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CONSISTENT ESTIMATORS OF PARAMETERS OF STATISTICAL STRUCTURES

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Abstract. In this paper, the necessary and sufficient conditions for the existence of consistent estimators of the parameters of statistical structures are given.

1. Introduction

Let (E, S) be a measurable space with a given family of probability measures $\{\mu_i, i \in I\}$. Let I be the set of parameters. An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

A. Skorokhod determined consistent estimators of parameters (see, [2]). Z. Zerakidze defined and studied consistent criteria for hypotheses testing (for Zerakidze's criteria for hypotheses testing see [8]).

By (ZFC) we denote the formal system of Zermelo–Fraenkel with the addition of axiom of choice (AC), i.e., (ZFC)=(ZF)&(AC). By (ZFC)&(CH) we denote the theory with the addition of a continuum hypothesis (CH): $2^{\aleph_0} = c$, where c denotes the first uncountable cardinal number. By (ZFC)&(MA) we denote the theory with the addition of Martin's axiom (MA). It is known that Martin's axiom (MA) is much weaker than the continuum hypothesis (CH). Moreover, the negation of the continuum hypothesis (¬CH) is compatible with Martin's axiom (see [4, 5]).

Z. Zerakidze proved (see [9]): 1) In the (ZFC)&(MA) theory, a Borel weakly separable statistical structure, whose cardinality is not greater than that of the comtinuum, is strongly separable; 2) On an arbitrary set E of continuum power, one can define an orthogonal statistical structure with a maximal possible power equal to 2^{2^c}, a weakly separable statistical structure with a maximal possible power equal to 2^c, and a strongly separable statistical structure with the maximal possible power equal to c.

In Section 2, we study a countable statistical structure that admits consistent criteria for hypotheses testing as well as a consistent estimators of the parameters. In Section 3, we study a continuum statistical structure that admit a consistent criteria for hypotheses testing as well as a consistent estimators of the parameters.

2. The Case of a Countable Statistical Structure

The following definitions are taken from [1–9].

Definition 2.1. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) (O) if the family of probability measures $\{\mu_i, i \in I\}$ consists of pairwise singular measures (i.e., $\mu_i \perp \mu_j$, $\forall i \neq j$).

Definition 2.2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable (WS) if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that

$$\mu_i(X_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases} \qquad (i, j \in I).$$

Definition 2.3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable (S) if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that

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$$1)\quad \mu_i(X_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases} \qquad (i, j \in I);$$

$$2)\quad \forall i, j \in I : \operatorname{card}(X_i \cap X_j) < c, \ \text{ if } i \neq j,$$

where c denotes the power of continuum.

Definition 2.4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable (SS) if there exists a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that the relations

$$\mu_i(X_i) = 1, \ \ \forall i \in I$$

is fulfilled.

Remark 2.1. It's obvious that (SS)⇒(S)⇒(WS)⇒(O), but not vice versa.

Example 2.1. Let $E = [0, 1] \times [0, 1]$, S be a Borel σ-algebra of subsets of E. Let's take the S-measurable sets

$$X_i = \begin{cases} 0 \le x \le 1, \quad y = i, & \text{if } i \in (0, 1]; \\ 0 \le x \le 1, \quad 0 \le y \le 1, & \text{if } i = 0 \end{cases}$$

and assume that $l_i,\ i \in (0, 1]$, are linear Lebesgue measures on X_i and l_0 is a plane Lebesgue measure on $[0, 1] \times [0, 1]$. Then the statistical structure $\{[0, 1] \times [0, 1],\ S,\ l_i,\ i \in [0, 1]\}$ is orthogonal, but not weakly separable.

Let I be the set of parameters and $B(I)$ be the σ-algebra of subsets of I which contains all finite subsets of I.

Definition 2.5. We say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimator of parameter $i \in I$ (CE) if there exists at least one measurable mapping $f : (E, S) \longrightarrow (I, B(I))$ such that

$$\mu_i(\{x : f(x) = i\}) = 1, \quad \forall\, i \in I.$$

Let H be the set of hypotheses and $B(H)$ be the σ-algebra of subsets of H which contains all finite subsets of h. Let $\{\mu_h, h \in H\}$ be the family of probability measures on (E, S).

