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INTERNAL HOM OF STABLE QUADRATIC MODULES

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Dedicated to the blessed memory of Edem Lagvilava

Abstract. Based on the our work on q-maps [9], we construct a kind of internal hom for stable quadratic modules, which are algebraic models for 1-type spectra.

1. Introduction

A stable quadratic module is a group homomorphism $C_\mathrm{r} \xrightarrow{\partial} C_\mathrm{b}$ together with a bilinear map $\{-,-\} : C_\mathrm{b}^\mathrm{ab} \otimes C_\mathrm{b}^\mathrm{ab} \to C_\mathrm{r}$ satisfying certain identities (see Section 3). Here, $G^\mathrm{ab} = G/[G,G]$ is the abelisation of a group G. The category of such objects is denoted by SQM. It is an easy consequence of the definition that for any stable quadratic module the both groups C_r and C_b are nilpotent groups of class two and $\mathsf{Im}(\partial)$ is a normal subgroup of C_b. Moreover,

$$\pi_1^C := \mathsf{Ker}(\partial) \quad \text{and} \quad \pi_0^C := \mathsf{Coker}(\partial)$$

are abelian. A morphism of stable quadratic modules is called a *weak equivalence* if it induces an isomorphism on π_i, $i = 0, 1$. Let $\mathbf{Ho}(\mathsf{SQM})$ denote the localization of SQM with respect to weak equivalences. According to [1]*Appendix C, the category $\mathbf{Ho}(\mathsf{SQM})$ is equivalent to the full subcategory of the homotopy category of spectra with objects of those X with $\pi_i X = 0$, $i \neq 0, 1$. This fundamental fact shows an importance of stable quadratic modules. The main novelty of this work is to construct a kind of internal hom for stable quadratic modules. Our construction is based on the notion of quadratic maps. The topological significance of our construction will appear elsewhere.

Recall that in the work of Hans-Joachim Baues and his coauthors the quadratic maps between nilpotent groups of class two play an important role (see [1, 3–7]). Continuing these ideas, in [9] we showed that there is a subclass of quadratic (i.e., of degree two polynomial) maps called q-maps which turn out to be closed under the composition and valuewise addition (we write groups additively). In this way one obtains a category Niq whose objects are all class two nilpotent groups, while the morphisms are all q-maps between them. The important fact is that hom's in this category are groups. In this paper, we exploit Niq to construct internal hom's for stable quadratic modules.

The paper is organised as follows. In Section 2, we recall the main definitions and some results from our paper [9], which are most important for our investigation. In Section 3, we recall the definition of stable quadratic modules and provide several important examples. In Sections 4 and 5, we describe the main construction of the paper a sqm $\mathbb{H}\mathrm{om}(C, D)$ and in the last section we study the extended functorial properties of $\mathbb{H}\mathrm{om}(C, D)$.

2. Preliminaries on Nilpotent Groups of Class Two and Nonabelian Quadratic Maps

2.1. **The category Nil.** We fix some notation. Groups usually are written additively. For a group G and elements $a, b \in G$, let $[a, b] = -a - b + a + b$ be the commutator of a and b. If G_1 and G_2 are subgroups of G, then $[G_1, G_2]$ denotes the subgroup generated by elements $[a, b]$, where $a \in G_1$ and $b \in G_2$. For any group G, we denote by G^ab the abelianisation of G, that is, the quotient

$$G^\mathrm{ab} := G/[G, G].$$

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For an element $x \in G$, let $\hat{x}$ denote the class of x in G^{ab}. An element $a \in G$ is called *central* if $[a, x] = 0$ for all $x \in G$. We denote by $\mathsf{Z}(G)$ the centre of G, which is the subgroup consisting of all central elements of G. A subgroup A of a group G is called *central* when $[G, A] = 0$, in other words, $A \subset \mathsf{Z}(G)$.

Recall also that for an abelian group B, the second exterior power $\Lambda^2(B)$ is the quotient of $B \otimes B$ by the subgroup generated by $b \otimes b$, $b \in B$. The class of $a \otimes b$ in $\Lambda^2(B)$ is denoted by $a \wedge b$.

A group G is of *nilpotence class two*, or is a *nil*$_2$*-group*, if all triple commutators of G vanish, $[[G, G], G] = 0$. The smallest nonabelian groups of nilpotence class two are the quaternion group Q and the dihedral group $\mathbf{Z}/4\mathbf{Z} \ltimes \mathbf{Z}/2\mathbf{Z}$, both of order 8. We denote by **Nil** the category of groups of nilpotence class two.

Lemma 2.1.1 ([9, Lemma 1])**.** *For any $G \in$ **Nil**, one has:*

i) *There is a well-defined homomorphism $\mu(G) : \Lambda^2(G^{\mathrm{ab}}) \to G$ given by $\hat{a} \wedge \hat{b} \mapsto [a, b]$.*
ii) *For any $a, b \in G$, $[a, b] = a + b - a - b$.*
iii) *There is the inclusion $[G, G] \subset \mathsf{Z}(G)$.*
iv) *For any $a, b \in G$ and any $n \in \mathbf{Z}$, it follows that*

$$na + nb = n(a + b) + \frac{n(n-1)}{2}[a, b].$$

It is clear that $G \in$ **Nil** iff the commutator subgroup $[G, G]$ is central. In this case the homomorphism $\mu(G) : \Lambda^2(G^{\mathrm{ab}}) \to [G, G]$ is surjective.

Recall that the inclusion functor **Nil** $\subset$ **Groups** has a left adjoint given by

$$G \mapsto G^{\mathrm{nil}} := G/[[G, G], G].$$

Since left adjoints preserve all the existing colimits, one can obtain coproducts in **Nil** as $(-)^{\mathrm{nil}}$ of coproducts in **Groups**. But, in fact, coproducts in **Nil** are much easier to construct directly than those in **Groups**. Namely, one has

Proposition 2.1.2 ([9, Proposition 1])**.** *For two nil*$_2$*-groups G, H let $G \vee H$ be the set $G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \times G \times H$. The equalities*

$$\begin{aligned}
&(\xi, g, h) + (\xi', g', h') = (\xi + \xi' - \hat{g}' \otimes \hat{h}, g + g', h + h'),\\
&-(\xi, g, h) = (-\xi - \hat{g} \otimes \hat{h}, -g, -h),\\
&0 = (0, 0, 0)
\end{aligned}$$

equip this set with a nil$_2$*-group structure such that there is a central extension*

$$0 \to G^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \to G \vee H \to G \times H \to 0.$$

Moreover, the maps $i_G : G \to G \vee H$, $i_H : H \to G \vee H$ given by $i_G(g) = (0, g, 0)$ and $i_H(h) = (0, 0, h)$ form a coproduct diagram in **Nil***.*

The forgetful functor **Nil** $\to$ **Sets** has a left adjoint, whose value on a set S is known as *the free nilpotent group of class two generated by S* and is denoted by $\mathbf{Z}_{\mathsf{Nil}}[S]$. If F_S is the free group spanned by S, then $\mathbf{Z}_{\mathsf{Nil}}[S] = (F_S)^{\mathrm{nil}}$. Moreover, since left adjoints preserve coproducts, and S is the coproduct of S copies of a singleton in **Sets**, one has

$$\mathbf{Z}_{\mathsf{Nil}}[S] = \bigvee_S \mathbf{Z}$$

in **Nil**. Using 2.1.2, one obtains the following particular case of the famous result of Witt, which asserts that the graded Lie ring obtained by the lower central series of a free group is a free Lie ring.

Corollary 2.1.3 ([9, Corollary])**.** *For a free nil*$_2$*-group G, one has a central extension of the groups*

$$0 \to \Lambda^2(G^{\mathrm{ab}}) \xrightarrow{\mu(G)} G \to G^{\mathrm{ab}} \to 0.$$

2.2. **Quadratic maps.** We recall some of our results proved in [9].

