Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 1, 111–118

INDEXED ABSOLUTE ALMOST CONVOLUTED NÖRLUND SUMMABILITY

SMITA SONKER[1] AND ALKA MUNJAL[2]

Abstract. The indexed almost absolute Nörlund summability $\phi - |N, p, q; m|_k$ factor of orthogonal series has been studied for a least set of the sufficient conditions and a set of moderated theorems have been developed. This result is very useful as absolute summability is used for Bounded Input Bounded Output (BIBO) stability of the system. By applying certain conditions to the main result, the sufficient conditions for $|N, p, q; m|_k$ summability have been obtained (a previous result), which validate the application and importance of the present work.

1. Introduction

The absolute convergence of orthogonal series is closely linked to the quantitative measurement of the uniform continuity and the bounded variation of the functions. For the study of absolute convergence of orthogonal series, several classical criteria have been established. A methodical proof for these criteria can be achieved by various summability methods. Out of these criteria, the best way to achieve a methodical proof is to determine a least set of sufficient conditions for absolute summable factor of orthogonal series.

Non-absolute convergent factor of orthogonal series can be studied using the concept of absolute summability. So, absolute summability is extremely contributive in understanding the concept of the absolute convergence of orthogonal series. Consequently, several criteria can be used systematically to get a non-absolute convergent factor for the orthogonal series.

Okuyama [15,17] and Leindler [11–13] studied the orthogonal series with the help of various absolute summability factors. Bor [1–6] and Rhoades [18, 19] have also derived a number of theorems on absolute Nörlund summability. Many results have been established on absolute Nörlund summability [7–9, 20–23]. In the present study, an almost everywhere absolute convoluted Nörlund summability factor for orthogonal series has been worked out and a set of new and well-known results has been deduced from the presented theorems. Furthermore, it has been shown that an orthogonal series can be made absolute Nörlund summable with some sufficient conditions.

Consider a series $\sum a_n$ with its partial sums s_n. Let p and q be given as the sequences

$$P_n := \sum_{v=0}^{n} p_v = p_0 + p_1 + p_2 + \cdots + p_n \neq 0 \ \ (n \geq 0),$$
$$Q_n := \sum_{v=0}^{n} q_v = q_0 + q_1 + q_2 + \cdots + q_n \neq 0 \ \ (n \geq 0).$$

The convolution $(p * q)_n$ is defined by

$$(p * q)_n := \sum_{v=0}^{n} p_{n-v} q_v = \sum_{v=0}^{n} p_v q_{n-v}. \tag{1.1}$$

Apparently, $P_n := (p * 1)_n = \sum\limits_{v=0}^{n} p_v$ and $Q_n := (1 * q)_n = \sum\limits_{v=0}^{n} q_v$.

2020 *Mathematics Subject Classification.* 40G05, 40F05, 42C15.

Key words and phrases. Absolute Summability; Convoluted Nörlund Mean; Almost Everywhere Summable; Orthogonal Series.

If $(p*q)_n \neq 0$ at all values of n, the convoluted Nörlund transform sequence of $\{s_n\}$ is $\{t_n^{p,q}\}$, which is given by

$$t_n^{p,q} := \frac{1}{(p*q)_n} \sum_{v=0}^{n} p_{n-v} q_v s_v. \tag{1.2}$$

The series $\sum a_n$ is then said to be $|N, p_n, q_n|_k$ summable of order k for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n^{p,q} - t_{n-1}^{p,q}|^k \tag{1.3}$$

converges. If $k = 1$ in this summability, then it reduces to $|N, p_n, q_n|$ summability given by Tanaka [23]. Also, Krasniqi [10] introduced and studied the almost summability $|N, p, q; m|_k$, $k \geq 1$ using

$$\sum_{n=0}^{\infty} n^{k-1} |t_{n,m}^{p,q} - t_{n-1,m}^{p,q}|^k \tag{1.4}$$

uniformly converges with respect to m, instead of condition 1.3, where

$$t_{n,m}^{p,q} := \frac{1}{(p*q)_n} \sum_{v=0}^{n} p_{n-v} q_v s_{v,m} \tag{1.5}$$

and

$$s_{n,m} = \frac{1}{n+1} \sum_{v=m}^{n+m} s_v. \tag{1.6}$$

We note that for $q_n = 1$ and $p_n = 1$, the almost summability $|N, p, q; m|_k$ reduces to almost summabilities $|N, p; m|_k$ and $|\bar{N}, q; m|_k$ (see [10]).

