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ON FACTORIZATION OF PARTLY NON-RATIONAL 2×2 MATRIX-FUNCTIONS

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Dedicated to the memory of Prof. Edem Lagvilava

Abstract. It is proposed a generalization of the method developed by the authors for the solution of the $\mathbb{R}$-linear conjugation problem on the unit circle with rational coefficients. It is done an equivalent reduction of the matrix-functions to the triangular form using the structure of the considered matrix-functions (namely, rationality of one of the rows or the columns). Then the obtained triangular matrix-function is factorized by applying G. N. Chebotarev's approach.

1. Introduction

The paper deals with an explicit solution of the factorization problem for a class of partly non-rational matrix-functions.

In the classical framework, the factorization problem means a representation of a square non-singular matrix-function $G \in \mathcal{G}(\mathcal{M}(\Gamma))^{n\times n}$ defined on a simple closed smooth curve Γ on the complex plane $\mathbb{C}$ in the following form:

$$G(t) = G^+(t)\Lambda(t)G^-(t), \tag{1}$$

where non-singular matrices $G^+(t), G^-(t)$ possess an analytic continuation into D^+, D^-, respectively. Here, D^+, D^- are the domains on the Riemann sphere lying, respectively, to the left and to the right of the curve Γ, with respect to the orientation chosen on Γ. $\Lambda(t)$ is the $n \times n$ diagonal matrix,

$$\Lambda(t) = \mathrm{diag}\left\{\left(\frac{t-t^+}{t-t^-}\right)^{æ_1}, \dots, \left(\frac{t-t^+}{t-t^-}\right)^{æ_n}\right\}, \tag{2}$$

and $t^+ \in D^+, t^- \in D^-$ are certain (fixed) points. In particular, if $\Gamma = \mathbb{R}$, then one can choose

$$t^+ = i, t^- = -i,$$

and if Γ is a bounded curve and $0 \in D^+$, then

$$\frac{t-t^+}{t-t^-} = t.$$

The integers $æ_1, \dots, æ_n$ are called partial indices, and the matrices $G^-(t)$, $G^+(t)$ are known as *minus-, plus-factors.* The factorization of type (1) is called left-factorization or classical left-factorization. If it exists, then the partial indices are uniquely determined up to their order, i.e., they are invariants of the factorization problem. Upon interchanging $G^+(t)$ and $G^-(t)$ in (1), we arrive at the right-factorization. In the above definition, by $\mathcal{M}(\Gamma))^{n\times n}$ is denoted a class of all square $n \times n$ matrix-functions defined on Γ, and $\mathcal{G}$ stands for invertible matrix-functions.

Factorization problem has a long and interesting history (see, e.g., [7, 10, 14, 24] and references therein). Initially, it is linked to the name by Bernhardt Riemann or, more precisely, with two problems formulated by him, known as the Riemann boundary value problem (or Riemann–Hilbert boundary value problem, or $\mathbb{C}$-linear conjugation problem, see [9]), and the Riemann monodromy problem (or 21st Hilbert problem, or Riemann–Hilbert problem, see [5]). Importance of the factorization

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problem is due to its relation to several mathematical problems of different nature and to its numerous applications.

At present, the factorization problem is interesting due to its connections with the notable mathematical problems (vector-matrix boundary value problems, systems of singular integral equations, the Wiener–Hopf and other convolution type equations, the Riemann–Hilbert problem, classification of vector bundles on the Riemann sphere, non-linear evolution equations, the Toeplitz operators, etc.), as well as with the applied problems (elasticity and elasto-plasticity, radiation and neutron transport, wave diffraction, fracture mechanics, geomechanics, signal processing, financial mathematics, etc.). Several monographs and extended surveys on the theory, on specific approaches and applications of the factorization of matrix-functions have been published (see an extended list of references on the theme in the survey paper [12]).