Let G and H be arbitrary groups. We call a map $f : G \to H$ *weakly quadratic* if for any $a, b \in G$, the *cross-effect*

$$(a \mid b)_f := -(f(a) + f(b)) + f(a + b)$$

commutes with $f(c)$ for all $c \in G$ and is linear in a and b. Thus we have

$$f(a + b) = f(a) + f(b) + (a \mid b)_f,$$

and the equalities

$$(a_1 + a_2 \mid b)_f = (a_1 \mid b)_f + (a_2 \mid b)_f,$$
$$(a \mid b_1 + b_2)_f = (a \mid b_1)_f + (a \mid b_2)_f$$

hold for any $a, a_1, a_2, b_1, b_2, b \in G$.

A weakly quadratic map $f : G \to H$ is *quadratic* if in fact $(a \mid b)_f \in \mathsf{Z}(H)$ for all $a, b \in G$.

If G and H are abelian groups, this definition coincides with the one given in [3]. In fact, obviously, every weakly quadratic map to an abelian group is quadratic. We denote by $\mathsf{wQuad}(G, H)$ the set of all weakly quadratic maps from G to H and by $\mathsf{Quad}(G, H)$ that of quadratic maps. It is clear that a map $f : G \to H$ is a homomorphism iff $(- \mid -)_f = 0$. Thus

$$\mathsf{Hom}(G, H) \subseteq \mathsf{Quad}(G, H) \subseteq \mathsf{wQuad}(G, H).$$

Lemma 2.2.1 ([9, Lemma 2])**.** *For $f \in \mathsf{wQuad}(G, H)$, the following assertions are true:*

i) *The cross-effect yields a well-defined homomorphism $(- \mid -)_f : G^{\mathrm{ab}} \otimes G^{\mathrm{ab}} \to H$.*
ii) $f(0) = 0$.
iii) $f(-a) = -f(a) + (a \mid a)_f$.
iv) *If $c \in [G, G]$, then for any $a \in G$, one has $f(a+c) = f(a)+f(c)$. In particular, the restriction of f to the commutator subgroup is a homomorphism.*
v) *For any $a, b \in G$, one has*

$$f([a, b]) = -f(b + a) + f(a + b) = [f(a), f(b)] + (a \mid b)_f - (b \mid a)_f.$$

vi) *For any $a, b, c \in G$, one has $f([a, [b, c]]) = [f(a), [f(b), f(c)]]$.*

We will also need the following observation.

Lemma 2.2.2. *For a weakly quadratic map $f : G \to H$ and a subgroup $H' \subseteq H$, the following assertions are equivalent:*

- *$f^{-1}(H')$ is a subgroup of G;*
- *$(a \mid b)_f \in H'$ for all $a, b \in f^{-1}(H')$.*

Moreover, in these circumstances, if H' contains $[H, H]$, then $f^{-1}(H')$ contains $[G, G]$.

Proof. The first statement is immediate in view of $f(a + b) = f(a) + f(b) + (a \mid b)_f$. The second follows from v) of 2.2.1. □

The set of quadratic maps for nilpotent groups of class two has some remarkable properties. First of all, unlike $\mathsf{Hom}(G, H)$ or $\mathsf{wQuad}(G, H)$ the set $\mathsf{Quad}(G, H)$ is a group with respect to the pointwise addition of maps. This is the subject of the following Lemma.

Lemma 2.2.3 ([9, Lemma 3])**.** *Let G be a group and let H be a nilpotent group of class two. If the maps $f, g : G \to H$ are quadratic, then $f + g$ and $-f$ are also quadratic and*

$$(a \mid b)_{f+g} = (a \mid b)_f + (a \mid b)_g + [f(b), g(a)],$$
$$(a \mid b)_{-f} = [f(b), f(a)] - (a \mid b)_f.$$

Note that the above proof also shows that the map

$$\mathsf{Quad}(G, H) \xrightarrow{f \mapsto (-|-)_f} \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, \mathsf{Z}(H))$$

is quadratic, too.

Lemma 2.2.4 ([9, Lemma 5]). *For any groups $(G_i)_{i\in I}$ and H, one has the natural bijections*

$$\mathsf{Quad}\Big(H, \prod_i G_i\Big) \approx \prod_i \mathsf{Quad}(H, G_i).$$

If, moreover, $H \in \mathbf{Nil}$, then there is a central extension

$$0 \to \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H)) \xrightarrow{\alpha} \mathsf{Quad}(G_1 \times G_2, H) \to \mathsf{Quad}(G_1, H) \times \mathsf{Quad}(G_2, H) \to 0,$$

where $(\alpha(\xi))(g_1, g_2) = \xi(\hat{g}_1, \hat{g}_2)$ for $\xi \in \mathsf{Hom}(G_1^{\mathrm{ab}} \otimes G_2^{\mathrm{ab}}, \mathsf{Z}(H))$ and $g_k \in G_k$, $k = 1, 2$.

In the rest of the paper, we assume that all groups under consideration are nilpotent of class two.

Lemma 2.2.5 ([9, Lemma 6]). *Let $f : G \to H$ be a weakly quadratic map. For any homomorphism $h : G_1 \to G$, the composite $fh : G_1 \to H$ is also weakly quadratic and*

$$(a \mid b)_{fh} = (h(a) \mid h(b))_f, \quad a, b \in G_1;$$

moreover, if f is quadratic, then so is fh.

For any homomorphism $g : H \to H_1$, the composite $gf : G \to H_1$ is also weakly quadratic and

$$(a \mid b)_{gf} = g((a \mid b)_f).$$

If, moreover, f is quadratic, then gf will be quadratic provided g carries central elements to central elements.

Thus for any $N \in \mathbf{Nil}$, one obtains the functors

$$\mathsf{wQuad}(-, N) : \mathbf{Nil}^{\mathrm{op}} \to \mathbf{Sets},$$
$$\mathsf{Quad}(-, N) : \mathbf{Nil}^{\mathrm{op}} \to \mathbf{Nil},$$
$$\mathsf{wQuad}(N, -) : \mathbf{Nil} \to \mathbf{Sets}.$$

However, the mapping $\mathsf{Quad}(N, -)$ is NOT functorial.

Examples 2.2.6. i) For a fixed $n \in \mathbf{Z}$, consider the map $n : G \to G$ given by $a \mapsto na$. Then

$$(a \mid b)_n = -\frac{n(n-1)}{2}[a, b].$$

Thus $n \in \mathsf{Quad}(G, G)$.

ii) Let $+ : G \times G \to G$ be the map given by $(a, b) \mapsto a + b$. Then

$$((a, b) \mid (c, d))_+ = [c, b].$$

In particular, $+ \in \mathsf{Quad}(G \times G, G)$.

iii) For any elements $a \in G$ and $b \in \mathsf{Z}(G)$, we put

$$f_{a,b}(n) = na + \frac{n(n-1)}{2}b.$$

The map $f_{a,b} : \mathbf{Z} \to G$ is a quadratic map with $(n \mid m)_{f_{a,b}} = nmb$ for any $n, m \in \mathbf{Z}$. Owing to [9], any quadratic map $f : \mathbf{Z} \to G$ is of this form.

2.3. **The category Niq and q-maps.**

Definition 2.3.1. A weakly quadratic map $f : G \to H$ between nil$_2$-groups is a *q-map* (see [9, Definition 1]) if $(a \mid b)_f \in [H, H]$ for all $a, b \in G$. We denote by $\mathsf{Q}(G, H)$ the collection of all q-maps from G to H, so that

$$\mathsf{Hom}(G, H) \subseteq \mathsf{Q}(G, H) \subseteq \mathsf{Quad}(G, H).$$

In other words, $\mathsf{Q}(G, H)$ is determined by the pullback square

$$\begin{array}{ccc}
\mathsf{Q}(G, H) & \longrightarrow & \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, [H, H]) \\
\big\downarrow & & \big\downarrow \\
\mathsf{Quad}(G, H) & \xrightarrow{f \mapsto (-|-)_f} & \mathsf{Hom}(G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}, \mathsf{Z}(H)),
\end{array}$$

where the right vertical map is induced by the inclusion $[H,H] \rightarrowtail \mathsf{Z}(H)$.

