Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 1, 131–133

ABSOLUTELY NEGLIGIBLE SETS AND THEIR ALGEBRAIC SUMS

MARIAM BERIASHVILI[1], MARIKA KHACHIDZE[2] AND ALEKS KIRTADZE[3]

Abstract. For invariant (quasi-invariant) σ-finite measures on an uncountable group, the behaviour of absolutely negligible sets with respect to the algebraic sums is studied.

In the paper by Sierpiński [8], it was proved that there exist two subsets X and Y of $\mathbf{R}$ such that $\lambda(X) = \lambda(Y) = 0$ and $X + Y = \mathbf{R}$, where λ is the standard Lebesgue measure on the real line $\mathbf{R}$.

The above-mentioned result can be extended to a wide class of uncountable topological groups equipped with σ-finite invariant (quasi-invariant) Borel measures (in this connection, cf., also [7].

It is reasonable to ask whether similar statements hold in more general situations when no topology is considered on a given group. Namely, it is natural to pose the following question:

Let $(G, \cdot)$ be an uncountable group equipped with a nonzero σ-finite G-invariant (G-quasiinvariant) measure μ and let $\mathcal{I}(\mu)$ be a σ-ideal of all μ-measure zero sets.

Do there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ whose algebraic sum $X \cdot Y$ is equal to G?

The formulation of the question posed above should be replaced by another one. Namely, the following problem is of interest from the measure-theoretical point of view.

Let $(G, \cdot)$ be an uncountable group and let μ be a nonzero σ-finite left G-invariant (left G-quasiinvariant) measure on G.

Does there exist a left G-invariant (left G-quasiinvariant) measure μ' on G extending μ and such that for some sets $X \in I(\mu')$ and $Y \in I(\mu')$, the relation

$$X \cdot Y = G$$

is satisfied?

Let us introduce one notion from the general theory of invariant (quasi-invariant) measures, which plays a crucial role in our further constructions.

Let $(G, \cdot)$ be an arbitrary group and let X be a subset of G. We say that X is G-absolutely negligible in G if for every σ-finite left G-invariant (respectively, left G-quasi-invariant) measure μ on G, there exists a left G-invariant (respectively, left G-quasi-invariant) measure μ' on G extending μ and satisfying the relation $\mu'(X) = 0$.

Example 1. In 1914, S. Mazurkiewicz presented transfinite constructions of a subset A of the Euclidian plane $\mathbf{R}^2$, having the following extraordinary property: every straight line in $\mathbf{R}^2$ meets A at exactly two points. The descriptive structure of a Mazurkiewicz set turned out to be rather complicated. In general, one cannot assert that a Mazurkiewicz set is necessarily nonmeasurable with respect to a standard Lebesgue measure in the Euclidean plane λ_2 measure. Indeed, there are Mazurkiewicz subsets of the plane which have λ_2 -measure zero. Moreover, there exists a Mazurkiewicz set which is λ_2-thick. In general:

- there exists a Mazurkiewicz set which is absolutely negligible with respect to $M(\mathbf{R}^2)$;
- there exists a Mazurkiewicz set which is not-absolutely negligible with respect to $M(\mathbf{R}^2)$ (see [4]).

Example 2. In 1905, Hamel considered $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers and proved the existence of a basis in this space (a Hamel basis). It is known that every Hamel basis of the space $\mathbf{R}^n$ is an absolutely negligible subset of $\mathbf{R}^n$ [1,4].

2020 *Mathematics Subject Classification.* 28A05, 28D05.

Key words and phrases. Invariant measure; Absolutely negligible set; Extension of measure; Surjective homomorphism.

The following lemma is true.

Lemma 1. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be two groups, $\varphi : G_1 \longrightarrow G_2$ be a surjective homomorphism and let Y be a G_2-absolutely negligible subset of G_2. Then the set $X = \varphi^{-1}(Y)$ is G_1-absolutely negligible in G_1.*

Various properties of absolutely negligible sets are considered in [2,3,5].

In the above-mentioned question, for an uncountable commutative group $(G, +)$, the following statements are valid.

Theorem 1. *For any uncountable commutative group $(G, +)$, there exist two G-absolutely negligible sets X and Y in G such that $X + Y = G$.*

Remark 1. It immediately follows from Theorem 1 that if $(G, +)$ is an arbitrary uncountable commutative group, then there exists a G-absolutely negligible subset Z of G such that

$$Z + Z = G.$$

Theorem 2. *Let $(G, +)$ be an uncountable commutative group and let μ be a σ-finite G-invariant (respectively, G-quasi-invariant) measure on G. There exists a G-invariant (respectively, G-quasi-invariant) extension μ' of μ such that*

$$\mu'(X) = \mu'(Y) = 0, \quad X + Y = G,$$

for some G-absolutely negligible subsets A and B of G which do not depend on μ.