Definition 2.6. We say that the statistical structure $\{E, S, \mu_h, h \in H\}$ admits a consistent criterion (Zerakidze's criterion) (CC) for hypothesis testing if there exists at least one measurable mapping $\delta : (E, S) \longrightarrow (H, B(H))$ such that

$$\mu_h(\{x : \delta(x) = i\}) = h, \quad \forall\, h \in H.$$

Remark 2.2. It's evident that: 1) (CE)⇒(SS)⇒(S)⇒(WS)⇒(O); 2) (CC)⇒(SS) ⇒(S)⇒(WS)⇒(O), but not vice versa.

In the example below, we give the construction of a strongly separable statistical structure that does not admit a consistent estimators of parameters.

Example 2.2. As a set of parameters, we consider the set $I = R = (-\infty, +\infty)$ and let $B(I) = L(I)$ be a Lebesgue σ-algebra on R. Let $\delta : R \longrightarrow R$ denote some bijective mapping at the axis R which is Lebesgue non-measurable. We divide the segment $[-\frac{1}{2}, \frac{1}{2}]$ into classes as follows: points x and y are included in a certain class if and only if the difference $x - y$ is a rational number. It is evident that the different classes are disjoint. Take one point from each class and denote the set of these points by A. It is obvious that the set A is not $L(R)$-measurable and its cardinality is continuum $\operatorname{card} A = c$. Therefore there is one-to-one mapping $f_1 : A \longrightarrow [0, 1]$ such that $f_1(A) = [0, 1]$. As $A \subset [-\frac{1}{2}, \frac{1}{2}] \subset [-1, 1]$, it is obvious that $\operatorname{card}\{[-1, 1] \setminus A\} = c$ and there exists the bijective reflection $f_2 : [-1, 1] \setminus A \longrightarrow [-1, 0]$ such that $f_2([-1, 1] \setminus A) = [-1, 0]$. Let

$$\delta(x) = \begin{cases} x, & \text{if } x \in R \setminus [-1, 1]; \\ f_1, & \text{if } x \in A; \\ f_2, & \text{if } x \in [-1, 1] \setminus A. \end{cases}$$

$\delta(x)$ is Lebesgue non-measurable because $\delta^{-1}[0,1] = f_1^{-1}[0,1] = A$. Hence, the inverse mapping δ^{-1} will also be Lebesgue non-measurable. Let

$$\mu_i(X) = \begin{cases} 1, & \text{if } \delta(i) \in X; \\ 0, & \text{if } \delta(i) \notin X, \end{cases}$$

for $i \in R$ and $X \in L(R)$. It is easy to see that the statistical structure $\{R, L(R), \mu_i, i \in R\}$ is a strongly separable statistical structure that does not admit a consistent estimators of parameters. It means that there exists the measurable mapping

$$\widetilde{\delta} : (R, L(R)) \longrightarrow (R, L(R))$$

such that $\mu_i(\{x : \widetilde{\delta}(x) = i\}) = 1$, $\forall i \in R$. Therefore $\delta(i) \in \{x : \widetilde{\delta}(x) = i\}$. Hence, we have $\widetilde{\delta}(\delta(i)) = i$, $\forall i \in R$. On the other hand, $\delta^{-1}(\delta(i)) = i$, $\forall ih \in R$. Consequently, $\delta^{-1} \circ \delta = \widetilde{\delta} \circ \delta$, and therefore $\delta^{-1} = \widetilde{\delta} \circ \delta \circ \delta^{-1} = \widetilde{\delta}$. Thus we get that δ^{-1} is measurable, which contradicts the fact that the inverse function of a non-measurable function δ is non-measurable.