Due to the above-said applications, special attention is paid to constructive approaches, meaning exact formulas or explicit algorithms to the exact calculations (see [24] and references therein). One of a widely used approach is due to Chebotarev [6], who proposed an explicit algorithm for determination of factors and partial indices for 2×2 triangular matrix-function. This algorithm is based on exploiting of the representation of a ratio of functions in a continued fraction. Generalizations of Chebotarev's algorithm is proposed by many authors (see [1, 12, 15, 21]).

Stability analysis is one of the important directions in the study of the factorization problem (see pioneering works [4, 11], as well as [13]). Investigations of the proposed algorithms from the stability point of view is very important especially when we are aimed at getting an approximate solution to the factorization problem (see, [2, 3, 16, 17]). We have also to mention here the Janashia–Lagvilava method (see, e.g., [8]) the essential components of which are the triangular factorization followed by the appropriate approximation and the construction of unitary matrix-functions of a special form.

The factorization problem is tightly related to the study of vector-matrix boundary value problems. In fact, the factorization of a given matrix $G(t)$ is equivalent to the construction of the canonical matrix for the homogeneous Riemann boundary value problem ($\mathbb{C}$-linear conjugation problem) (see [20]) with $G(t)$, being its matrix coefficient

$$\Phi^{+}(t) = G(t)\Phi^{-}(t), t \in \Gamma. \tag{3}$$

The so-called $\mathbb{R}$-linear conjugation problem [18, 19]

$$\varphi^{+}(t) = a(t)\varphi^{-}(t) + b(t)\overline{\varphi^{-}(t)} + c(t), t \in \Gamma, \tag{4}$$

is also reduced to problem (3). Factorization of the corresponding matrix coefficient of the reduced problem plays an essential role [22, 23]. Thus in [22] is proposed a new algorithm of factorization consisting of two-fold application of Chebotarev's approach, namely, an equivalent reduction of the considered matrix to the triangular form followed by the direct use of Chebotarev's approach. This method is applied to the polynomial [22] and to the rational [23] 2×2 matrices related to the $\mathbb{R}$-linear conjugation problem.

In the present paper, we generalize this method and use it to the factorization of partly rational 2×2 matrices, i.e., the matrices with one rational row or one rational column. Such matrices may be either related to $\mathbb{R}$-linear conjugation problem or to have an arbitrary structure.

2. Reduction of the Matrix to the Triangular Form

Let us consider a square non-singular matrix-function $G \in \mathcal{G}(\mathcal{M}(\Gamma))^{2\times 2}$ defined on a simple closed smooth curve Γ on the complex plane $\mathbb{C}$,

$$G(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}. \tag{5}$$

Let $D^{+} = int\Gamma$ and $D^{-} = ext\Gamma$ stand for the interior and exterior parts, respectively, of Γ and assume $D^{+} \ni 0$, $D^{-} \ni \infty$. Suppose that one of the rows ((a, b) or (c, d)) or one of the columns ((a, c) or (b, d)) is a rational function, but the remaining terms are arbitrary functions. Without loss of generality, we may assume that rational are the functions $b(t), d(t)$. All other situations can be reduced to this one by multiplication of the matrix $G(t)$ by a constant matrices of the unit determinant $T_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ from the left/right or from both sides.

2.1. **Reduction to the triangular form: the case of the unit determinant.** In this subsection, we consider the reduction to the triangular form of the matrices with the unit determinant

$$\det G(t) = a(t)d(t) - b(t)c(t) \equiv 1, \quad t \in \Gamma. \tag{6}$$

Such an assumption is rather technical and we will discuss it in the next subsection.