We now extend this concept with any factors as follows: let $\varphi = (\varphi_n)$ be a sequence of positive real numbers, then the series $\sum a_n$ is indexed absolute almost convoluted Nörlund summable of order k for $k \geq 1$, if the series

$$\sum_{n=1}^{\infty} |\varphi_n (t_{n,m}^{p,q} - t_{n-1,m}^{p,q})|^k, \tag{1.7}$$

or

$$\sum_{n=1}^{\infty} |\varphi_n \Delta t_{n,m}^{p,q}|^k \tag{1.8}$$

uniformly converges with respect to m, and we write briefly $\sum_{n=0}^{\infty} a_n \in \varphi - |N, p, q; m|_k$. We need the notations:

$$R_n := (p*q)_n, \quad R_n^j := \sum_{v=j}^{n} p_{n-v} q_v,$$

$$R_{n-1}^n = 0, \quad R_n^0 = R_n,$$

$$\hat{R}_n^j := \sum_{v=j}^{n} \frac{p_{n-v} q_v}{v+1}, \quad \hat{R}_{n-1}^n := 0.$$

Let $f(x)$ be a periodic function integrable over (a, b) and $\Phi_n(x)$ be an orthonormal system defined in the interval (a, b). The orthogonal series of $f(x)$ belongs to $L^2(a, b)$ and is given by

$$f(x) := \sum_{n=0}^{\infty} a_n \Phi_n(x), \tag{1.9}$$

where

$$a_n := \int_a^b f(x) \Phi_n(x) dx. \tag{1.10}$$

2. Known Results

Okuyama's [16] previous findings are as follows.

Theorem 2.1. *In order to make an orthogonal series* $\sum_{n=0}^{\infty} c_n \Phi_n(x)$ *almost everywhere* $|N, p, q|$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{1/2}$$

must converge.

Theorem 2.2. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of non-negative numbers. In order to make an orthogonal series* $\sum_{n=0}^{\infty} c_n \Phi_n(x)$ *almost everywhere* $|N, p, q|$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega_n}$$

and

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega_n w_n^{(1)}$$

must converge, where
i) $\{\Omega_n\}$ *is a sequence of positive numbers,*
ii) $\{\Omega_n/n\}$ *is a sequence of non-increasing numbers and*
iii) $w_n^{(1)}$ *is defined by*

$$w_j^{(1)} := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 . \tag{2.1}$$

3. Main Results

The established theorems are as follows.

Theorem 3.1. *In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *almost everywhere* $\varphi - |N, p, q; m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ \varphi_n^2 \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\hat{R}_n^j}{R_n} - \frac{\hat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{3.1}$$

must uniformly converge with respect to m *for* $1 \leq k \leq 2$.

Theorem 3.2. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of non-negative numbers. In order to make an orthogonal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$ *almost everywhere* $\varphi - |N, p, q; m|_k$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\frac{k}{1-k}}}{\Omega_n}$$

must converge and

$$\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega_n^{2/k-1} \Re_n^{(k)}$$

must uniformly converge with respect to m *for* $1 < k \leq 2$, *where*
i) $\{\Omega_n\}$ *is a sequence of positive numbers,*
ii) $\left\{\Omega_n/\varphi_n^{\frac{k}{k-1}}\right\}$ *is a sequence of non-increasing numbers and*

iii) $\Re_n^{(k)}$ *is defined by*

$$\Re_j^{(k)} := \frac{1}{\varphi_j^{\frac{1}{k-1}-1}} \sum_{n=j}^{\infty} \varphi_n^{\frac{2}{k-1}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\hat{R}_n^j}{R_n} - \frac{\hat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{3.2}$$

4. Lemma

Beppo Levi [14] established the Lemma and used it to prove the results based on the functions of series and integrals. Lemma has also been used to prove the presented theorems.