Theorem 2.1. *A countable statistical structure $\{E, S, \mu_i, i \in N\}$ $(N = 1, 2, \dots)$ admits both a consistent criterion δ for hypothesis testing and a consistent estimator of parameter f if and only if this statistical structure is orthogonal in the theory (ZFC).*

Proof. Necessity. Since a countable statistical structure $\{E, S, \mu_i, i \in N\}$ admits a consistent estimator of parameter, there exists a measurable mapping $f : (E, S) \longrightarrow (I, B(I))$ such that

$$\mu_i(\{x : f(x) = i\}) = 1, \quad \forall\, i \in N.$$

Let $X_i = \{x : f(x) = i\}$, then it is evident that $X_i \cap X_j = \emptyset$ $\forall i \neq j$ and $\mu_i(X_i) = 1$ $\forall i \in N$. Therefore the statistical structure $\{E, S, \mu_i, i \in N\}$ is strongly separable. Thus, by Remark 2.1, the necessity is proved.

Sufficiency. Let a countable statistical structure $\{E, S, \mu_i, i \in N\}$ be orthogonal. The singularity of probability measures $\{\mu_i, i \in N\}$ implies the existence of a family of S-measurable sets X_{ij} such that for any $i \neq j$, we have $\mu_j(X_{ij}) = 0$ and $\mu_i(E \setminus X_{ij}) = 0$. If we now consider the sets $X_i = \cup_{j \neq i}(E \setminus X_{ij})$, we can see that $\mu_i(X_i) = 0$ and $\mu_j(E \setminus X_i)$, $\forall j \neq i$. It means that the statistical structure $\{E, S, \mu_i, i \in N\}$ is weakly separable and there exists a family of S-measurable sets $\{\widetilde{X}_i, i \in N\}$ such that

$$\mu_j(\widetilde{X}_i) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Let us consider the sets $\overline{X}_i = \widetilde{X}_i \setminus (\widetilde{X}_i \cap (\cup_{j \neq i} \widetilde{X}_j))$, $i \in N$. It is clear that $\overline{X}_i \cap \overline{X}_j = \emptyset$ $\forall i \neq j$ and $\mu_i(\overline{X}_i) = 1$, $\forall i \in N$. Define the mapping $f : (E, S) \to (N, B(N))$ as follows: $f(\overline{X}_i) = i$, $i \in N$. Then we have $\mu_i(\{x : f(x)\}) = 1$, $\forall i \in N$, i.e., the statistical structure $\{E, S, \mu_i, i \in N\}$ admits a consistent estimator of parameter. □

3. The Case of a Strongly Separable Statistical Structure

According to Remark 2.2, the implication (SS)⇒(CE), as well as (SS)⇒ (CC), is not true, in general. Therefore we have to consider narrower classes of statistical structures. In addition, we also need to move on to completions of probability measures.

Let $\{\mu_h, h \in H\}$ be the Charlier probability measures defined on the measurable space (E, S). For each $h \in H$, we denote by $\overline{\mu}_h$ the completion of the measure μ_h, and by $\mathrm{dom}(\overline{\mu}_h)$ – the σ-algebra of all $\overline{\mu}_h$-measurable subsets of E. Let

$$S_1 = \cap_{h \in H} \mathrm{dom}(\overline{\mu}_h).$$

Definition 3.1. A statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is called strongly separable if there exists a family of S_1-measurable sets $\{Z_h, h \in h\}$ such that the relations

1) $\mu_h(Z_h) = 1$, $\forall h \in H$;
2) $Z_{h_1} \cap Z_{h_2} = \emptyset$, $\forall h_1 \neq h_2$,;
3) $\cup_{h \in H} Z_h = E$ are fulfilled.

Definition 3.2. We say that the orthogonal statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ admits a consistent criterion (Zerakidze's criterion) for hypothesis testing if there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (H, B(H))$ such that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = 1, \quad \forall\, h \in H.$$

Definition 3.3. We say that the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits a consistent estimator of parameter if there exists at least one measurable mapping $f : (E, S_1) \longrightarrow (I, B(I))$ such that

$$\overline{\mu}_i(\{x : f(x) = i\}) = 1, \quad \forall\, i \in I.$$

Theorem 3.1. *In order for the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, $\operatorname{card} H = c$, to admit a consistent criterion (Zerakidze's criterion) for hypothesis testing, it is necessary and sufficient that this statistical structure be strongly separable by Definition* 3.1.

