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A NOTE ON THE CONVERGENCE OF WAVELET FOURIER SERIES

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Abstract. In this paper, we discuss the rate of convergence of Wavelet Fourier series of periodic functions. Our result generalizes the results of M. Skopina [13] [Localisation Principle for wavelet expansion, self seminar system, Proceedings of the International Workshop, Dubna, (1999), 125-133] and V. Karanjgaokar [5] *et al.* [On the rate of Convergence of Wavelet Fourier Series, *Jñãnãbha*, 51(1) (2021), 12-18], by introducing a general monotonically decreasing function $P_n(x)$, satisfying certain conditions.

1. Introduction

The concept of wavelet has been viewed as a synthesis of various ideas originated from different disciplines including mathematics (see Loknath and Debnath [9]). It was observed that the computational efficiency of wavelet expansions is related to their multiresolution form and other well-studied properties. Wavelets are local in time and frequency, and a wavelet basis for $L^2(\mathbb{R})$ consist of translations and dilations of one or more functions (see M. A. Kon [7]) and then wavelet becomes a very important tool for signal analysis. Wavelet Fourier Series is a special type of wavelet expansion which is a Fourier Series with wavelet bases. In this paper, we discuss the convergence of wavelet Fourier Series.

The main aim of discovery of wavelets is to study the time-frequency signal analysis. Wavelets have been introduced by A. Grossmann and J. Morlet [2], as functions whose translations and dilations could be used for expansions in $L^2(\mathbb{R})$. The prototype of wavelets can be found in the works of A. Haar [3]. S. Mallat [10] and Y. Meyer, both independently developed the framework of multiresolution analysis to generate orthonormal bases for $L^2(\mathbb{R})$. P. G. Lamarie and Y. Meyer [8] constructed wavelets in $S(\mathbb{R}^n)$, the space of rapidly decreasing smooth functions.

In this paper, we are going to study the rate of convergence of Wavelet Fourier Series of periodic functions, i.e., we analyse the convergence rate of Periodic Multiresolution Analysis (PMRA) of functions $f \in L_p(\mathbb{R})$, $(1 \leq p \leq \infty)$. We generalize the results of M. Skopina [12] and V. Karanjgaokar *et al.* [5] by introducing a general monotonically decreasing function of x and n, satisfying certain specific conditions. For this purpose, first let us have a look on the following definitions.

1.1. **Periodic multiresolution analysis (PMRA).** The concept of PMRA has been defined and used in Deng Feng and Si Long [1], Prestin and Selig [11] and Skopina [12]. Let $\phi \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$ be respectively a scaling function of MRA and a wavelet function given by

$$\hat{\phi}(x) = m_0\Big(\frac{x}{2}\Big)\hat{\phi}\Big(\frac{x}{2}\Big)$$

and

$$\hat{\psi}(x) = m_0\Big(\frac{x+1}{2}\Big)\hat{\phi}\Big(\frac{x}{2}\Big)e^{i\pi x},$$

where $m_0 \in L^2(\mathbb{T})$ is a low pass filter. The normalized integer shifts and scales of ψ given by

$$\psi_{j,n}(x) = 2^{\frac{j}{2}}\psi(2^j x + n), \quad j, n \in \mathbb{Z}$$

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constitute an orthonormal basis in $L^2(\mathbb{R})$. If both the functions ϕ and ψ have sufficient decay, then the functions

$$\Phi_{j,n}(x) = 2^{\frac{j}{2}} \sum_{l\in\mathbb{Z}} \phi(2^j x + 2^j l + n)$$

and

$$\Psi_{j,n}(x) = 2^{\frac{j}{2}} \sum_{l\in\mathbb{Z}} \psi(2^j x + 2^j l + n)$$

are in $L^2(\mathbb{T})$ and the systems $\{\Phi_{j,n}\}_{n=0}^{2^j-1}$ and $\{\Psi_{j,n}\}_{n=0}^{2^j-1}$ are orthonormal for each $j = 0, 1, 2, \ldots$. The spaces