Lemma 4.1. *Let a non-negative function* $V_n(t) \in L(U)$ *if*

$$\sum_{n=1}^{\infty} \int_U V_n(t) < \infty, \tag{4.1}$$

then the series

$$\sum_{n=1}^{\infty} V_n(t) \tag{4.2}$$

almost (*absolutely*) *converges everywhere to a function* $V(t) \in L(U)$ *over* U.

5. Proof of the Theorems

Let the v^{th} partial sum of the series be given by $s_v(x) = \sum_{j=0}^{v} a_j \Phi_j(x)$ for $1 < k < 2$.

$$\begin{aligned} s_{v,m}(x) &= \frac{1}{v+1} \sum_{k=0}^{v} s_{k+m}(x) \\ &= \frac{1}{v+1} \sum_{k=0}^{v} \sum_{j=0}^{k+m} a_j \Phi_j(x) \\ &= \sum_{j=0}^{v} \left(1 - \frac{j}{v+1}\right) a_{m+j} \Phi_{m+j}(x) + s_{m-1}(x). \end{aligned} \tag{5.1}$$

Almost Nörlund transform $t_{n,m}^{p,q}(x)$ is given by

$$\begin{aligned} t_{n,m}^{p,q}(x) &= \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_{v,m}(x) \\ &= s_{m-1}(x) + \frac{1}{R_n} \sum_{j=0}^{n} a_{m+j} \Phi_{m+j}(x) \sum_{v=j}^{n} p_{n-v} q_v \\ &- \frac{1}{R_n} \sum_{j=0}^{n} j a_{m+j} \Phi_{m+j}(x) \sum_{v=j}^{n} \frac{p_{n-v} q_v}{v+1} \\ &= s_{m-1}(x) + \frac{1}{R_n} \sum_{j=0}^{n} \left(R_n^j - j \widehat{R}_n^j \right) a_{m+j} \Phi_{m+j}(x). \end{aligned} \tag{5.2}$$

$$\begin{aligned} \Delta t_{n,m}^{p,q}(x) &= t_{n,m}^{p,q}(x) - t_{n-1,m}^{p,q}(x) \\ &= \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right] a_{m+j} \Phi_{m+j}(x). \end{aligned} \tag{5.3}$$

Proof of Theorem 3. The series

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_a^b |\varphi(n)\Delta t_{n,m}^{p,q}(x)|^k dx\\
&\quad\le \sum_{n=1}^{\infty}|\varphi(n)|^k\int_a^b |\Delta t_{n,m}^{p,q}(x)|^k dx\\
&\quad= O(1)\sum_{n=1}^{\infty}|\varphi(n)|^k(b-a)^{1-\frac{k}{2}}\left(\int_a^b |\Delta t_{n,m}^{p,q}(x)|^2 dx\right)^{\frac{k}{2}}\\
&\quad= O(1)\sum_{n=1}^{\infty}\left\{\varphi^2(n)\sum_{j=1}^{n}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\widehat{R}_n^j}{R_n}-\frac{\widehat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}\\
&\quad= O(1).
\end{aligned}\tag{5.4}$$

It has been observe that $|\Delta t_{n,m}^{p,q}(x)|$ is a non-negative function due to which (5.4) converges because

$$\sum_{n=1}^{\infty}\left\{\varphi^2(n)\sum_{j=1}^{n}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\widehat{R}_n^j}{R_n}-\frac{\widehat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}$$

uniformly converges with respect to m by the assumption.