We start with the following equality:

$$X_{0,L}(t) = G(t)X_{0,R}(t), \quad t \in \Gamma, \tag{7}$$

where $X_{0,L}(t) = E_2$, $X_{0,R}(t) = G^{-1}(t) = \begin{pmatrix} d(t) & -b(t) \\ -c(t) & a(t) \end{pmatrix}$. By the assumption, $\frac{d(t)}{b(t)}$ is a ratio of polynomials $\frac{d(t)}{b(t)} = \frac{Q(t)}{P(t)}$. We also suppose, without loss of generality, that $\deg Q(t) \geq \deg P(t)$ (if not, we interchange columns of $G^{-1}(t)$ by multiplying (7) by T_0 from the right).

Our further aim is to expand $\frac{Q(t)}{P(t)}$ in a finite continued fraction

$$\frac{Q(t)}{P(t)} = S_0(t) + \frac{1}{S_1(t) + \frac{1}{S_2(t) + \cdots \frac{1}{S_l(t)}}}. \tag{8}$$

First, we divide polynomial $Q(t)$ by the polynomial $P(t)$ and obtain the representation

$$\frac{d(t)}{b(t)} = \frac{Q(t)}{P(t)} = S_0(t) + \frac{R_1(t)}{P(t)} \Leftrightarrow 1 - S_0(t)\frac{P(t)}{Q(t)} = \frac{R_1(t)}{Q(t)}. \tag{9}$$

Next, we divide polynomial $P(t)$ by the polynomial $R_1(t)$ and obtain the representation

$$\frac{P(t)}{R_1(t)} = S_1(t) + \frac{R_2(t)}{R_1(t)} \Leftrightarrow R_1(t)S_1(t) - P(t) = -R_2(t), \tag{10}$$

then we divide the polynomial $R_1(t)$ by the polynomial $R_2(t)$ and get

$$\frac{R_1(t)}{R_2(t)} = S_2(t) + \frac{R_3(t)}{R_2(t)} \Leftrightarrow R_1(t) - R_2(t)S_2(t) = R_3(t), \tag{11}$$

and continue this process up to the l-th step. The last division of the polynomials in our representation of $\frac{Q(t)}{P(t)}$ in the continued fraction yields two possible situations:

(a) $R_{l-1}(t)$ is divided by $R_l(t)$ with the vanishing remainder ($R_{l+1}(t) \equiv 0$), i.e.,

$$\frac{R_{l-1}(t)}{R_l(t)} = S_l(t). \tag{12}$$

(b) $R_{l-1}(t)$ is divided by $R_l(t)$ with a constant remainder ($R_{l+1}(t) \equiv C$), i.e.,

$$\frac{R_{l-1}(t)}{R_l(t)} = S_l(t) + \frac{C}{R_l(t)}. \tag{13}$$

Now, we transform relation (7). First, we multiply the both sides of (7) from the right by the polynomial matrix

$$T_1(t) = \begin{pmatrix} 1 & 0 \\ S_0(t) & 1 \end{pmatrix}. \tag{14}$$

Then we obtain a new right-component $X_{1,R}(t)$ of the solution to (7),

$$\begin{aligned} X_{1,R}(t) = X_{0,R}(t)T_1(t) &= \begin{pmatrix} d(t)\left(1 - S_0(t)\frac{P(t)}{Q(t)}\right) & -d(t)\frac{P(t)}{Q(t)} \\ F_1(t) & a(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{d(t)}{Q(t)}R_1(t) & -\frac{d(t)}{Q(t)}P(t) \\ F_1(t) & a(t) \end{pmatrix}, \end{aligned} \tag{15}$$

where $F_1(t) = -c(t) + a(t)S_0(t)$. Next, we multiply from the right both sides of the relation $X_{1,L}(t) = G(t)X_{1,R}(t)$ by the polynomial matrix

$$T_2(t) = \begin{pmatrix} 1 & S_1(t) \\ 0 & 1 \end{pmatrix}. \tag{16}$$

Then we obtain a new right-component $X_{2,R}(t)$ of the solution to (7),

$$\begin{aligned} X_{2,R}(t) = X_{0,R}(t)T_1(t)T_2(t) &= \begin{pmatrix} \frac{d(t)}{Q(t)}R_1(t) & \frac{d(t)}{Q(t)}\left(R_1(t)S_1(t) - P(t)\right) \\ F_1(t) & F_2(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{d(t)}{Q(t)}R_1(t) & -\frac{d(t)}{Q(t)}R_2(t) \\ F_1(t) & F_2(t) \end{pmatrix}, \end{aligned} \tag{17}$$

where $F_2(t) = F_1(t)S_1(t) + a(t)$.