Actually, by 2.2.2 and 2.2.3 the set $\mathsf{Q}(G,H)$ is a normal subgroup of $\mathsf{Quad}(G,H)$ [9, Lemma 8]. In particular, we see that any linear combination of homomorphisms is a q-map.

Obviously also, a weakly quadratic map is in $\mathsf{Q}(G,H)$ iff its composite with $H \twoheadrightarrow H^{\mathrm{ab}}$ is a homomorphism.

Moreover, one has also an embedding

$$\mathsf{Quad}(G,[H,H]) \subset \mathsf{Q}(G,H)$$

as a central subgroup.

Lemma 2.3.2. *For an abelian group H, one has*

$$\mathsf{Q}(G,H) = \mathsf{Hom}(G,H)$$

for any $G \in \mathbf{Nil}$.

Proof. Since $[H,H]=0$, a map $f : G \to H$ is a q-map iff $(- \mid -)_f = 0$. □

Examples 2.3.3. The first two quadratic maps considered in Examples 2.2.6 are actually q-maps. Also, the map

$$\delta = i_1 + i_2 : G \to G \vee G$$

is a q-map, with $(x \mid y)_\delta = [i_1(y), i_2(x)]$.

On the other hand, the quadratic map $f_{a,b} : \mathbf{Z} \to G$, associated to the elements $a \in G$ and $b \in \mathsf{Z}(G)$ as in iii) of 2.2.6, is a q-map iff $b \in [G,G]$. Thus for any $G \in \mathbf{Nil}$, one has the following central extension:

$$0 \to [G,G] \to \mathsf{Q}(\mathbf{Z},G) \xrightarrow{\mathsf{ev}(1)} G \to 0.$$

A 2-cocycle $G \times G \to [G,G]$ corresponding to this central extension is given by the commutator map.

Lemma 2.3.4 ([9, Proposition 3])**.** *For q-maps $f : G \to H$ and $g : G_1 \to G$, the cross-effect of their composite is given by*

$$(a \mid b)_{fg} = f((a \mid b)_g) + (g(a) \mid g(b))_f, \quad a, b \in G_1.$$

It follows that the composite of q-maps is a q-map. Hence there is a well-defined category **Niq** whose objects are nil_2-groups and morphisms are all q-maps between them. The hom-sets

$$\mathsf{Hom}_{\mathbf{Niq}}(G,H) = \mathsf{Q}(G,H)$$

are equipped with structures of nilpotent groups of class two. **Nil** is a subcategory of **Niq**, with the same objects. The hom-functor of **Niq** (with values in sets) gives rise to a well-defined bifunctor

$$\mathsf{Q}(-,-) : \mathbf{Nil}^{\mathrm{op}} \times \mathbf{Nil} \to \mathbf{Nil}.$$

Moreover, the functors $[-,-], (-)^{\mathrm{ab}} : \mathbf{Nil} \to \mathbf{Ab}$ extend along the inclusion $\mathbf{Nil} \hookrightarrow \mathbf{Niq}$ to the functors on **Niq**.

Composition in **Niq** is left distributive,

$$(f + f')g = fg + f'g,$$

but not right distributive; rather it is right quadratic in the sense that one has

$$f(g+g') = fg + fg' + (g \mid g')_f, \quad f \in \mathsf{Q}(G,H), \quad g, g' \in \mathsf{Q}(G_1,G),$$

where $(g \mid g')_f$ lies in the center of $\mathsf{Q}(G_1,H)$, is bilinear in g, g' and quadratic in f; more precisely, one has

$$(g \mid g')_{f+f'}(x) = (g \mid g')_f(x) + (g \mid g')_{f'}(x) + [fg'(x), f'g(x)].$$

The category **Niq** possesses all products and both the inclusion $\mathbf{Nil} \hookrightarrow \mathbf{Niq}$ and the forgetful functor $\mathbf{Niq} \to \mathbf{Sets}$ respect products.

Every object in **Niq** has a canonical internal group structure. However, a morphism in **Niq** is compatible with the corresponding internal group structures iff it is lies in **Nil**, i. e., is a homomorphism.

It should be noted that bijective morphisms in **Niq** need not be isomorphisms in **Niq** (see [9, Example 1.5.9]).

Proposition 2.3.5. *For any nil$_2$-groups G_1, G, H, the composite*

$$\circ_H^{G_1,G} : \mathsf{Q}(G_1,G) \to \mathsf{Hom}_{\mathbf{Nil}}(\mathsf{Q}(G,H),\mathsf{Q}(G_1,H)) \hookrightarrow \mathsf{Q}(\mathsf{Q}(G,H),\mathsf{Q}(G_1,H))$$

sending $g : G_1 \to G$ to the map given by $f \mapsto fg$, is a quadratic map. Moreover, the image of the map sending $f : G \to H$ to the map given by $g \mapsto fg$ defines a homomorphism

$$\circ_{G,H}^{G_1} : \mathsf{Q}(G,H) \to \mathsf{Quad}(\mathsf{Q}(G_1,G),\mathsf{Q}(G_1,H)).$$

Proof. We have already seen that for any q-map g (in fact, for any map whatsoever), $\circ_H^{G_1,G}(g)$ is a homomorphism. To see that this homomorphism varies quadratically in g, we calculate

$$(g \mid g')_{\circ_H^{G_1,G}}(f)(x) = (-(fg + fg') + f(g+g'))(x) = (g(x) \mid g'(x))_f,$$

which is obviously linear in g and g'. Moreover, $f \mapsto (g(-) \mid g'(-))_f$ is, obviously, a central element of $\mathsf{Q}(\mathsf{Q}(G,H),\mathsf{Q}(G_1,H))$.

For the second map we first have to show that for any fixed $f \in \mathsf{Q}(G,H)$, the map

$$\circ_{G,H}^{G_1}(f) : \mathsf{Q}(G_1,G) \to \mathsf{Q}(G_1,H)$$

is quadratic. To this end, one may have that assigning to $g, g' : G_1 \to G$ the q-map given by

$$x \mapsto -(fg(x) + fg'(x)) + f(g(x) + g'(x)) = (g(x) \mid g'(x))_f$$

is linear in g and g'—which is essentially the same statement as before; the centrality is also clear.

Finally, the fact that $\circ_{G,H}^{G_1}$ is a homomorphism just means that $(f+f')g = fg + f'g$. □

3. Stable Quadratic Modules

3.1. The category of stable quadratic modules.

Definition 3.1.1 ([1, 2])**.** A *stable quadratic module*, shortly sqm, consists of the following data of groups and homomorphisms:

$$C = \left(C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} \xrightarrow{\{-,-\}} C_{\mathrm{r}} \xrightarrow{\partial} C_{\mathrm{b}} \right),$$

where $C_{\mathrm{b}} \in \mathbf{Nil}$, and for any $c, c' \in C_{\mathrm{b}}$ and $r, r' \in C_{\mathrm{r}}$, the following identities hold:

$$\begin{aligned} \partial\{\hat{c}, \hat{c}'\} &= [c, c'], \\ \left\{\widehat{\partial r}, \widehat{\partial r'}\right\} &= [r, r'], \\ \{\hat{c}, \hat{c}'\} + \{\hat{c}', \hat{c}\} &= 0. \end{aligned} \tag{1}$$

Thus the following diagram

$$\begin{array}{ccc} C_{\mathrm{r}}^{\mathrm{ab}} \otimes C_{\mathrm{r}}^{\mathrm{ab}} & \xrightarrow{[-,-]} & C_{\mathrm{r}} \\ {\scriptstyle \partial\otimes\partial}\downarrow & \nearrow{\scriptstyle \{-,-\}} & \downarrow{\scriptstyle \partial} \\ C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow[{[-,-]}]{} & C_{\mathrm{b}} \end{array}$$

commutes. To simplify the notation, we mostly write $\{c, c'\}$ instead of $\{\hat{c}, \hat{c}'\}$. Moreover, for a sqm C we denote

$$C_{\mathrm{c}} := \mathsf{Im}\left(C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} \xrightarrow{\{-,-\}} C_{\mathrm{r}} \right).$$

A morphism of stable quadratic modules, called also a *strict map*, is a pair $f = (C_{\mathrm{b}} \xrightarrow{f_{\mathrm{b}}} D_{\mathrm{b}}, C_{\mathrm{r}} \xrightarrow{f_{\mathrm{r}}} D_{\mathrm{r}})$ of group homomorphisms making the diagram

$$\begin{array}{ccccc} C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & C_{\mathrm{r}} & \xrightarrow{\partial} & C_{\mathrm{b}} \\ {\scriptstyle f_{\mathrm{b}}\otimes f_{\mathrm{b}}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{r}}} & & \downarrow{\scriptstyle f_{\mathrm{b}}} \\ D_{\mathrm{b}}^{\mathrm{ab}} \otimes D_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & D_{\mathrm{r}} & \xrightarrow{\partial} & D_{\mathrm{b}} \end{array}$$

commute. With this notion of a morphism, the stable quadratic modules form a category which we denote by **SQM**.