Hence, by Lemma 4.1,

$$\sum_{n=1}^{\infty}|\varphi(n)\Delta t_{n,m}^{p,q}(x)|^k \tag{5.5}$$

almost converges everywhere. Schwartz's inequality is applicable for $k=1$ and used upto $k=2$. Hence proof of the theorem is complete. □

Proof of the theorem 4. The series

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_a^b |\varphi_n\Delta t_{n,m}^{p,q}(x)|^k dx\\
&\quad\le \sum_{n=1}^{\infty}|\varphi_n|^k\left\{\sum_{j=1}^{n}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\hat{R}_n^j}{R_n}-\frac{\hat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}\\
&\quad= O(1)\sum_{n=1}^{\infty}\left(\frac{\varphi_n^{\frac{k}{1-k}}}{\Omega_n}\right)^{1-\frac{k}{2}}\left\{\frac{\Omega_n^{\frac{2}{k}-1}}{\varphi_n^{\frac{k}{1-k}}}\sum_{j=1}^{n}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\hat{R}_n^j}{R_n}-\frac{\hat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}\\
&\quad= O(1)\left(\sum_{n=1}^{\infty}\frac{\varphi_n^{\frac{k}{1-k}}}{\Omega_n}\right)^{1-\frac{k}{2}}\left\{\sum_{n=1}^{\infty}\frac{\Omega_n^{\frac{2}{k}-1}}{\varphi_n^{\frac{k}{1-k}}}\sum_{j=1}^{n}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\hat{R}_n^j}{R_n}-\frac{\hat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}\\
&\quad= O(1)\left\{\sum_{j=1}^{\infty}|a_{m+j}|^2\sum_{n=j}^{\infty}\frac{\Omega_n^{\frac{2}{k}-1}}{\varphi_n^{\frac{k}{1-k}}}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\hat{R}_n^j}{R_n}-\frac{\hat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2\right\}^{\frac{k}{2}}\\
&\quad= O(1)\left\{\sum_{j=1}^{\infty}|a_{m+j}|^2\left(\frac{\Omega_j}{\varphi_j^{\frac{k}{k-1}}}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}\varphi_n^{\frac{2}{k-1}}\left[\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}-j\left(\frac{\hat{R}_n^j}{R_n}-\frac{\hat{R}_{n-1}^j}{R_{n-1}}\right)\right]^2\right\}^{\frac{k}{2}}\\
&\quad= O(1)\left\{\sum_{j=1}^{\infty}|a_{m+j}|^2\Omega^{\frac{2}{k}-1}(j)\Re_j^{(k)}\right\}^{\frac{k}{2}}=O(1).
\end{aligned}\tag{5.6}$$

where

$$\Re_j^{(k)} := \frac{1}{\varphi_j^{\frac{k-2}{1-k}}} \sum_{n=j}^{\infty} \varphi_n^{\frac{2}{k-1}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\hat{R}_n^j}{R_n} - \frac{\hat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{5.7}$$

Based on the assumption, equation (5.6) is uniformly finite with respect to m. This can be proved using the same concept as used for proving Theorem 3.1. The detail of the proof is out of the scope and hence omitted. □

6. Corollaries

Case I [10]. For $\varphi_n = n^{(1-1/k)}$, Theorem 3.1 and Theorem 3.2 give some well-known results of Krasniqi [10] on $|N,p,q;m|_k$ summability as follows.

Corollary 6.1. *In order to make an orthogonal series* $\sum\limits_{n=0}^{\infty} a_n \Phi_n(x)$ *almost everywhere* $|N,p,q;m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ n^{2\left(1-\frac{1}{k}\right)} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\hat{R}_n^j}{R_n} - \frac{\hat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{6.1}$$

must uniformly converge with respect to m *for* $1 \le k \le 2$.