Proof. Necessity. The existence of a consistent criterion (Zerakidze's criterion) for hypothesis testing means that there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (H, B(H))$ such that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = 1, \quad \forall\, h \in H.$$

Denoting $Z_h = \{x : \delta(x) = h\}$ for $h \in H$, we get:
1) $\overline{\mu}(Z_h) = \overline{\mu}(\{x : \delta(x) = h\}) = 1$, $\forall h \in H$;
2) $Z_{h_1} \cap Z_{h_2} = \emptyset$, $\forall h_1 \neq h_2$,
3) $\cup_{h\in H} Z_h = \{x : \delta(x) \in H\} = E$.

Hence the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is strongly separable by Definition 3.1.

Sufficiency. Since the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, $\operatorname{card} H = c$, is strongly separable, there exists a family $\{Z_h,\ h \in H\}$ of elements of the σ-algebra $S_1 = \cap_{h\in H} \operatorname{dom}(\overline{\mu}_h)$ such that:
1) $\overline{\mu}(Z_h) = 1$, $\forall h \in H$;
2) $Z_{h_1} \cap Z_{h_2} = \emptyset$, $\forall h_1 \neq h_2$,
3) $\cup_{h\in H} Z_h$ =E.

For $x \in E$, we put $\delta(x) = h$, where h is a unique hypothesis from the set H for which $x \in Z_h$. The existence and uniqueness of such hypothesis h can be proved by using conditions 2) and 3).

Take now $Y \in B(H)$. Then $\{x : \delta(x) \in Y\} = \cup_{h\in Y} Z_h$. We have to show that $\{x : \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{h_0})$ for each $h_0 \in H$.

If $h_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{h\in Y} Z_h = Z_{h_0} \cup (\cup_{h\in Y\setminus\{h_0\}} Z_h).$$

On the one hand, from conditions 1), 2) and 3) it follows that

$$Z_{h_0} \in S_1 = \cap_{h\in H} \operatorname{dom}(\overline{\mu}_h) \subseteq \operatorname{dom}(\overline{\mu}_{h_0}).$$

On the other hand, the inclusion

$$\cup_{h\in Y\setminus\{h_0\}} Z_h \subseteq (E \setminus Z_{h_0})$$

implies that $\overline{\mu}_{h_0}(\cup_{h\in Y\setminus\{h_0\}} Z_h) = 0$, and hence

$$\cup_{h\in Y\setminus\{h_0\}} Z_h \in \operatorname{dom}(\overline{\mu}_{h_0}).$$

Since $\operatorname{dom}(\overline{\mu}_{h_0})$ is a σ-algebra, we conclude that

$$\{x : \delta(x) \in Y\} = Z_{h_0} \cup (\cup_{i\in Y\setminus\{h_0\}} Z_h) \in \operatorname{dom}(\overline{\mu}_{h_0}).$$

If $h_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \cup_{h\in Y} Z_h \subseteq (E \setminus Z_{h_0})$ and we conclude that $\overline{\mu}_{h_0}(\{x : \delta(x) \in Y\}) = 0$. The last relation implies that

$$\{x : \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{h_0}), \quad \forall Y \in B(H).$$

Thus we have proved the validity of the relation $\{x : \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{h_0})$ for any $h_0 \in H$. Hence

$$\{x : \delta(x) \in Y\} \in \cap_{h\in H} \operatorname{dom}(\overline{\mu}_h) = S_1.$$

Therefore the mapping $\delta : (E, S_1) \longrightarrow (H, B(H))$ is a measurable mapping.

Since $B(H)$ contains all finite subsets of H, we ascertain that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = \overline{\mu}_h(Z_h) = 1, \quad \forall\, h \in H,$$

i.e., this statistical structure admits Zerakidze's criterion. □

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