Lemma 3.1.2. *For any sqm C, the subgroup $C_{\mathrm{c}} \subseteq C_{\mathrm{r}}$ is central; moreover. $[C_{\mathrm{r}}, C_{\mathrm{r}}] \subseteq C_{\mathrm{c}}$ and $\partial C_{\mathrm{c}} = [C_{\mathrm{b}}, C_{\mathrm{b}}]$.*

Proof. For any $c, c' \in C_{\mathrm{b}}$ and $r \in C_{\mathrm{r}}$, one has

$$[r, \{c, c'\}] = \left\{\widehat{\partial r}, \partial \widehat{\{c, c'\}}\right\} = \left\{\widehat{\partial r}, \widehat{[c, c']}\right\}.$$

But by the definition, $\widehat{[c, c']} = 0$ in $C_{\mathrm{b}}^{\mathrm{ab}}$, so the last term is zero, i. e., C_{c} is indeed central in C_{r}. The remaining statements follow immediately from (1) in 3.1.1. □

Corollary 3.1.3. *For any sqm C, the group C_{r} belongs to* **Nil**.

It follows moreover that in a sqm C the group $\mathsf{Im}(\partial)$ is a normal subgroup of C_{b}, whereas $\mathsf{Ker}(\partial)$ and $\mathsf{Coker}(\partial)$ are abelian. We define

$$\pi_0^C := \mathsf{Coker}(\partial), \quad \pi_1^C := \mathsf{Ker}(\partial).$$

Furthermore, we have a well-defined map

$$k^C : \pi_0^C \to \pi_1^C$$

given by $k^C(x) = \{x, x\}$. Note that, in fact, by the last identity in (1), one has

$$k^C \in {}_2\,\mathsf{Hom}(\pi_0^C, \pi_1^C) \left(\cong \mathsf{Hom}(\pi_0^C \otimes \mathbf{Z}/2\mathbf{Z}, \pi_1^C) \cong \mathsf{Hom}(\pi_0^C, {}_2\pi_1^C)\right).$$

An sqm C with $k^C = 0$ will be called *split*. The subset $\mathrm{Spl}(C)_{\mathrm{b}} = \{x \in C_{\mathrm{b}} \mid \{x, x\} = 0\}$ is easily seen to be a normal subgroup containing $\mathsf{Im}(\partial)$, so that each sqm C has a largest split subobject $\mathrm{Spl}(C)$, with $\mathrm{Spl}(C)_{\mathrm{r}} = C_{\mathrm{r}}$.

A morphism of sqm's is called a *weak equivalence* if it induces an isomorphism on π_i, $i = 0, 1$. Let **Ho**(**SQM**) denote the localization of **SQM** with respect to weak equivalences. It is well known that the category **Ho**(**SQM**) is equivalent to the full subcategory of the homotopy category of spectra with the objects of those X with $\pi_i X = 0$, $i \neq 0, 1$ (see, e. g., [1]*Appendix C).

Here, there are some important examples of sqm's.

Examples 3.1.4. i) For any group N from **Nil**, we denote by $\mathbb{L}(N)$ the sqm with

$$\mathbb{L}(N)_{\mathrm{b}} = N = \mathbb{L}(N)_{\mathrm{r}}, \quad \partial = \mathsf{Id}_N, \quad \{a, b\} = [a, b].$$

It is clear that $\mathbb{L}(N)$ is weakly equivalent to 0. Here, obviously, $0 := \mathbb{L}(0)$ is a zero object for **SQM**. In fact, one sees easily that for a sqm C, the conditions

a) C is isomorphic to $\mathbb{L}(N)$ for some N;
b) C is weakly equivalent to 0;
c) $\partial : C_{\mathrm{r}} \to C_{\mathrm{b}}$ is an isomorphism

are equivalent. ii) For a nil$_2$-group N and its subgroup N' with $[N, N] \subseteq N'$, we denote by $\mathbb{K}(N, N')$ the sqm with

$$\mathbb{K}(N, N')_{\mathrm{b}} = N, \quad \mathbb{K}(N, N')_{\mathrm{r}} = N', \quad \partial = \text{inclusion}, \quad \{a, b\} = [a, b],$$

so that, in particular, $\mathbb{L}(N) = \mathbb{K}(N, N)$. We also denote $\mathbb{A}(N) = \mathbb{K}(N, [N, N])$.

It is easy to see that the canonical map $\mathbb{K}(N, N') \to \mathbb{K}(N/N', \{0\})$ is a weak equivalence. In fact, the following conditions are equivalent:

a) C is isomorphic to $\mathbb{K}(N, N')$ for some $[N, N] \subseteq N'$;
b) C is weakly equivalent to $\mathbb{K}(A, 0)$ for some abelian group A;
c) $\partial : C_{\mathrm{r}} \to C_{\mathrm{b}}$ is a monomorphism.

iii) For a nil$_2$-group N and its central subgroup A, we denote by $\mathbb{K}(N, N/A)$ the sqm with

$$\mathbb{K}(N, N/A)_{\mathrm{b}} = N/A, \quad \mathbb{K}(N, N/A)_{\mathrm{r}} = N, \quad \partial = \text{projection}, \quad \{a + A, b + A\} = [a, b],$$

so that, in particular, $\mathbb{L}(N) = \mathbb{K}(N, N/\{0\})$. We also denote $\mathbb{B}(N) = \mathbb{K}(N, N/\mathsf{Z}(N))$.

Note that sqm's C, isomorphic to ones of the form $\mathbb{K}(A,1) := \mathbb{K}(A, A/A)$, are precisely those with $C_{\mathrm{b}} = 0$. Also, it can be easily seen that there is a canonical weak equivalence $\mathbb{K}(A,1) \to \mathbb{K}(N, N/A)$. In fact, the following conditions are equivalent:

a) C is isomorphic to $\mathbb{K}(N, N/A)$ for some $A \subseteq \mathsf{Z}(N)$;
b) C is weakly equivalent to $\mathbb{K}(A,1)$ for some abelian group A;
c) $\partial : C_{\mathrm{r}} \to C_{\mathrm{b}}$ is onto.

iv) There is a "universal way to turn a nil$_2$-group into an sqm". For a nil$_2$-group N, we denote by $\mathbb{S}[N]$ the following sqm:

$$\mathbb{S}[N]_{\mathrm{b}} = N, \quad \mathbb{S}[N]_{\mathrm{r}} = \tilde{\Lambda}^2 N^{\mathrm{ab}}, \quad \partial(\hat{a}\tilde{\wedge}\hat{b}) = [a,b], \quad \{a,b\} = \hat{a}\tilde{\wedge}\hat{b}.$$

Here, for an abelian group A,

$$\tilde{\Lambda}^2 A = A \otimes A/(b \otimes a \sim -a \otimes b)$$

is the antisymmetrization of the tensor square of A, with the class of $a \otimes b$ denoted by $a\tilde{\wedge}b$.