Corollary 6.2. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of non-negative numbers. In order to make an orthogonal series* $\sum\limits_{n}^{\infty} a_n \Phi_n(x)$ *almost everywhere* $|N,p,q;m|_k$ *summable, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega_n}$$

must converge and

$$\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega^{2/k-1}(n) \Re_n^{(k)}$$

must uniformly converge with respect to m *for* $1 \le k \le 2$, *where*

i) $\{\Omega_n\}$ *is a sequence of positive numbers,*
ii) $\{\Omega_n/n\}$ *is a sequence of non-increasing numbers and*
iii) $\Re_n^{(k)}$ *is defined by*

$$\Re_j^{(k)} := \frac{1}{j^{\left(\frac{2}{k}-1\right)}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\hat{R}_n^j}{R_n} - \frac{\hat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2. \tag{6.2}$$

Case II. For $p_v = 1$ and $q_v = 1$, together with $\varphi_n = n^{\left(1-\frac{1}{k}\right)}$, Theorem 3.1 and Theorem 3.2 are reduced other some results of Krasniqi [10] on summability $|\bar{N},q;m|_k$ and $|N,p;m|_k$ as follows.

Corollary 6.3. *In order to make an orthogonal series* $\sum\limits_{n=0}^{\infty} a_n \Phi_n(x)$ *almost everywhere* $|\bar{N},q;m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ \left(\frac{n^{\left(1-\frac{1}{k}\right)} q_n}{Q_n Q_{n-1}} \right)^2 \sum_{j=1}^{n} \left[Q_{j-1} + j \left(\frac{Q_n}{n+1} - \sum_{v=j}^{n} \frac{q_v}{v+1} \right) \right]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{6.3}$$

must uniformly converge with respect to m *for* $1 \le k \le 2$.

Corollary 6.4. *In order to make an orthogonal series* $\sum\limits_{n=0}^{\infty} a_n \Phi_n(x)$ *almost everywhere* $|N,p;m|_k$ *summable, the following series*

$$\sum_{n=1}^{\infty} \left\{ \left(\frac{n^{\left(1-\frac{1}{k}\right)} p_n}{P_n P_{n-1}} \right)^2 \sum_{j=1}^{n} p_{n-j}^2 [\Im_n^j]^2 |a_{m+j}|^2 \right\}^{k/2} \tag{6.4}$$

must uniformly converge with respect to m for $1 \leq k \leq 2$, where

$$\Im_n^j = 1 - \frac{P_{n-j-1}}{p_{n-j}} + j \sum_{v=0}^{n-j} \frac{P_n - (n-v+1)p_n p_v}{(n-v)(n+1-v)p_n p_{n-j}}. \tag{6.5}$$

7. Conclusions

The paper focuses on the study of absolute convoluted Nörlund $\varphi - |N, p, q; m|_k$ summability factor which is useful for achieving the stability of the system. The BIBO stability of the system can be achieved by the condition of absolute summability, as absolute summable is necessary and sufficient condition, *i.e.*,

$$\text{BIBO stability} \;\Leftrightarrow\; \sum_{M=-\infty}^{\infty} |I(M)| < \infty,$$

where $I(M)$ is input impulse response of the system.

The absolute summability plays an important role in signal processing as a double digital filter in finite and infinite impulse response (FIR & IIR). Employing the convoluted Nörlund $\varphi - |N, p, q; m|_k$ summability (a generalized summability), the functions of the filters (like removal of unwanted frequency components, enhancement of the required frequency components, permanent unit power factor, overcoming unbalancing situation, *etc.*) have been improved. The results are also useful in engineering, for example, the load signal can be represented as a summation of orthonormal functions (orthogonal series).

Based on this investigation, it can be concluded that our theorem is a generalized version which can be reduced to well known summabilities as shown in the corollaries. Under certain suitable conditions $\varphi_n = n^{\left(1-\frac{1}{k}\right)}$(Case III), the main theorems render the result of Krasniqi [14] on $|N, p, q; m|_k$ summability, which explains the importance and validation of the presented work.

Acknowledgement

This work has been financially supported by Science and Engineering Research Board (SERB) through Project No. EEQ/2018/000393. The authors offer their true thanks to the Science and Engineering Research Board for giving financial support.