Thus, for k even, $k \le l-1$, we have the representation

$$X_{k,R}(t) = X_{0,R}(t)T_1(t)T_2(t)\ldots T_k(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_{k-1}(t) & -\frac{d(t)}{Q(t)}R_k(t) \\ F_{k-1}(t) & F_k(t) \end{pmatrix} \tag{18}$$

and for k odd, $k \le l-1$, we have

$$X_{k,R}(t) = X_{0,R}(t)T_1(t)T_2(t)\ldots T_k(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_k(t) & -\frac{d(t)}{Q(t)}R_{k+1}(t) \\ F_k(t) & F_{k+1}(t) \end{pmatrix}. \tag{19}$$

Therefore in the case **(a)**, we obtain two different representations of $X_{l,R}(t) = X_{l-1,R}(t)T_l(t)$ in the triangular form:

$$X_{l,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_{l-1}(t) & -\frac{d(t)}{Q(t)}R_l(t) \\ F_{l-1}(t) & F_l(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S_l(t) & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{d(t)}{Q(t)}R_l(t) \\ F_{l+1}(t) & F_l(t) \end{pmatrix}, \quad l \ \text{ is even,} \tag{20}$$

$$X_{l,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_l(t) & -\frac{d(t)}{Q(t)}R_{l-1}(t) \\ F_l(t) & F_{l-1}(t) \end{pmatrix} \begin{pmatrix} 1 & S_l(t) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{d(t)}{Q(t)}R_l(t) & 0 \\ F_l(t) & F_{l+1}(t) \end{pmatrix}, \quad l \ \text{ is odd.} \tag{21}$$

The first form of representation (20) can be transformed to the second one (21) if we multiply the relation $X_{l,L}(t) = G(t)X_{l,R}(t)$ by the matrix $T_{l+1}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e., we have

$$X_{l+1,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_l(t) & 0 \\ -F_l(t) & F_{l+1}(t) \end{pmatrix}. \tag{22}$$

In case **(b)**, we obtain two different representations of $X_{l,R}(t) = X_{l-1,R}(t)T_l(t)$ in the triangular form:

$$X_{l,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_{l-1}(t) & -\frac{d(t)}{Q(t)}R_l(t) \\ F_{l-1}(t) & F_l(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S_l(t) & 1 \end{pmatrix} = \begin{pmatrix} \frac{d(t)}{Q(t)}C & -\frac{d(t)}{Q(t)}R_l(t) \\ F_{l+1}(t) & F_l(t) \end{pmatrix}, \quad l \ \text{ is even,} \tag{23}$$

$$X_{l,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}R_l(t) & -\frac{d(t)}{Q(t)}R_{l-1}(t) \\ F_l(t) & F_{l-1}(t) \end{pmatrix} \begin{pmatrix} 1 & S_l(t) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{d(t)}{Q(t)}R_l(t) & -\frac{d(t)}{Q(t)}C \\ F_l(t) & F_{l+1}(t) \end{pmatrix}, \quad l \ \text{ is odd.} \tag{24}$$