Of particular importance is $\mathbb{S} = \mathbb{S}[\mathbf{Z}]$; it looks like

$$\mathbb{S} = \left(\mathbf{Z} \otimes \mathbf{Z} \xrightarrow{\{-,-\}} \mathbf{Z}/2\mathbf{Z} \xrightarrow{\partial=0} \mathbf{Z} \right).$$

Lemma 3.1.5. *For any $C \in \mathsf{SQM}$, one has the natural bijections*

$$\mathsf{Hom}_{\mathsf{SQM}}(\mathbb{L}(N), C) \approx \mathsf{Hom}_{\mathsf{Nil}}(N, C_{\mathrm{r}}),$$
$$\mathsf{Hom}_{\mathsf{SQM}}(C, \mathbb{L}(N)) \approx \mathsf{Hom}_{\mathsf{Nil}}(C_{\mathrm{b}}, N),$$
$$\mathsf{Hom}_{\mathsf{SQM}}(\mathbb{S}[N], C) \approx \mathsf{Hom}_{\mathsf{Nil}}(N, C_{\mathrm{b}}).$$

In particular, there are adjunctions

$$\mathbb{S}[-] \dashv (-)_{\mathrm{b}} \dashv \mathbb{L} \dashv (-)_{\mathrm{r}},$$

that is, in the above sequence, each functor is left adjoint to the next one.

Moreover, one has

$$\mathsf{Hom}_{\mathsf{SQM}}(\mathbb{A}(N), C) \approx \mathsf{Hom}_{\mathsf{Nil}}(N, \mathrm{Spl}(C)_{\mathrm{b}}),$$
$$\mathsf{Hom}_{\mathsf{SQM}}(C, \mathbb{A}(N)) \approx \{\ f \in \mathsf{Hom}_{\mathsf{Nil}}(C_{\mathrm{b}}, N) \ \mid\ \mathsf{Im}(f\partial) \subseteq [N,N]\ \},$$

so that, in particular, $\mathsf{Hom}_{\mathsf{Nil}}(N, N') \approx \mathsf{Hom}_{\mathsf{SQM}}(\mathbb{A}(N), \mathbb{A}(N'))$, *and*

$$\mathsf{Hom}_{\mathsf{SQM}}(\mathbb{K}(N, N/A), C) \approx \left\{\ f \in \mathsf{Hom}_{\mathsf{Nil}}(N, C_{\mathrm{r}}) \ \mid\ f(A) \subseteq \pi_1^C\ \right\},$$
$$\mathsf{Hom}_{\mathsf{SQM}}(C, \mathbb{K}(A,1)) \approx \mathsf{Hom}(C_{\mathrm{r}}/\mathsf{Im}(\{-,-\}), A).$$

Proof. For $\mathsf{Hom}_{\mathsf{SQM}}(\mathbb{L}(N), C)$, consider the diagram

$$\begin{array}{ccccc} N^{\mathrm{ab}} \otimes N^{\mathrm{ab}} & \xrightarrow{[-,-]} & N & = & N \\ \big\downarrow{\scriptstyle \partial f^{\mathrm{ab}} \otimes \partial f^{\mathrm{ab}}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \partial f} \\ C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & C_{\mathrm{r}} & \xrightarrow{\partial} & C_{\mathrm{b}}; \end{array}$$

one can easily see that it commutes for any given homomorphism f.

For $\mathsf{Hom}_{\mathsf{SQM}}(C, \mathbb{L}(N))$, consider the diagram

$$\begin{array}{ccccc} C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & C_{\mathrm{r}} & \xrightarrow{\partial} & C_{\mathrm{b}} \\ \big\downarrow{\scriptstyle f\partial^{\mathrm{ab}} \otimes f\partial^{\mathrm{ab}}} & & \big\downarrow{\scriptstyle f\partial} & & \big\downarrow{\scriptstyle f} \\ N^{\mathrm{ab}} \otimes N^{\mathrm{ab}} & \xrightarrow{[-,-]} & N & = & N; \end{array}$$

again, it commutes for any homomorphism $f : C_{\mathrm{b}} \to N$.

For $\mathsf{Hom}_{\mathsf{SQM}}(\mathbb{S}[N], C)$, it is easy to see that for any given homomorphism $f : N \to C_{\mathrm{b}}$, there is a unique f_{r} making the diagram

$$\begin{array}{ccccc} N^{\mathrm{ab}} \otimes N^{\mathrm{ab}} & \twoheadrightarrow & \tilde{\Lambda}^2 N^{\mathrm{ab}} & \xrightarrow{[-,-]} & N \\ \downarrow{\scriptstyle f^{\mathrm{ab}} \otimes f^{\mathrm{ab}}} & & \downarrow{\scriptstyle f_{\mathrm{r}}} & & \downarrow{\scriptstyle f} \\ C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & C_{\mathrm{r}} & \xrightarrow{\partial} & C_{\mathrm{b}} \end{array}$$

which commutes, namely, the one given by $f_{\mathrm{r}}(\hat{a}\tilde{\wedge}\hat{b}) = \{f(a), f(b)\}$.

The remaining proofs are similar. $\square$

3.2. **Limits in SQM.** Since by 3.1.5 the functors $(-)_{\mathrm{b}}$ and $(-)_{\mathrm{r}}$ are both representable, a standard argument shows that any limits that might exist in **SQM** must be computed "componentwise". It is then easy to see that defining them in this way indeed works. In particular, the product of a family (C_i) of sqm's is given by

$$\prod_i C_i = \left(\Big(\prod_i (C_{i\mathrm{b}}) \Big)^{\mathrm{ab}} \otimes \Big(\prod_i (C_{i\mathrm{b}}) \Big)^{\mathrm{ab}} \xrightarrow{\prod_i \{-,-\}_i} \prod_i (C_{i\mathrm{r}}) \xrightarrow{\prod_i \partial_i} \prod_i (C_{i\mathrm{b}}) \right),$$

and the kernel of a morphism $f : C \to D$ is given by

$$\mathsf{Ker}(f) = \left(\mathsf{Ker}(f_{\mathrm{b}})^{\mathrm{ab}} \otimes \mathsf{Ker}(f_{\mathrm{b}})^{\mathrm{ab}} \xrightarrow{\{-,-\}} \mathsf{Ker}(f_{\mathrm{r}}) \xrightarrow{\partial} \mathsf{Ker}(f_{\mathrm{b}}) \right).$$

Similarly, since by 3.1.5 the functor $\mathsf{Hom}_{\mathsf{Nil}}((-)_{\mathrm{b}}, -)$ is representable, any colimit of a diagram in **SQM** must necessarily have its b-part equal to the colimit in **Nil** of the b-parts of that diagram. This is however not anymore true for the r-parts, as we will now see.

3.3. **Coproduct in SQM.** To construct coproducts in **SQM**, denote, for sqm's C and C', by $(C \sqcup C')_{\mathrm{r}}$ the set $C_{\mathrm{b}}^{\mathrm{ab}} \otimes {C'}_{\mathrm{b}}^{\mathrm{ab}} \times C_{\mathrm{r}} \times {C'}_{\mathrm{r}}$ with the group structure determined by

$$(\xi, r, r') + (\eta, s, s') = (\xi + \eta - \partial s \otimes \partial r', r + s, r' + s').$$

Since $(r, r') \mapsto -\partial r \otimes \partial r'$ is a cocycle, this indeed gives a group fitting in a central extension

$$0 \to C_{\mathrm{b}}^{\mathrm{ab}} \otimes {C'}_{\mathrm{b}}^{\mathrm{ab}} \to (C \sqcup C')_{\mathrm{r}} \to C_{\mathrm{r}} \times {C'}_{\mathrm{r}} \to 0.$$