References

1. H. Bor, Absolute Nörlund summability factors. *Utilitas Math.* **40** (1991), 231–236.
2. H. Bor, On the absolute Nörlund summability factors. *Glas. Mat. Ser. III* **27(47)** (1992), no. 1, 57–62.
3. H. Bor, A note on absolute Nörlund summability factors. *Real Anal. Exchange* **18** (1992/93), no. 1, 82–86.
4. H. Bor, Absolute Nörlund summability factors of infinite series. *Bull. Calcutta Math. Soc.* **85** (1993), no. 3, 223–226.
5. H. Bor, On absolute Nörlund summability factors. *Comput. Math. Appl.* **60** (2010), no. 7, 2031–2034.
6. H. Bor, Absolute Nörlund summability factors involving almost increasing sequences. *Appl. Math. Comput.* **259** (2015), 828–830.
7. G. C. H. Güleç, M. A. Sarigöl, Hausdorff measure of noncompactness of certain matrix operators on absolute Nörlund spaces. *Trans. A. Razmadze Math. Inst.* **175** (2021), no. 2, 205–214.
8. G. C. Hazar, M. A. Sarigöl, On absolute Nörlund spaces and matrix operators. *Acta Math. Sin. (Engl. Ser.)* **34** (2018), no. 5, 812–826.
9. G. C. Hazar, M. A. Sarigöl, On factor relations between weighted and Nörlund means. *Tamkang J. Math.* **50** (2019), no. 1, 61–69.
10. Xh. Z. Krasniqi, On absolute almost generalized Nörlund summability of orthogonal series. *Kyungpook Math. J.* **52** (2012), no. 3, 279–290.
11. L. Leindler, On the absolute Riesz summability of orthogonal series. *Acta Sci. Math. (Szeged)* **46** (1983), no. 1-4, 203–209.
12. L. Leindler, On the newly generalized absolute Riesz summability of orthogonal series. *Anal. Math.* **21** (1995), no. 4, 285–297.
13. L. Leindler, K. Tandori, On absolute summability of orthogonal series. *Acta Sci. Math. (Szeged)* **50** (1986), no. 1-2, 99–104.
14. I. P. Natanson, *Theory of Functions of a Real Variable.* Vol. II. Translated from the Russian by Leo F. Boron Frederick Ungar Publishing Co., New York 1961 265 pp.
15. Y. Okuyama, On the absolute Nörlund summability of orthogonal series. *Proc. Japan Acad. Ser. A Math. Sci.* **54** (1978), no. 5, 113–118.

16. Y. Okuyama, On absolute generalized Nörlund summability of orthogonal series. *Tamkang J. Math.* **33** (2002), no. 2, 161–165.
17. Y. Okuyama, T. Tsuchikura, On the absolute Riesz summability of orthogonal series. *Anal. Math.* **7** (1981), no. 3, 199–208.
18. B. E. Rhoades, On the total inclusion for Nörlund methods of summability. *Mathematische Zeitschrift* **96** (1971), no. 3, 183–188.
19. B. E. Rhoades, E. Savaş, On absolute Nörlund summability of Fourier series. *Tamkang J. Math.* **33** (2002), no. 4, 359–364.
20. M. A. Sarigöl, On some absolute summability methods. *Bull. Calcutta Math. Soc.* **83** (1991), no. 5, 421–426.
21. M. A. Sarigöl, On $|\mathbf{T}|_k$ summability and absolute Nörlund summability. *Math. Slovaca* **42** (1992), no. 3, 325–329.
22. M. A. Sarigöl, M. Mursaleen, Almost absolute weighted summability with index k and matrix transformations. *J. Inequal. Appl.* **2021**, Paper no. 108, 11 pp.
23. M. Tanaka, On generalized Nörlund methods of summability. *Bull. Austral. Math. Soc.* **19** (1978), no. 3, 381–402.

(Received 02.10.2020)

[1]Department of Mathematics, National Institute of Technology Kurukshetra, Kurukshetra-136119, INDIA

[2]Department of Mathematics, Akal University, Talwandi Sabo-151302, Bathinda, INDIA
Email address: smita.sonker@gmail.com
Email address: alka_math@auts.ac.in