$$V_j = \text{span}\{\Phi_{j,n},\ n = 0, 1, 2, \ldots, 2^j - 1\}$$

and

$$W_j = \text{span}\{\Psi_{j,n},\ n = 0, 1, 2, \ldots, 2^j - 1\}$$

satisfy the properties:

$$V_0 = \{\text{const}\}, V_j \subset V_{j+1}, V_{j+1} = V_j \oplus W_j$$

and

$$\bigcup_{j=0}^{\infty} V_j = L^2(\mathbb{T}),$$

for all $j = 0, 1, 2, \ldots$. The collection $\{V_j\}_{j=0}^{\infty}$ is called a periodic multiresolution analysis generated by Φ.

1.2. **Wavelet Fourier series (Skopina [13]).** If $f \in L^2(\mathbb{T})$, then

$$\langle f, \Phi_{0,0}\rangle \Phi_{0,0} + \sum_{j=0}^{\infty} \sum_{n=0}^{2^j-1} \langle f, \Psi_{j,n}\rangle \Psi_{j,n} \tag{1.1}$$

is called wavelet Fourier series. The double sum in (1.1) can be transformed into a single sum by redenoting periodic wavelets as

$$w_0 = \Phi_{0,0},\quad w_{2^j+L} = \Psi_{j,L},\quad 0 \le L \le 2^j - 1,$$

and the series (1.1) can be rewritten as

$$\sum_{k=0}^{\infty} \langle f, w_k\rangle w_k. \tag{1.2}$$

Let $S_N(f)$ denote the N^{th} partial sum of (1.2), with $N = 2^j + L$, $0 \le L < 2^j - 1$ and let $S_{2^j-1}(f)$ be an orthogonal projection of f onto V_j with $\{\Phi_{j,n}\}_{n=0}^{2^j-1}$ as orthonormal basis in V_j, then

$$S_{2^j-1}(f) = \sum_{n=0}^{2^j-1} \langle f, \Phi_{j,n}\rangle \Phi_{j,n},$$

$$S_N(f) = \sum_{n=0}^{2^j-1} \langle f, \Phi_{j,n}\rangle \Phi_{j,n} + \sum_{n=0}^{L} \langle f, \Psi_{j,n}\rangle \Psi_{j,n}.$$

Set $f = w_0 = 1$ in (1.2) and since $\langle f, w_k\rangle = \delta_{0,k}$, we have $S_N(f) = 1$ for all N, $j = 0, 1, 2, \ldots$. Hence

$$\int_0^1 \sum_{k=0}^{N} w_k(x)\overline{w_k(t)}dt \equiv 1, \qquad \int_0^1 \sum_{k=0}^{2^j-1} \Phi_{j,k}(x)\overline{\Phi_{j,k}(t)}dt \equiv 1.$$

2. Theorems and Lemmas

This section includes the following Theorem 2.1 and Theorem 2.2, which will be generalized by our Main Theorem. This section also includes the Lemmas used in the proof of our theorem.

Theorem 2.1 (Skopina [12]). *Let $\phi, \psi \in L^2(\mathbb{R})$ and $n > 1$ such that*

$$|\phi(x)|.|\psi(x)| \le C/(1+|x|^n),$$

$f(x) = 0$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Then

$$S_N(f, x_0) = O(N^{1-n}),\ N \to \infty.$$

Theorem 2.2 (V. Karanjgaokar *et al.* [5]). *Let $\phi,\psi \in L^2$ $(\mathbb{R})$ and $n > 1$ such that.*

$$|\phi(x)|, |\psi(x)| \le C/(1+|x|)^n,$$

and if $f(x) = 0\ \forall x \in [x_0 - \delta, x_0 + \delta]$, where $0 < \delta < 1/2$, $x_0 \in R$ and C is a constant , then

