Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 1, 143–146

ON SOME VERSION OF RANDOM VARIABLES

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Abstract. The notion of a generalized random variable is introduced in terms of extensions of a given probability measure. Some properties of generalized random variables are considered.

Let Ω be an uncountable set and let P be a continuous probability measure on Ω, i.e., P vanishes at all singletons in Ω. Denoting $\mathcal{S} = \mathrm{dom}(P)$, we thus have a probability space $(\Omega, \mathcal{S}, P)$ such that $P(\{\omega\}) = 0$ for every $\omega \in \Omega$.

Let $f : \Omega \to \mathbf{R}$ be a function, where $\mathbf{R}$ is the real line.

We shall say that f is a generalized random variable (or a quasi-random variable, or a weak random variable) if there exists a measure P' on Ω which extends P and for which f becomes a random variable.

Accordingly, for a natural number $n > 0$, we shall say that a mapping $F : \Omega \to \mathbf{R}^n$ is a generalized random vector if there exists a measure P' on Ω which extends P and for which F becomes a random vector (the latter means that F is a measurable mapping from $(\Omega, \mathrm{dom}(P'), P')$ into $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, where $\mathcal{B}(\mathbf{R}^n)$ denotes, as usual, the Borel σ-algebra of $\mathbf{R}^n$).

Example 1. Clearly, if a probability P is defined on the family of all subsets of Ω, then any function $f : \Omega \to \mathbf{R}$ is a random variable (hence, a quasi-random variable). In this case, the concept of a generalized random variable becomes superfluous. However, such a case is very problematic, because, as is known, the statement that $\mathrm{dom}(P)$ always differs from the family of all subsets of Ω does not contradict the axioms of the contemporary **ZFC** set theory.

Example 2. Every real-valued function f on Ω whose range is at most countable (i.e., every real-valued step-function f) can be considered as a generalized random variable on Ω. Indeed, let

$$\mathrm{ran}(f) = \{r_0, r_1, \ldots, r_k, \ldots\}$$

and let $\Omega_k = f^{-1}(r_k)$ for each natural number k. Then the family $\{\Omega_k : k = 0, 1, \ldots\}$ forms a partition of Ω and, in view of the result obtained in [1], there exists a measure P' on Ω such that P' extends P and all sets Ω_k become P'-measurable. This immediately implies that f turns out to be a P'-measurable function and so, f is a generalized random variable. Observe that the real-valued step-functions on Ω form an algebra of functions and, simultaneously, a lattice of functions. This circumstance is sometimes useful in applications. However, the above-mentioned family is not closed under the standard limit operations of analysis.

The argument presented in Example 2 works for mappings $F : \Omega \to \mathbf{R}^n$, where n is a nonzero natural number. Namely, if the range of F is at most countable, then F can be considered as a generalized random vector.

In connection with Example 2, it makes sense to formulate the following statement which slightly strengthens the result of [1].

Theorem 1. *Let $(\Omega, \mathcal{S}, P)$ be a probability space and let $\{A_i : i \in I\}$ be a family of subsets of Ω such that $P(A_i \cap A_j) = 0$ for any two distinct indices i and j from I.*

Then there exists a probability P' on Ω which extends P and for which all sets A_i $(i \in I)$ are P'-measurable.

Example 3. Let Ω be an uncountable set and let P be a probability on Ω such that, for every set $A \subset \Omega$ with $\mathrm{card}(A) < \mathrm{card}(\Omega)$, one has $P(A) = 0$. Observe that if $\mathrm{card}(\Omega)$ is not cofinal with $\mathrm{card}(\mathbf{N})$,

2020 *Mathematics Subject Classification.* 28A05, 28D05.
Key words and phrases. Generalized random variable; Extension of measure; Absolutely nonmeasurable function.

where $\mathbf{N}$ denotes the set of all natural numbers, then such P can always be defined. According to one of Sierpiński's results (see, e.g., [10]), there exists a family $\{A_i : i \in I\}$ of subsets of Ω satisfying the following relations:

(1) $\mathrm{card}(I) > \mathrm{card}(\Omega)$;

(2) $\mathrm{card}(A_i) = \mathrm{card}(\Omega)$ for each index $i \in I$;

(3) $\mathrm{card}(A_i \cap A_j) < \mathrm{card}(\Omega)$ for any two distinct indices i and j from I.

In view of Theorem 1, there exists a probability measure P' on Ω which extends P and for which all sets A_i $(i \in I)$ become P'-measurable.

Clearly, for a vector function $F : \Omega \to \mathbf{R}^n$, the following two assertions are equivalent:

(i) F is a random vector in the standard sense;

(ii) all functions $\mathrm{pr}_k \circ F$, where $k = 1, 2, \dots, n$, are random variables in the standard sense.

For the concept of generalized random vectors, the above-mentioned equivalence fails to be true (cf. Theorem 3 below). At the same time, it can easily be seen that if $F : \Omega \to \mathbf{R}^n$ is a generalized random vector, then all functions $\mathrm{pr}_k \circ F$, where $k = 1, 2, \dots, n$, are generalized random variables.

In the sequel, we shall say that a function $f : \Omega \to \mathbf{R}$ is absolutely nonmeasurable if f is not a generalized random variable for the trivial continuous probability measure P_0 on Ω, whose domain consists of all countable and co-countable subsets of Ω.

In other words, the absolute nonmeasurability of $f : \Omega \to \mathbf{R}$ means that there exists no nonzero σ-finite measure μ on Ω, vanishing at all singletons of Ω and such that f is measurable with respect to μ (cf. [2, 5, 6]).

