The collection $\tau_N = \{U' = U \bigcap N : U \in \tau\}$ is an M-topology on N, called the subspace M-topology. With this M-topology, N is called a subspace of M and its open msets consisting of all mset intersections of open msets of M with N.

Example 2.1. Let $M = \{3/a, 4/b, 2/c, 5/d, 3/e\}$ and $\tau = \{\tau, M, \{2/c\}, \{2/a\}, \{3/a, 2/b\}, \{2/a, 3/d\}, \{2/a, 2/c\}, \{3/a, 3/b, 3/d\}, \{3/a, 4/b, 2/c\}, \{3/a, 4/b, 2/c, 1/e\}, \{2/a, 2/c, 3/d\}\}$ be an M-topology on M. If $N = \{2/a, 2/b, 3/d\} \subseteq M$, then $\tau_N = \{\phi, \{2/a, 2/b, 3/d\}, \{2/a\}, \{2/a, 2/b\}, \{2/a, 3/d\}\}$ is an M-topology on N and it is the subspace M-topology on N.

Example 2.2. Let $X = \{x, y, z, t\}$ and $M = \{3/x, 2/y, 4/z, 6/t\}$ be an mset in $[X]^6$. Then $\tau = \{\emptyset, M, \{2/x, 1/y\}, \{2/y, 3/z\}, \{2/x, 2/y, 3/z\}, \{2/x, 2/y, 3/z\}, \{2/x, 1/y, 2/z, 3/t\}, \{2/x, 2/y, 3/z, 3/t\}, \{1/y\}, \{2/x\}, \{2/z\}, \{1/y, 2/z\}, \{2/x, 1/y, 2/z\}, \{2/x, 2/y, 2/z\}, \{2/y, 2/z\}, \{2/x, 2/y, 2/z, 3/t\}\}$, is an M-topology.

Lemma 2.1 (Shravan and Tripathy (see [3]), Proposition 3.2)**.** *Let (M, τ) be an M-topological space. A multipoint $m = k/x \in cl(A)$ if and only if each q-nbhd of k/x is quasi-coincident with A.*

Lemma 2.2 (Shravan and Tripathy (see [4]), Proposition 3.3)**.** *A multipoint $m = k/x \in \text{int}(A)$ if and only if m has a q-nbhd contained in A.*

3. Main Results

In the case of a multiset topological space, quasi-coincidence can be defined based on the multiplicity function, which is the foundation of a multiset. In the literature, there are different types of quasi-coincidence based on different notions of sets. Those helped in the introduction of different types of mixed topological spaces.

In the case of fuzzy topological spaces, the notion of quasi-coincidence has been defined based on the membership function, the fundamental property of fuzzy topological space. For a detailed account of quasi-coincidence in fuzzy topological space, one may refer to Tripathy and Ray [8–11], and others. The base for the notion of multiset is the multiplicity of an element. Shravan and Tripathy [5] have introduced the notion of a mixed multiset topology based on the quasi-coincidence between an element and a set. In this article, we introduce the notion of quasi-coincidence considering between multisets. We also introduce the notions of multiset quasi-coincidence, multiset weak quasi-coincidence and multiset strong quasi-coincidence between multisets defining a multiset mixed topological space.

Definition 11. Let A, B be any two multisets in the space $[X]^w$. Then A is said to be quasi-coincident with B if and only if $C_A(x) + C_B(x) > w$, for some x in A and B.

Example 3.1. $X = \{x, y, z\}$, $A = \{3/x, 2/y, 4/z\}$ and $B = \{5/y, 4/z\}$ are M-sets in $[X]^6$. It can be easily verified that the mset A is quasi-coincident with the mset B.

Definition 12. Let A, B be any two multisets in the space $[X]^w$. Then A is said to be strong quasi-coincident with B if and only if $C_A(x) + C_B(x) > w$, for some x in A.

Example 3.2. $X = \{x, y, z\}$, $A = \{3/x, 1/y, 4/z\}$ and $B = \{5/x, 6/y, 4/z\}$ are M-sets in $[X]^6$ Here, $C_A(x) + C_B(x) = 8 > 6$, $C_A(y) + C_B(y) = 7 > 6$ and $C_A(z) + C_B(z) = 8 > 6$, hence the mset A is strong quasi-coincident with a mset B.

Definition 13. A multiset N in a multiset topological space (M, τ) is said to be a quasi-open set of a multiset B if and only if there exists an open mset P such that BqP and PqN.

Definition 14. A submultiset A is said to be open multimixed set, if $\text{int}(A) = A \in \tau_1(\tau_2)$.

Definition 15. A submultiset A is said to be closed multimixed set if $\text{closure}(A) = A$. We formulate the following two results, the proofs of these results are straightforward.

The following results are straightforward from the definitions.

Lemma 3.1. *Every strong multi quasi-coincident set is a multi quasi-coincident set. But the converse is not true in general.*

Lemma 3.2. *The super set of a strong q-coincident multiset is also a strong q-coincident.*

Theorem 3.1. *The intersection of two strong quasi-coincident sets is either a strong quasi-coincident set or an empty multiset.*

Proof. Let $B \subset [X]^w$ be a multiset and B_1, B_2 be any two strong quasi-coincident sets of B. We prove that the intersection of B_1 and B_2 is either an empty set or a strong quasi-coincident with a set B. Since B is strong quasi-coincident with a set $B_{1,}$ therefore

$$C_{B_1}(x) + C_B(x) > w, \text{ for all } x \in B_1.$$

Similarly, in the case of B_2, we have

$$C_{B_2}(x) + C_B(x) > w, \text{ for all } x \in B_2.$$

Now, two possible relations between B_1 and B_2 are either $B_1 \bigcap B_2 = \emptyset$ or $B_1 \bigcap B_2 \neq \emptyset$. If $B_1 \bigcap B_2 = \emptyset$, then the theorem is proven, but if not, then $C_{B_1 \bigcap B_2}(x) + C_B(x) > w$, for all $x \in B_1 \bigcap B_2$ by (1) and (2). Hence the theorem is established. □