$$S_N(f, x_0) = O(N^{1-n}) \text{ as } N \to \infty.$$

Lemma 2.3. *Let g and h be the functions defined on $\mathbb{R}$, with $\max(|g(x)|, |h(x)|) = O(P_n(x))$, where $P_n(x)$ is a function of x for each fixed positive integer n and is a positive monotonic decreasing function of $|x|$, with the series $\sum_{k=0}^{\infty} P_n(x)$ converging for fixed $n > 1$. Then*

$$\int_0^1 f(t) \sum_{k=0}^{L} \sum_{l' \in \mathbb{Z}} g(2^j x + 2^j l' + k) \sum_{l \in \mathbb{Z}} \overline{h(2^j t + 2^j l + k)} dt = \int_{-\infty}^{\infty} f(t) \sum_{v \in Z(j,L)} g(2^j x + v)\overline{h(2^j t + v)} dt,$$

where $Z(j, L) = \{v \in \mathbb{Z} : v = 2^j l + k, l \in \mathbb{Z}, k = 0, 1, \ldots, L\}$. The proof of this lemma is trivial and one can see the lemma for $P_n(x) = \frac{C}{1+|x|^n}$, $n > 1$ in M. Skopina [13].

Lemma 2.4 (Kelly *et al.* [6]). *Let μ be a bounded decreasing and integrable function in $[0, \infty)$. Then for all $x, y \in \mathbb{R}$,*

$$\sum_{k \in \mathbb{Z}} |\mu(x+k)||\mu(y+k)| \le C\mu\Big(\frac{|x-y|}{4}\Big),$$

where C is the constant depending only on μ.
The proof of this lemma is simple, its proof can be seen in M. Skopina [12] and Kelly et al. [6]. The proof of this lemma for $\mu(x) = \frac{1}{(1+|x|)^{1+\epsilon}}$ can be seen in V. Karanjgaokar [4].

3. Main Theorem

Theorem 3.1. *Let $\phi, \psi \in L^2(\mathbb{R})$ and let the inequalities*

$$|\phi(x)| = O(P_n|x|),$$
$$|\psi(x)| = O(P_n|x|),$$

hold, where for each fixed positive integer n, $P_n(x)$ is a function of x, which is positive, integrable and monotonic decreasing with $|x|$ and that

$$\sum_{k=0}^{\infty} 2^{j+k} P_n(2^{j+k})$$

converges for each fixed $n > 1$ and all $j = 0, 1, 2, \ldots$.

If for each fixed $n > 1$, there exists an $F(n, j) > 0$ such that

$$\sum_{k=0}^{\infty} 2^{j+k} P_n(2^{j+k}) = \sum_{k=j}^{\infty} 2^k P_n(2^k) = F(n, j) \quad (j = 0, 1, \ldots),$$

then if $f(x) = 0$, $\forall x \in [x_0 - \delta, x_0 + \delta]$, $0 < \delta < \frac{1}{2}$, $x_0 \in \mathbb{R}$, we have

$$S_N(f, x_0) = O(F(n, N)) \text{ as } N \to \infty.$$

Note.

(1) Our result generalizes the result of M. Skopina [12] for $P_n(x) = \frac{C}{1+|x|^n}$ for fixed $n > 1$ with $F(n,j) = 2^{j(1-n)}$.
(2) Our result also generalizes the result of V. Karanjgaokar *et al.* [5] for $P_n(x) = \frac{C}{(1+|x|)^n}$ for fixed $n > 1$ with $F(n,j) = 2^{j(1-n)}$.
(3) Four corollaries are given in Section 5, where we establish the results for different values of $P_n(x)$, in some of them the rate of convergence is found to be faster than that existing in the results of M. Skopina [12] and V. Karanjgaokar *et al.* [5]