In [2], a certain characterization of absolutely nonmeasurable real-valued functions was given (see also [5, 6]). This characterization is based on the notion of an absolute null subset of $\mathbf{R}$.

Recall that a set $X \subset \mathbf{R}$ is absolute null (of universal measure zero) if for any σ-finite continuous Borel measure ν on $\mathbf{R}$, one has $\nu^*(X) = 0$, where ν^* denotes the outer measure produced by ν.

There are very nontrivial examples of uncountable absolute null subsets of $\mathbf{R}$ (see, for instance, [5, 7–9]). In particular, any Luzin set in $\mathbf{R}$ is absolute null. The existence of Luzin subsets of $\mathbf{R}$ or of generalized Luzin subsets of $\mathbf{R}$ needs additional set-theoretic hypotheses (cf. [7–9]). On the other hand, the existence of an uncountable absolute null sets in $\mathbf{R}$ can be established within **ZFC** theory (see, e.g., [5], where a slightly more general result is presented).

Theorem 2. *For a function $f : \Omega \to \mathbf{R}$, these two assertions are equivalent:*

(1) *f is absolutely nonmeasurable;*

(2) *$\mathrm{ran}(f)$ is an absolute null subset of $\mathbf{R}$ and the set $f^{-1}(t)$ is at most countable for every point $t \in \mathbf{R}$.*

The proof of Theorem 2 is not difficult and can be found in [5] and [6].

Suppose that a natural number $n \geq 2$ and a vector function $F : \Omega \to \mathbf{R}^n$ are given. This F can be written as $F = (f_1, f_2, \dots, f_n)$, where each f_i is a real-valued function on Ω. Obviously, F produces exactly n vector functions $F_1, F_2, \dots, F_n$, where

$$F_i = (f_1, f_2, \dots, f_{i-1}, f_{i+1}, \dots, f_n) \quad (i = 1, 2, \dots, n).$$

Using Theorem 2, one can obtain the following statement.

Theorem 3. *Assume Martin's Axiom* (**MA**). *Let $\Omega = [0, 1]$ and let P be the standard probability Lebesgue measure on $[0, 1]$.*

There exists a vector function $F : [0, 1] \to \mathbf{R}^n$ such that:

(1) *any F_i $(i = 1, 2, \dots, n)$ is a random vector with respect to some measure P_i which extends P;*

(2) *the real-valued function $f_1 + f_2 + \dots + f_n$ associated with F is injective and its range is a generalized Luzin subset of $\mathbf{R}$ (so, this function is absolutely nonmeasurable).*

Consequently, every F_i $(i = 1, 2, \dots, n)$ is a generalized random vector, but F itself is not a generalized random vector.

Note that a result similar to Theorem 3 is formulated and proved in [5, Chapter 17]. Moreover, the mentioned result does not need any additional set-theoretical axioms so, it is provable within the **ZFC** set theory.

Observe that if $F : \Omega \to \mathbf{R}^n$ is a random vector and $G : \Omega \to \mathbf{R}^n$ is a generalized random vector, then their sum

$$F + G = (f_1 + g_1, f_2 + g_2, \dots, f_n + g_n)$$

and their product

$$F \cdot G = (f_1 \cdot g_1, f_2 \cdot g_2, \dots, f_n \cdot g_n)$$

are generalized random vectors.

A function $h : \mathbf{R}^n \to \mathbf{R}^m$ is called universally measurable if, for every Borel subset B of $\mathbf{R}^m$, the pre-image $h^{-1}(B)$ belongs to the domain of the completion of any σ-finite Borel measure on $\mathbf{R}^n$.

If h is universally measurable and $F : \Omega \to \mathbf{R}^n$ is a generalized random vector, then the composition $h \circ F : \Omega \to \mathbf{R}^m$ is also a generalized random vector.

Remark 1. In [3] and [4], the notion of an almost measurable function $f : \Omega \to \mathbf{R}$ was introduced and examined, where $\Omega = [0, 1]$ and P is the standard probability Lebesgue measure on Ω. It was also proved in those works that any almost measurable function turns out to be a generalized random variable.

In the analogous manner, the concept of an almost measurable vector function $F : \Omega \to \mathbf{R}^n$ can be defined and it can be proved that such F is a generalized random vector.

Theorem 4. *Let $\Omega = [0, 1]$ and let P be again the standard Lebesgue probability measure on Ω. Under* **MA***, there exists a vector function*

$$G : \Omega \to \mathbf{R}^{\mathbf{N}}$$

satisfying the following relations:

(1) *for any nonempty finite set $K \subset \mathbf{N}$, the vector function $\mathrm{pr}_K \circ G$ is a generalized random vector;*

(2) *for any infinite set $K \subset \mathbf{R}$, the vector function $\mathrm{pr}_K \circ G$ is absolutely nonmeasurable, i.e., there exists no continuous probability measure P' on Ω, for which $\mathrm{pr}_K \circ G$ becomes a measurable mapping acting from $(\Omega, \mathrm{dom}(P'), P')$ into $(\mathbf{R}^K, \mathcal{B}(\mathbf{R}^K))$.*

Remark 2. The proof of the existence of G is based on the fact that under **MA** every generalized Luzin subset of $\mathbf{R}$ is of universal measure zero. Moreover, taking into account Example 2, one can additionally assert in the formulation of Theorem 4 that:

(a) the vector function G is injective;

(b) for each natural number n, the range of the function $\mathrm{pr}_n \circ G$ is finite;

(c) all functions $\mathrm{pr}_n \circ G$, where $n \in \mathbf{N}$, are measurable with respect to some countably generated σ-algebra of subsets of Ω, which contains $\mathrm{dom}(P)$ and does not admit any continuous probability measure.

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(Received 18.02.2023)

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