The first one is reduced to the triangular form by multiplying $X_{l,L}(t) = G(t)X_{l,R}(t)$ by the matrix $T_{l+2}(t) = \begin{pmatrix} 1 & \frac{R_l(t)}{C} \\ 0 & 1 \end{pmatrix}$,

$$X_{l+2,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}C & 0 \\ F_{l+1}(t) & F_{l+2}(t) \end{pmatrix}, \quad l \ \text{ is even,} \tag{25}$$

and the second one by multiplying $X_{l,L}(t) = G(t)X_{l,R}(t)$ by the product of two matrices $T_{l+3}(t) = \begin{pmatrix} 1 & 0 \\ \frac{R_l(t)}{C} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence

$$X_{l+3,R}(t) = \begin{pmatrix} \frac{d(t)}{Q(t)}C & 0 \\ -F_l(t) & F_{l+2}(t) \end{pmatrix}, \quad l \quad \text{is even.} \tag{26}$$

Taking into account that the matrix G, as well as all transformation matrices T_j are polynomials and have the unit determinant, we can conclude that the following statement holds.

Lemma 1. *Let the matrices $X_{s,R}(t)$, $s = l, l+1, l+2, l+3$ be determined by formulas* (21), (22), (25), (26), *respectively, and*

$$X_{l,L}(z) = T_1(z)T_2(z)\dots T_l(z);\ X_{s,L}(z) = T_1(z)T_2(z)\dots T_l(z)T_s(z), s = l+1, l+2, l+3.$$

Then the following properties hold for matrices $X_{l,L}(z)$, $X_{s,L}(z)$, $X_{s,R}(t)$:

(i) *The matrices $X_{s,L}(t), X_{s,R}(t)$, $s = l, l+1, l+2, l+3$ satisfy the boundary relation*

$$X_{s,L}(t) = G(t)X_{s,R}(t), \quad t \in \Gamma. \tag{27}$$

(ii) *The matrix $X_{s,L}(t)$ is a polynomial and thus possesses an analytic continuation $X^+_{s,L}(z)$ into the domain D^+.*

(iii) *The matrix $X_{s,R}(t)$ has no analyticity property, it is defined only on the contour Γ and is represented there in the triangular form*

$$X_{s,R}(t) = \Delta(t) = \begin{cases} \begin{pmatrix} \frac{d(t)}{Q(t)}R_l(t) & 0 \\ (-1)^{l+1}F_l(t) & \frac{Q(z)}{d(t)R_l(t)} \end{pmatrix} & in\ case(\mathbf{a}), \\ \begin{pmatrix} C\frac{d(t)}{Q(t)} & 0 \\ (-1)^{l}F_{l+1}(t) & \frac{Q(t)}{Cd(t)} \end{pmatrix} & in\ case(\mathbf{b}). \end{cases} \tag{28}$$

In (28), *the number s is equal either to l, or to $l+1$, or to $l+2$, or to $l+3$.*

It follows that the matrix $G(t)$ is represented in the form

$$G(t) = X^+_{s,L}(t)\Delta^{-1}(t), t \in \Gamma. \tag{29}$$

Here,

$$\Delta^{-1}(t) = \begin{cases} \begin{pmatrix} \frac{Q(z)}{d(t)R_l(t)} & 0 \\ (-1)^{l}F_l(t) & \frac{d(t)}{Q(t)}R_l(t) \end{pmatrix} & in\ case(\mathbf{a}), \\ \begin{pmatrix} \frac{Q(t)}{Cd(t)} & 0 \\ (-1)^{l+1}F_{l+1}(t) & C\frac{d(t)}{Q(t)} \end{pmatrix} & in\ case(\mathbf{b}). \end{cases} \tag{30}$$

Note that the above formulation depends on the parity of the number l. The obtained representation can be interchanged accordingly.

2.2. **Reduction to the triangular form: the case of an arbitrary determinant.** In this subsection, we consider the case of a nonsingular matrix with a non-unit determinant and two rational entries. As before, without loss of generality, we can suppose that the second column is a rational vector. Applying the same transformations as in Subsection 2.1, we can obtain the following result, similar to that of Lemma 1.