Proof of Theorem 3.1. Since

$$\begin{aligned} S_{2^j-1}(f,x_0) &= \sum_{n=0}^{2^j-1} \langle f, \Phi_{j,n}\rangle \Phi_{j,n}(x_0) \\ &= \int_0^1 f(t) \sum_{n=0}^{2^j-1} \overline{\Phi_{j,n}(t)}\Phi_{j,n}(x_0)dt \\ &= 2^j \int_0^1 f(t) \sum_{n=0}^{2^j-1} \sum_{l\in\mathbb{Z}} \overline{\phi(2^jt+2^jl+n)}\phi(2^jx_0+2^jl+n)dt, \end{aligned}$$

therefore, using Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned} |S_{2^j-1}(f,x_0)| &\leq 2^j \int_{-\infty}^{\infty} |f(t)| \sum_{v\in Z(j,l)} |\phi(2^jt+v)||\phi(2^jx_0+v)|dt \\ &\leq 2^j \int_{-\infty}^{\infty} |f(t)|P_n|2^j(t-x_0)|dt \\ &\leq 2^j 2 \int_0^{\infty} |f(t)|P_n|2^j(t-x_0)|dt. \end{aligned}$$

Let j_0 denote the largest integer $\log_2 \delta$. Using the hypothesis that P_n is monotonic, we have

$$\begin{aligned} |S_{2^j-1}(f,x_0)| &\leq 2^{j+1} \sum_{k=j_0}^{\infty} \int_{2^k\leq|t-x_0|\leq 2^{k+1}} |f(t)|P_n|2^j(t-x_0)|dt \\ &\leq 2^{j+1} \sum_{k=j_0}^{\infty} P_n(2^{j+k}) \int_{|t-x_0|\leq 2^k} |f(t)|dt. \end{aligned}$$

If

$$I(h) = \frac{1}{h} \int_{|t-x_0|<h} |f(t)|dt,$$

then $I(h)$ is bounded on $(0,\infty)$. Indeed, $I(h) = 0$ for $h < \delta$.

$$I(h) \leq \delta^{-1}||f||_1 \quad \text{for} \quad \delta < h < 1/2$$

and also

$$
\begin{aligned}
I(h) \le & \frac{1}{h} \int\limits_{|t-x_0|\le[h]+1} |f(t)|dt \\
\le & \frac{2(h+1)}{h}||f||_1 \\
\le & 6||f||_1 \quad for\ h \ge \frac{1}{2}.
\end{aligned}
$$

Thus

$$I(h) = O(1),$$

$$|S_{2^j-1}(f,x_0)| = O(1)\sum_{k=j_0}^{\infty} 2^{j+k} P_n(2^{j+k}) = O(F(n,j)). \tag{3.1}$$

This proves the theorem for $N=2^j-1$. In particular, we have proved that the sequence $\{S_{2^j-1}(f,x_0)\}$ converges to 0. Now, for any positive integer N,

$$-S_N(f,x_0) = (S_{2^j-1}(f,x_0) - S_N(f,x_0)) + \sum_{i=j}^{\infty}(S_{2^{i+1}-1}(f,x_0) - S_{2^i-1}(f,x_0)).$$

The result will be proved for arbitrary N, if we are able to establish the relation

$$S_{2^j+L}(f,x_0) - S_{2^j-1}(f,x_0) = O(F(n,j)) \tag{3.2}$$

for all $j = 0,1,\ldots,\quad L = 0,1,\ldots,2^j-1$. Using the definition of $S_N(f,x_0)$, the left-hand side of equation (3.2) can be represented by

$$\int_0^1 f(t)\sum_{k=0}^{L}\overline{\Psi_{j,k}(t)}\Psi_{j,k}(x_0)dt,$$

thus equation (3.2) can be proved similarly to equation (3.1) which completes the proof of the theorem.

4. Corollaries

Here, we present four corollaries for different values of $P_n(x)$ two of which give faster rate of convergence.