Theorem 3.2. *The arbitrary union of strong q-coincident multiset is a strong q-coincident multiset.*

Proof. Let $B \subset [X]^w$ be a multiset. To prove that the arbitrary union of strong quasi-multiset is also a strong quasi-multiset, consider a set $U = \bigcup_{\alpha \in \bigwedge} B_\alpha$, where each B_α is a strong quasi-coincident with the set B.

Therefore $C_{B_\alpha}(x) + C_B(x) > w$, for all $x \in B_\alpha$. $C_{\bigcup_{\alpha \in \bigwedge} B_\alpha}(x) + C_{B_\alpha}(x) > w$, for all $x \in \bigcup_{\alpha \in \bigwedge} B_\alpha$ Hence $\bigcup_{\alpha \in \bigwedge} B_\alpha$ is a strong quasi-coincident with a set B. □

Multiset strong quasi-coincident topological space.

We define a topology on a multiset, quasi-strong in an alternative way, by considering the notion of multiset quasi-coincident sets. Consider an mset space $[X]^w$. Let M be any mset in X. Suppose U is the collection of all strong quasi-coincident multisets. Then the topology τ on M can be defined as follows.

Theorem 3.3. *Let (M, τ_1) and (M, τ_2) be two M-topological spaces in $[X]^w$. Define $\tau_1(\tau_2)$ by $\tau_1(\tau_2)$ $= \{A \subset [X]^w$: for every multiset B_1 quasi-coincident with A, there exists a τ_2 quasi-open set B of B_1 and $\tau_1 - \overline{B} \subseteq A\}$.*

Proof.

i) Clearly, we have that empty multiset and whole multiset are in $\tau_1(\tau_2)$.

ii) Let $A_1, A_2 \tau_1(\tau_2)$. Then we have two multisets B_1 and B_2 which are quasi-open sets of A_1 and A_2 such that a multiset BqB_1 and BqB_2 with $B_1 \subset A_1$ and $B_2 \subset A_2$. Hence we have $\overline{B_1 \bigcap B_2} \subseteq \overline{B_1} \bigcap \overline{B_2} \subseteq A_1 \bigcap A_2$. Further, we have BqA_1 and BqA_2 implying $BqA_1 \bigcap A_2$. Therefore $A_1 \bigcap A_2 \in \tau_1(\tau_2)$.

iii) Let $\{A_i$: $i \in \bigwedge\}$ be a collection of elements of $\tau_1(\tau_2)$. Then there exists $\{B_i\}$ a class of quasi-open sets of $A_i, i \in \bigwedge$ such that BqB_i, which implies $Bq \bigcup_{i \in \bigwedge} B_i$. Further, we have $\bar{B}_i \subseteq A_i$, for $i \in \bigwedge$, implies $\bigcup_{i \in \bigwedge} \bar{B}_i \subseteq \bigcup_{i \in \bigwedge} A_i \in \tau_1(\tau_2)$. Thus we have $\bigwedge A_i \in \tau_1(\tau_2)$. Hence $\tau_1(\tau_2)$ is a multiset mixed topological space. □

Note 3.1. The topology $\tau_1(\tau_2)$ defined in the above Theorem is called a mixed multiset topological space using the multiset quasi-coincident on a multiset to a multiset. In view of Lemma 2.1 and Lemma 2.2, we state the following two results without proof.

Proposition 3.1. *Let (M, τ) be an M-topological space. A multiset $N \subseteq cl(A)$ if and only if each quasi-nbd of N is quasi-coincident with A.*

Proposition 3.2. *Let (M, τ) be an M-topological space. A multiset $N \subseteq \text{int}(A)$ if and only if N has quasi-nbd which is contained in A.*

Proposition 3.3. *For any two multiset topologies τ_1 and τ_2 on a set X, the multiset mixed topology $\tau_1(\tau_2)$ is coaser than τ_2.*

Proof. To prove this theorem, we have to show that every open set from the multiset mixed topology $\tau_1(\tau_2)$ is open in τ_2 or there exists an open set which is a quasi-open set of $\tau_1(\tau_2)$. From Theorem 3.3 it is clear that if $A \in \tau_1(\tau_2)$, then for any open set B with the condition AqB, there exists always an open set from τ_2 which is quasi-nbd of B. Hence we get always an open quasi-nbd of τ_2 which shows that the multiset mixed topology $\tau_1(\tau_2)$ is coaser than τ_2. □

Conclusion. In this article, we introduce new type of multiset quasi-coincident in the multiset theory on the basis of multiset to another multiset, since a set may contain a singleton element, therefore this process can be applied for a set to a point. On the basis of multiset quasi-coincident we define a multiset mixed topology and discuss some example and its some topological property.

Conflict of Interest/Competing interests. The authors declare that the article is free from conflicts of interest.

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[1]Department of Mathematics, Tripura University, Agartala-799022; Tripura, INDIA

[2]Department of Mathematics, ICFAI University, Kamalghat (Near Agartala), PIN.799210,West Tripura, India

Email address: `tripathybc@yahoo.com; binodtripathy@tripurauniv.in`
Email address: `rakhaldas@iutripura.edu.in; rakhaldas95@gmail.com`