Lemma 2. *Let $G(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $t \in \Gamma$, be a nonsingular matrix on Γ (i.e. $\lambda(t) := \det G(t) = a(t)d(t) - b(t)c(t) \neq 0$, $t \in \Gamma$). Let the second column of $G(t)$ be a rational vector.*

Let the matrices $X_{s,R}(t)$, $s = l, l+1, l+2, l+3$ be determined by formulas (21), (22), (25), (26), *respectively, and*

$$X_{l,L}(z) = T_1(z)T_2(z)\cdots T_l(z);\ X_{s,L}(z) = T_1(z)T_2(z)\cdots T_l(z)T_s(z), \quad s = l+1, l+2, l+3.$$

Then the following properties hold for matrices $X_{l,L}(z), X_{s,L}(z), X_{s,R}(t)$:

(i) *The matrices* $X_{s,L}(t), X_{s,R}(t)$, $s = l, l+1, l+2, l+3$ *satisfy boundary relation*

$$X_{s,L}(t) = G(t)X_{s,R}(t), \quad t \in \Gamma. \tag{31}$$

(ii) *The matrix* $X_{s,L}(t)$ *is a polynomial one and thus it possesses an analytic continuation* $X^+_{s,L}(z)$ *into domain* D^+.

(iii) *The matrix* $X_{s,R}(t)$ *has no analyticity property, it is defined only on the contour* Γ *and is represented there in the triangular form*

$$X_{s,R}(t) = \Delta(t) = \begin{cases} \begin{pmatrix} \frac{d(t)}{Q(t)} R_l(t) & 0 \\ (-1)^{l+1} F_l(t) & \frac{\lambda(t)Q(t)}{d(t)R_l(t)} \end{pmatrix} & \textit{in case } (\mathbf{a}), \\ \begin{pmatrix} C\frac{d(t)}{Q(t)} & 0 \\ (-1)^{l} F_{l+1}(t) & \frac{\lambda(t)Q(t)}{Cd(t)} \end{pmatrix} & \textit{in case } (\mathbf{b}). \end{cases} \tag{32}$$

In (32)*, the number* s *is equal either to* l*, or to* $l+1$*, or to* $l+2$*, or to* $l+3$.

It follows that the matrix $G(t)$ *is represented in the form*

$$G(t) = X^+_{s,L}(t)\Delta^{-1}(t), t \in \Gamma. \tag{33}$$

Here

$$\Delta^{-1}(t) = \begin{cases} \begin{pmatrix} \frac{Q(t)}{d(t)R_l(t)} & 0 \\ (-1)^{l}\frac{F_l(t)}{\lambda(t)} & \frac{d(t)}{\lambda(t)Q(t)} R_l(t) \end{pmatrix} & \textit{in case } (\mathbf{a}), \\ \begin{pmatrix} \frac{Q(t)}{Cd(t)} & 0 \\ (-1)^{l+1}\frac{F_{l+1}(t)}{\lambda(t)} & C\frac{d(t)}{\lambda(t)Q(t)} \end{pmatrix} & \textit{in case } (\mathbf{b}). \end{cases} \tag{34}$$

2.3. **Factorization of matrix** $G(t)$**.** Now, we are ready to factorize the matrix coefficient $G(t)$ of problem (3). To avoid additional technical difficulties, let us consider in this section only the case of unit determinant. More general case will be a subject for further publications.

We start with the representation of the matrix $G(t)$ in the form (29), where the triangular matrix $\Delta^{-1}(t)$ has representation (30).

Further, we have to factorize the matrix $\Delta^{-1}(t)$ following Chebotarev's approach [6]. The matrix $\Delta^{-1}(t)$ has the following form:

$$\Delta^{-1}(t) = \begin{pmatrix} x(t) & 0 \\ a(t) & y(t) \end{pmatrix},$$

where the components are described in Lemma 1. First, we factorize the diagonal elements $x(t), y(t)$, i.e., represent them in the form $x(t) = \frac{x^+(t)}{x^-(t)}, y(t) = \frac{y^+(t)}{y^-(t)}$.