We define the homomorphism $\partial : (C \sqcup C')_{\mathrm{r}} \to C_{\mathrm{b}} \vee {C'}_{\mathrm{b}}$ by the equality

$$\partial(\xi, r, r') = [-,-](\xi) + \partial r + \partial r'$$

and the homomorphism $\{-,-\} : (C_{\mathrm{b}} \vee {C'}_{\mathrm{b}})^{\mathrm{ab}} \otimes (C_{\mathrm{b}} \vee {C'}_{\mathrm{b}})^{\mathrm{ab}} \cong (C_{\mathrm{b}}^{\mathrm{ab}} \oplus {C'}_{\mathrm{b}}^{\mathrm{ab}}) \otimes (C_{\mathrm{b}}^{\mathrm{ab}} \oplus {C'}_{\mathrm{b}}^{\mathrm{ab}}) \to (C \sqcup C')_{\mathrm{r}}$ by the equality

$$\{a, b\} = \begin{cases} (0, \{a, b\}, 0), & a \otimes b \in C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}}, \\ (0, 0, \{a, b\}), & a \otimes b \in {C'}_{\mathrm{b}}^{\mathrm{ab}} \otimes {C'}_{\mathrm{b}}^{\mathrm{ab}}, \\ (a \otimes b, 0, 0), & a \otimes b \in C_{\mathrm{b}}^{\mathrm{ab}} \otimes {C'}_{\mathrm{b}}^{\mathrm{ab}}, \\ (-b \otimes a, 0, 0), & a \otimes b \in {C'}_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}}. \end{cases}$$

It is then straightforward to check that this defines a sqm $C \sqcup C'$ with $(C \sqcup C')_{\mathrm{b}} = C_{\mathrm{b}} \vee {C'}_{\mathrm{b}}$ and with morphisms $i_C : C \to C \sqcup C'$, $i_{C'} : C' \to C \sqcup C'$ given by $(i_C)_{\mathrm{b}} = i_{C_{\mathrm{b}}}$, $(i_C)_{\mathrm{r}}(r) = (0, r, 0)$ and, similarly, for $i_{C'}$. One then has

Proposition 3.3.1. *For any C, C', the sqm $C \sqcup C'$ with the morphisms i_C, $i_{C'}$ is the coproduct of C and C' in* **SQM**.

Proof. Given morphisms $f : C \to D$, $f' : C' \to D$, we define $(f, f') : C \sqcup C' \to D$ by the equalities

$$(f, f')_{\mathrm{b}} = (f_{\mathrm{b}}, {f'}_{\mathrm{b}}) : C_{\mathrm{b}} \vee {C'}_{\mathrm{b}} \to D_{\mathrm{b}}$$

and

$$(f, f')_{\mathrm{r}}(\xi, r, r') = \{f_{\mathrm{b}}(-), {f'}_{\mathrm{b}}(-)\}(\xi) + f_{\mathrm{r}}(r) + {f'}_{\mathrm{r}}(r').$$

This is then the unique extension of f and f' along i_C and $i_{C'}$, in view of the equality

$$(\hat{c} \otimes \hat{c}', 0, 0) = \{(i_C)_{\mathrm{b}}(c), (i_{C'})_{\mathrm{b}}(c')\}. \qquad \square$$

In particular, we obtain immediately

Corollary 3.3.2. *The coproduct in* $\mathbf{SQM}$ *fits in a central extension of the form*

$$0 \to \mathbb{L}(C_{\mathrm{b}}^{\mathrm{ab}} \otimes D_{\mathrm{b}}^{\mathrm{ab}}) \to C \sqcup D \to C \times D \to 0.$$

4. 2-Linear Maps and the Category $\mathbf{SQM}_2$

The goal of this section is to contruct a category $\mathbf{SQM}_2$, which will be related to the category $\mathbf{SQM}$ as $\mathbf{Niq}$ relates to $\mathbf{Nil}$. We start with analogues of q-maps for SQM's.

4.1. 2-linear maps and the group $\mathbb{H}\mathrm{om}_{\mathrm{b}}(C, D)$.

Definition 4.1.1. Let C and D be stable quadratic modules. A *2-linear map* $\boldsymbol{f} : C \to D$ is a triple $\boldsymbol{f} = (f_{\mathrm{b}}, f_{\mathrm{r}}, f_{\mathrm{c}})$, with the maps $f_{\mathrm{b}} : C_{\mathrm{b}} \to D_{\mathrm{b}}$, $f_{\mathrm{r}} : C_{\mathrm{r}} \to D_{\mathrm{r}}$ and a homomorphism

$$f_{\mathrm{c}} : C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} \to D_{\mathrm{c}}, \quad \hat{c} \otimes \hat{c}' \mapsto f_{\mathrm{c}}(\hat{c}, \hat{c}')$$

such that the following conditions (for simplicity we write $f(c.c')$ instead $f_{\mathrm{c}}(\hat{c}, \hat{c}')$) hold:

i) $\partial f_{\mathrm{r}}(r) = f_{\mathrm{b}}\partial(r)$,
ii) $\partial f_{\mathrm{c}}(c, c') = (c \mid c')_{f_{\mathrm{b}}}$,
iii) $f_{\mathrm{c}}(\partial^{\mathrm{ab}}(r), \partial^{\mathrm{ab}}(r')) = (r \mid r')_{f_{\mathrm{r}}}$,
iv) $f_{\mathrm{r}}(\{c, c'\}) = f_{\mathrm{c}}(c, c') - f_{\mathrm{c}}(c', c) + \{f_{\mathrm{b}}c, f_{\mathrm{b}}c'\}$

for any $r, r' \in C_{\mathrm{r}}$ and $c, c' \in C_{\mathrm{b}}$. In particular, it follows from iii) that f_{r} is quadratic and from ii) that f_{b} is a q-map. Indeed, recall from 3.1.2 that $D_{\mathrm{c}} = \mathsf{Im}\left(\{-,-\} : D_{\mathrm{b}}^{\mathrm{ab}} \otimes D_{\mathrm{b}}^{\mathrm{ab}} \to D_{\mathrm{r}}\right)$ is central in D_{r} and one has $\partial D_{\mathrm{c}} = [D_{\mathrm{b}}, D_{\mathrm{b}}]$.

The set of all 2-linear maps from C to D will be denoted by $\mathbb{H}\mathrm{om}_{\mathrm{b}}(C, D)$.

We also need a weaker notion of *quadratic map* between sqm's. This is a triple $\boldsymbol{f} = (f_{\mathrm{b}}, f_{\mathrm{r}}, f_{\mathrm{c}})$ as above, where it is required only that f_{c} is a homomorphism to $\mathsf{Z}(D_{\mathrm{r}})$, not necessarily to $D_{\mathrm{c}} \subseteq \mathsf{Z}(D_{\mathrm{r}})$; moreover, it is still required that conditions i)-iv) above are satisfied and that f_{b} is quadratic. It then follows that f_{r} is quadratic, too.

The set of all quadratic maps from C to D will be denoted by $\mathbb{Q}\mathrm{uad}(C, D)$.

Thus the first three conditions assert that the following diagrams:

$$\begin{array}{ccc} C_{\mathrm{r}} & \xrightarrow{f_{\mathrm{r}}} & D_{\mathrm{r}} \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ C_{\mathrm{b}} & \xrightarrow{f_{\mathrm{b}}} & D_{\mathrm{b}} \end{array} \qquad\qquad \begin{array}{ccc} C_{\mathrm{r}}^{\mathrm{ab}} \otimes C_{\mathrm{r}}^{\mathrm{ab}} & \xrightarrow{(-|-)_{f_{\mathrm{r}}}} & D_{\mathrm{r}} \\ \downarrow{\scriptstyle\partial\otimes\partial} & \nearrow{\scriptstyle f_{\mathrm{c}}} & \downarrow{\scriptstyle\partial} \\ C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{(-|-)_{f_{\mathrm{b}}}} & D_{\mathrm{b}} \end{array}$$

commute. The fourth condition says that the measure of noncommutativity of the diagram

$$\begin{array}{ccc} C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & C_{\mathrm{r}} \\ \downarrow{\scriptstyle f_{\mathrm{b}}^{\mathrm{ab}}\otimes f_{\mathrm{b}}^{\mathrm{ab}}} & & \downarrow{\scriptstyle f_{\mathrm{r}}} \\ D_{\mathrm{b}}^{\mathrm{ab}} \otimes D_{\mathrm{b}}^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & D_{\mathrm{r}} \end{array}$$

is exactly the measure of nonsymmetry of the bilinear map f_{c}.