For an uncountable group $(G, \cdot)$ the following statements are valid.

Theorem 3. *Let $(G, \cdot)$ be an uncountable group such that*

$$(\mathrm{card}(G))^{\omega} = \mathrm{card}(G).$$

Then there exist two G-absolutely negligible sets X and Y in G for which

$$X \cdot Y = G.$$

The proofs of the above-mentioned statements can be found in [5].

Theorem 4. *Let $(G, \cdot)$ be an arbitrary group such that*

$$G = G' \times G'', \quad (G' \cap G'' = \{e\}),$$

where G' and G'' are the subgroups of G and $\mathrm{card}(G') = \omega_1$. Let μ be a nonzero σ-finite G-quasi-invariant measure on G. Then for each uncountable set $X \subset G'$, there exist a G-quasi-invariant measure μ' on G extending μ and a set $Y \in I(\mu')$ for which we have

$$X \cdot Y = G \notin I(\mu').$$

In particular, if $X \in I(\mu')$, then G is representable in the form of algebraic product of two μ'-measure zero sets.

For the proof of Theorem 4, see [6].

Let $(G, \cdot)$ be an arbitrary uncountable group.

Lemma 2. *Let $(H, \otimes)$ be an uncountable group (commutative or noncommutative) and let μ be a nonzero σ-finite H-invariant measure on H. If*

$$\varphi : G \to H$$

is a surjective homomorphism and there exist a nonzero σ-finite H-left invariant measure $\mu' \supset \mu$ and two sets $X \in I(\mu')$ and $Y \in I(\mu')$ on H such that

$$X \otimes Y = H,$$

then there exist the measures ν and ν' on G and two sets $X' \in I(\nu')$ and $Y' \in I(\nu')$ on G for which the following relations are satisfied:

(a) *$\nu' \supset \nu$;*

(b) *$X' \cdot Y' = G$;*

(c) *ν and ν' are G-left invariant measures on G.*

From the above lemma, we readily obtain the following statement.

Theorem 5. *Let $(G, \cdot)$ and $(H, \cdot)$ be arbitrary uncountable groups and let*

$$\varphi : G \to H$$

be a surjective homomorphism. Let μ be a nonzero σ-finite H-left invariant measure on H. If there exist a nonzero σ-finite H-left invariant (H-left-quasi-invariant) measure $\mu' \supset \mu$ on H and two absolutely negligible sets X and Y such that $X \cdot Y = H$, then there exist nonzero σ-finite G-left invariant (G-left-quasi-invariant) measures ν and ν' satisfying the following relations:

(1) *ν' is a nonzero σ-finite G-left invariant (G-left-quasi-invariant) measure on G;*
(2) *$\nu' \supset \nu$;*
(3) *there exist two absolutely negligible sets X' and Y' such that $X' \cdot Y' = G$.*

Remark 2. If $(G, \cdot)$ is an uncountable commutative group, then the existence of two absolutely negligible sets X and Y such that $X + Y = G$ is guaranteed by Theorem 2.

Acknowledgement

This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG), Grant YS-21-1667.

References

1. M. Beriashvili, Measurability properties of certain paradoxical subsets of the real line. *Georgian Math. J.* **23** (2016), no. 1, 25–32.
2. A. B. Kharazishvili, On vector sums of measure zero sets. *Georgian Math. J.* **8** (2001), no. 3, 493–498.
3. A. B. Kharazishvili, *Topics in Measure theory and Real Analysis.* Atlantis Studies in Mathematics, 2. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.
4. A. Kharazishvili, Measurability properties of Mazurkiewicz sets. *Bull. TICMI* **20** (2016), no. 2, 44–46.
5. A. B. Kharazishvili, A. P. Kirtadze, On algebraic sums of measure zero sets in uncountable commutative groups. *Proc. A. Razmadze Math. Inst.* **135** (2004), 97–103.
6. A. Kirtadze, On the method of direct products in the theory of quasi-invariant measures. *Georgian Math. J.* **12** (2005), no. 1, 115–120.
7. M. Kneser, Summenmengen in lokalkompakten abelschen Gruppen. (German) *Math. Z.* **66** (1956), 88–110.
8. W. Sierpiński, Sur la question de la mesurabilité de la base de M. Hamel. *Fund. Math.* **1** (1920), no. 1, 105–111.

(Received 12.12.2022)

[1]Department of Mathematics No. 63 of Georgian Technical University, 77 Kostava Str., Tbilisi, 0160, Georgia

[2]Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia

[3]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Email address: mariam_beriashvili@yahoo.com
Email address: m.khachidze1995@gmail.com
Email address: kirtadze2@yahoo.com