Then, following [6], we introduce the Cauchy type integral (for its properties one can consult e.g., [9])

$$\sigma^{\pm}(z) = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{a(\tau)x^-(\tau)d\tau}{y^+(\tau)(\tau - z)}, \quad z \in D^{\pm}. \tag{35}$$

Then the pair of matrices

$$Y^{\pm}(z) = \begin{pmatrix} x^{\pm}(z) & 0 \\ y^{\pm}(z)\sigma^{\pm}(z) & y^{\pm}(z) \end{pmatrix} \tag{36}$$

satisfies the boundary condition

$$Y^+(t) = \Delta^{-1}(t)Y^-(t), \tag{37}$$

the matrices are analytic in the corresponding domains and nonsingular everywhere except probably infinity.

Note that $Y^{\pm}(z)$ are not necessarily the canonical matrices of the boundary value problem (37) (see, e.g. [20]), i.e., the order of $\det Y^-(z)$ at infinity is not necessarily equal to the sum of orders of the columns of the matrix $Y^-(z)$ at infinity. To avoid this we expand the function $\frac{1}{\sigma^-(z)}$ into a continued fraction

$$\frac{1}{\sigma^-(z)} = U_0(z) + \cfrac{1}{U_1(z) + \cfrac{1}{U_2(z) + \cdots \cfrac{1}{U_r(z)}}},$$

where $U_0(z), U_1(z), \dots, U_r(z)$ are the polynomials of order $q_0, q_1, \dots, q_r$, respectively. At each step (except of the last one) of the expansion, we will have the identities

$$\begin{gathered} 1 = U_0(z)\sigma^-(z) + V_1(z), \\ \sigma^-(z) = U_1(z)V_1(z) + V_2(z), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots , \\ V_{s-3}(z) = U_{s-2}(z)V_{s-2}(z) + V_{s-1}(z), \end{gathered}$$

where the remainder terms $V_1(z), V_2(z), \dots, V_{s-1}(z)$ have at infinity the orders $\nu_1, \nu_2, \dots, \nu_{s-1}$, respectively. At the last step, we will have as before either

$$V_{s-1}(z) = U_s(z)V_s(z),$$

or

$$V_{s-1}(z) = U_s(z)V_s(z) + C.$$

Next, we perform multiplication of both sides of (37) by triangular polynomial matrices of type $\begin{pmatrix} 1 & 0 \\ U_i(t) & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & U_i(t) \\ 0 & 1 \end{pmatrix}$ in order to get the necessary property of the transformed matrix $\widetilde{Y}^-$. Whenever it is obtained, the orders of the columns æ$_1$, æ$_2$ of the transformed matrix $\widetilde{Y}^-(z)$ become partial indices of the matrix $\triangle^{-1}(t)$. Finally, we have the following result.

Theorem 1. *Let $G(t)$ be a square non-singular matrix-function $G \in \mathcal{G}(\mathcal{M}(\Gamma))^{2\times 2}$ defined on a simple closed smooth curve Γ on the complex plane $\mathbb{C}$. Suppose additionally that at least one row or one column of the matrix $G(t)$ is rational and* $\det G(t) \equiv 1$.

Then the matrix $G(t)$ possesses the factorization

$$G(t) = G^+(t)\Lambda(t)G^-(t), \quad t \in \Gamma, \tag{38}$$

where

$$G^+(t) = X^+_{s,L}(t)\widetilde{Y}^+(t),\ \Lambda(t) = \begin{pmatrix} t^{æ_1} & 0 \\ 0 & t^{æ_2} \end{pmatrix},\ G^-(t) = \Lambda^{-1}(t)\widetilde{Y}^-(t).$$

Here, the matrices $\widetilde{Y}^+(t), \widetilde{Y}^-(t)$ are those appeared as the result of the above described transformation of equality (37).

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