Lemma 4.1.2. *Quadratic maps* $\boldsymbol{f}$ *with* $f_{\mathrm{c}} = 0$ *are exactly the morphisms of* $\mathbf{SQM}$.

4.2. **Operations on 2-linear maps.** We introduce the addition of 2-maps.

Definition 4.2.1. Let $\boldsymbol{f}, \boldsymbol{g} : C \to D$ be quadratic maps. Define the sum $\boldsymbol{f}+\boldsymbol{g} : C \to D$ by

$$(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{b}} = f_{\mathrm{b}} + g_{\mathrm{b}}, \quad (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{r}} = f_{\mathrm{r}} + g_{\mathrm{r}}, \quad (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c,c') = f_{\mathrm{c}}(c,c') + g_{\mathrm{c}}(c,c') + \{f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)\}\,.$$

Lemma 4.2.2. *The sum of two quadratic maps is again a quadratic map. Moreover, the sum of 2−linear maps is 2-linear.*

Proof. We have to check conditions i)–iv). Since ∂ is a homomorphism, one has

$$\begin{aligned}\partial(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{r}} &= \partial(f_{\mathrm{r}}+g_{\mathrm{r}})\\ &= \partial f_{\mathrm{r}} + \partial g_{\mathrm{r}}\\ &= f_{\mathrm{b}}\partial + g_{\mathrm{b}}\partial\\ &= (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{b}}\partial\end{aligned}$$

and condition i) holds. We have also

$$\begin{aligned}\partial(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(\hat{c},\hat{c}') &= \partial f_{\mathrm{c}}(\hat{c},\hat{c}') + \partial g_{\mathrm{c}}(\hat{c},\hat{c}') + \partial\{f_{\mathrm{b}}(\hat{c}'), g_{\mathrm{b}}(\hat{c})\}\\ &= (c \mid c')_{f_{\mathrm{b}}} + (c \mid c')_{g_{\mathrm{b}}} + [f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)]\\ &= (c \mid c')_{f_{\mathrm{b}}+g_{\mathrm{b}}}.\end{aligned}$$

On the last step we have used Lemma 2.2.3. This shows that condition ii) holds. We can write

$$\begin{aligned}(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(\partial^{\mathrm{ab}}(\hat{r}), \partial^{\mathrm{ab}}((\hat{r}')) &= f_{\mathrm{c}}(\partial^{\mathrm{ab}}((\hat{r}), \partial^{\mathrm{ab}}((\hat{r}')) + g_{\mathrm{c}}(\partial^{\mathrm{ab}}((\hat{r}), \partial^{\mathrm{ab}}((\hat{r})) + \{f_{\mathrm{b}}\partial(r'), g_{\mathrm{b}}\partial(r)\}\\ &= (r \mid r')_{f_{\mathrm{r}}} + (r \mid r')_{g_{\mathrm{r}}} + \{\partial f_{\mathrm{r}}(r'), \partial g_{\mathrm{r}}(r)\}\\ &= (r \mid r')_{f_{\mathrm{r}}} + (r \mid r')_{g_{\mathrm{r}}} + [f_{\mathrm{r}}(r'), f_{\mathrm{r}}(r)]\\ &= (r \mid r')_{f_{\mathrm{r}}+g_{\mathrm{r}}}.\end{aligned}$$

This shows that condition iii) holds. Similarly, we have

$$\begin{aligned}(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{r}}(\{c,c'\}) =& f_{\mathrm{r}}(\{c,c'\}) + g_{\mathrm{r}}(\{c,c'\})\\ =& f_{\mathrm{c}}(c,c') - f_{\mathrm{c}}(c',c) + \{f_{\mathrm{c}}(c), f_{\mathrm{c}}(c')\} + g_{\mathrm{c}}(c,c') - g_{\mathrm{c}}(c',c) + \{g_{\mathrm{c}}(c), g_{\mathrm{c}}(c')\}\\ =& (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c,c') - \{f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)\} - (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c',c) + \{f_{\mathrm{b}}(c), g_{\mathrm{b}}(c')\}\\ &+ \{f_{\mathrm{b}}(c)+g_{\mathrm{b}}(c), f_{\mathrm{b}}(c')+g_{\mathrm{b}}(c')\} - \{f_{\mathrm{b}}(c), g_{\mathrm{b}}(c')\} - \{g_{\mathrm{b}}(c), f_{\mathrm{b}}(c')\}\\ =& (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c,c') - (\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c',c) + \{f_{\mathrm{b}}(c)+g_{\mathrm{b}}(c), f_{\mathrm{b}}(c')+g_{\mathrm{b}}(c')\}\,.\end{aligned}$$

Here, we have used the fact that images of f_{c}, g_{c} as well as those of $\{-,-\}$ are central in D_r.

Finally, if both $\boldsymbol{f}$ and $\boldsymbol{g}$ are 2-linear, then $(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}$ clearly takes the values in the image of $\{-,-\}$, so $\boldsymbol{f}+\boldsymbol{g}$ is 2-linear, too. □

Lemma 4.2.3. *The above sum equips both of the sets* $\mathbb{Hom}_{\mathrm{b}}(C,D) \subset \mathbb{Quad}(C,D)$ *with the group structures. The inverse of an* $\boldsymbol{f}$ *is given by*

$$(-\boldsymbol{f})_{\mathrm{b}} = -f_{\mathrm{b}}, \quad (-\boldsymbol{f})_{\mathrm{r}} = -f_{\mathrm{r}}, \quad (-\boldsymbol{f})_{\mathrm{c}}(c,c') = -f_{\mathrm{c}}(c,c') + \{f_{\mathrm{b}}(c'), f_{\mathrm{b}}(c)\}\,.$$

Moreover, for any $\boldsymbol{f}, \boldsymbol{g}$*, one has*

$$[\boldsymbol{f},\boldsymbol{g}]_{\mathrm{b}} = [f_{\mathrm{b}}, g_{\mathrm{b}}], \quad [\boldsymbol{f},\boldsymbol{g}]_{\mathrm{r}} = [f_{\mathrm{r}}, g_{\mathrm{r}}]$$

and

$$[\boldsymbol{f},\boldsymbol{g}]_{\mathrm{c}}(c,c') = \{f_{\mathrm{b}}(c), g_{\mathrm{b}}(c')\} + \{f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)\}\,.$$

Proof. One can easily check that for any $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : C \to D$, both $((\boldsymbol{f}+\boldsymbol{g})+\boldsymbol{h})_{\mathrm{c}}(c,c')$ and $(\boldsymbol{f}+(\boldsymbol{g}+\boldsymbol{h}))_{\mathrm{c}}(c,c')$ are equal to

$$f_{\mathrm{c}}(c,c') + g_{\mathrm{c}}(c,c') + h_{\mathrm{c}}(c,c') + \{f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)\} + \{f_{\mathrm{b}}(c'), h_{\mathrm{b}}(c)\} + \{g_{\mathrm{b}}(c'), h_{\mathrm{b}}(c)\} \tag{2}$$

(let us observe that the expression takes place in an abelian group $\mathsf{Z}(D_{\mathrm{r}})$, so an order of summation is irrelevant). Thus $+$ is associative. The rest follows from this fact. In fact, to check the formula for the commutator, the only nontrivial case is to compute $(-\boldsymbol{f}-\boldsymbol{g}+\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}$, which can be done by using (2):

$$\begin{aligned}(-\boldsymbol{f}-\boldsymbol{g}+\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c,c') &= (-\boldsymbol{f})_{\mathrm{c}}(c,c')+(-\boldsymbol{g})_{\mathrm{c}}(c,c')+(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c,c')\\ &\quad+\{f_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}-\{f_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)+g_{\mathrm{b}}(c)\}-\{g_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)+g_{\mathrm{b}}(c)\}\,.\end{aligned}$$