Corollary 4.1. *Let*

$$P_n(x) = \frac{x^{-n}}{(\log x)^n}.$$

Then for $N = 2^j - 1$,

$$S_N(f,x_0) = O\Big\{\frac{N^{1-n}}{(\log(N+1))^n}\Big\}, \quad (N\to\infty).$$

Proof. For $x = 2^{j+k}$,

$$P_n(2^{j+k}) = (2^{(j+k)(-n)})(\log 2^{j+k})^{-n}.$$

Hence

$$
\begin{aligned}
\sum_{k=j_0}^{\infty} 2^{j+k}(P_n(2^{j+k})) = & \sum_{k=j_0}^{\infty}(2^{j+k})^{1-n}(\log 2^{j+k})^{-n} \\
\le & 2^{j(1-n)}(\log 2^j)^{-n}\sum_{k=0}^{\infty}(2^k)^{1-n} \\
= & O(1)2^{j(1-n)}(\log 2^j)^{-n} \\
= & O(1)(N+1)^{1-n}(\log(N+1))^{-n}.
\end{aligned}
$$

Thus we take

$$F(n,N) = N^{1-n}(\log(N+1))^{-n}.$$

Substitute it into the main theorem to get the result. □

Corollary 4.2. *Let*

$$P_n(x) = x^{-n}\log^n(x).$$

Then for $N = 2^j - 1$,

$$S_N(f,x_0) = O\{N^{(1-n)}(\log 2)^n\}, \quad (N \to \infty).$$

Compare this $P_n(x)$ *with Theorem* 2.1 *of Skopina* [12].

Proof. For

$$x = 2^{j+k},$$
$$P_n(2^{j+k}) = (2^{(j+k)})^{-n}(\log 2^{(j+k)})^n$$

and

$$\begin{aligned}\sum_{k=j_0}^{\infty} 2^{j+k}P_n(2^{j+k}) =& \sum_{k=j_0}^{\infty}(2^{(j+k)})^{(1-n)}(j+k)^n(\log 2)^n\\ =& 2^{j(1-n)}(\log 2)^n\sum_{k=j_0}^{\infty}2^{k(1-n)}(j+k)^n\\ =& O(1)(2^j)^{(1-n)}(\log 2)^n.\end{aligned}$$

Since the series on the right-hand side converges by D'Alenbert's ratio test, we have

$$F(n,j) = (2^j)^{1-n}(\log 2)^n$$

and

$$F(n,N) = (N+1)^{(1-n)}(\log 2)^n.$$

Substitute it in the main theorem to get the result. □

Note: for fixed positive integer $n > 1$,

$$x^{-n} < x^{-n}(\log x)^n, \quad (x > 2).$$

Thus

$$P_n = O(x^{-n}) \Rightarrow P_n(x) = O(x^{-n}(\log x)^n),$$

i.e., a weaker condition but the ultimate $(N+1)^{(1-n)}(\log 2)^n$ is sharper than $(N+1)^{(1-n)}$.

Corollary 4.3. *In our theorem, if we take* $P_n(x) = e^{-nx}$, *(fixed* $n > 0$*), then*

$$F(n,j) = 2^j e^{-n2^j} + \frac{2^{j+1}}{e^{n2^{j+1}} - 2}\,. \tag{4.1}$$

Note. The rate of convergence in equation (4.1) is faster than $F(n,j) = 2^{j(1-n)}$, (fixed $n > 1$) in M. Skopina [12] and V. Karanjgaokar *et al.* [5].

Corollary 4.4. *In our theorem, if we take* $P_n(x) = \frac{1}{nx}log(1+\frac{n}{x})$, *(fixed* $n > 0$*), then*

$$F(n,j) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}\,2^i\,n^{i-1}}{2^{ij}\,(2^i-1)\,i} \le \frac{1}{2^{j-1}-n},$$

which tends to zero as j *tends to* ∞.

Note. If we take $P_n(x) = \frac{1}{nx}\log(1+\frac{1}{nx})$, (fixed $n>1$), then

$$F(n,j) \leq \frac{1}{2^{j-1}-1},$$

which tends to zero as j tends to ∞.

Our results in this paper not only generalize the existing results but also give sufficient examples of $P_n(x)$ in the form of corollaries in which the rate of convergence is faster than that existing in the results due to M. Skopina [12] and V. Karanjgaokar *et al.* [5]

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