Now,

$$\begin{aligned}((-\boldsymbol{f})_{\mathrm{c}}+(-\boldsymbol{g})_{\mathrm{c}}+(\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}})(c,c') =&(-f_{\mathrm{c}}-g_{\mathrm{c}}+f_{\mathrm{c}}+g_{\mathrm{c}})(c,c')\\ &+\{f_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)\}+\{g_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}+\{f_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}\end{aligned}$$

Since all terms lie in $\mathsf{Z}(G)$, we have $(-f_{\mathrm{c}}-g_{\mathrm{c}}+f_{\mathrm{c}}+g_{\mathrm{c}})(c,c')=0$, thus we get

$$\begin{aligned}(-\boldsymbol{f}-\boldsymbol{g}+\boldsymbol{f}+\boldsymbol{g})_{\mathrm{c}}(c,c') &=\{f_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)\}+\{g_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}+\{f_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}\\ &\quad-\{f_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)\}-\{g_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)+g_{\mathrm{b}}(c)\}\\ &=\{f_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}-\{g_{\mathrm{b}}(c'),f_{\mathrm{b}}(c)\}\\ &=\{f_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\}+\{f_{\mathrm{b}}(c),g_{\mathrm{b}}(c')\}\,.\end{aligned}$$

□

Another way to determine the set of 2-linear maps is by the pullback diagram

$$\begin{array}{ccc}\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D) & \longrightarrow & \mathsf{Hom}(C_{\mathrm{b}}^{\mathrm{ab}}\otimes C_{\mathrm{b}}^{\mathrm{ab}},D_{\mathrm{c}})\\ \downarrow & & \downarrow\\ \mathbb{Q}\mathrm{uad}(C,D) & \xrightarrow{\boldsymbol{f}\mapsto f_{\mathrm{c}}} & \mathsf{Hom}(C_{\mathrm{b}}^{\mathrm{ab}}\otimes C_{\mathrm{b}}^{\mathrm{ab}},\mathsf{Z}(D_{\mathrm{r}})),\end{array}$$

where the vertical map on the right is induced by the inclusion $D_{\mathrm{c}}\subseteq\mathsf{Z}(D_{\mathrm{r}})$. Then as in 2.3.1, it follows from 2.2.2 that $\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)$ is a normal subgroup in $\mathbb{Q}\mathrm{uad}(C,D)$.

Examples 4.2.4. Similarly to Lemma 3.1.5 above, one easily establishes the bijections (resp., group isomorphisms)

$$\begin{aligned}\mathbb{Q}\mathrm{uad}(C,\mathbb{L}(N))&\approx\mathsf{Quad}(C_{\mathrm{b}},N),\\ \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,\mathbb{L}(N))&\cong\mathsf{Q}(C_{\mathrm{b}},N)\end{aligned}$$

given by $\boldsymbol{f}\mapsto f_{\mathrm{b}}$ and

$$\begin{aligned}\mathbb{Q}\mathrm{uad}(\mathbb{L}(N),C)&\approx\mathsf{Quad}(N,C_{\mathrm{r}}),\\ \mathbb{H}\mathrm{om}_{\mathrm{b}}(\mathbb{L}(N),C)&\cong\{\ q\in\mathsf{Quad}(N,C_{\mathrm{r}})\ \mid\ \mathsf{Im}(-\mid-)_q\subseteq C_{\mathrm{c}}\ \}\end{aligned}$$

given by $\boldsymbol{f}\mapsto f_{\mathrm{r}}$.

Moreover, one has

$$\mathbb{H}\mathrm{om}_{\mathrm{b}}(\mathbb{S}[N],C)\approx\big\{\ (f_{\mathrm{b}},f_{\mathrm{c}})\in\mathsf{Quad}(N,C_{\mathrm{b}})\times\mathsf{Hom}(N^{\mathrm{ab}}\otimes N^{\mathrm{ab}},C_{\mathrm{c}})\ \mid\ (-\mid-)_{f_{\mathrm{b}}}=\partial f_{\mathrm{c}}\big\}\,;$$

in particular,

$$\mathbb{H}\mathrm{om}_{\mathrm{b}}(\mathbb{S},C)\approx C_{\mathrm{b}}\times C_{\mathrm{c}}.$$

4.3. **Composite of 2-linear maps.**

Definition 4.3.1. Let $\boldsymbol{f}:C\to D$ and $\boldsymbol{g}:D\to E$ be 2-linear maps. Their *composite* is defined by

$$(\boldsymbol{g}\boldsymbol{f})_{\mathrm{b}}=g_{\mathrm{b}}f_{\mathrm{b}},\quad(\boldsymbol{g}\boldsymbol{f})_{\mathrm{r}}=g_{\mathrm{r}}f_{\mathrm{r}},\quad(\boldsymbol{g}\boldsymbol{f})_{\mathrm{c}}(c,c')=g_{\mathrm{c}}(f_{\mathrm{b}}c,f_{\mathrm{b}}c')+g_{\mathrm{r}}(f_{\mathrm{c}}(c,c')).$$

Lemma 4.3.2. *The composite of 2-linear maps* $\boldsymbol{g}:D\to E$, $\boldsymbol{f}:C\to D$ *defines a 2-linear map* $\boldsymbol{g}\boldsymbol{f}:C\to E$.

Proof. First of all, it follows from iv) of 4.1.1 for $\boldsymbol{g}$ that $g_{\mathrm{r}}(f_{\mathrm{c}}(c,c'))$ lies in E_{c}, hence also $(\boldsymbol{g}\boldsymbol{f})_{\mathrm{c}}(c,c')$ lies there for any c, c', so that $\mathsf{Im}\,((\boldsymbol{g}\boldsymbol{f})_{\mathrm{c}})\subseteq E_{\mathrm{c}}$.

For condition i) of 4.1.1, we have

$$\partial(\boldsymbol{g}\boldsymbol{f})_{\mathrm{r}}=\partial(g_{\mathrm{r}}f_{\mathrm{r}})=g_{\mathrm{b}}(\partial f_{\mathrm{r}})=g_{\mathrm{b}}f_{\mathrm{r}}\partial=(\boldsymbol{g}\boldsymbol{f})_{\mathrm{b}}\partial.$$

For condition ii), we compute

$$\begin{aligned}
\partial(\boldsymbol{gf})_{\mathrm{c}}(c,c') &= \partial g_{\mathrm{c}}(f_{\mathrm{b}}c, f_{\mathrm{b}}c') + \partial g_{\mathrm{r}}(f_{\mathrm{c}}(c,c')) \\
&= (f_{\mathrm{b}}(c) \mid f_{\mathrm{b}}(c'))_{g_{\mathrm{b}}} + g_{\mathrm{b}}(\partial f_{\mathrm{c}}(c,c')) && \text{(by i) and ii) of 4.1.1)} \\
&= (f_{\mathrm{b}}(c) \mid f_{\mathrm{b}}(c'))_{g_{\mathrm{b}}} + g_{\mathrm{b}}((c \mid c')_{f_{\mathrm{b}}}) && \text{(by ii) of 4.1.1)} \\
&= (c \mid c')_{(\boldsymbol{gf})_{\mathrm{b}}} && \text{(by 2.3.4).}
\end{aligned}$$

For condition iii), we compute

$$\begin{aligned}
(r \mid r')_{g_{\mathrm{r}}f_{\mathrm{r}}} &= -(g_{\mathrm{r}}f_{\mathrm{r}}(r) + g_{\mathrm{r}}f_{\mathrm{r}}(r')) + g_{\mathrm{r}}f_{\mathrm{r}}(r+r') \\
&= -(g_{\mathrm{r}}f_{\mathrm{r}}(r) + g_{\mathrm{r}}f_{\mathrm{r}}(r')) + g_{\mathrm{r}}(f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r') + (r \mid r')_{f_{\mathrm{r}}}) \\
&= -(g_{\mathrm{r}}f_{\mathrm{r}}(r) + g_{\mathrm{r}}f_{\mathrm{r}}(r')) + g_{\mathrm{r}}(f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r')) + g_{\mathrm{r}}((r \mid r')_{f_{\mathrm{r}}}) + (f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r') \mid (r \mid r')_{f_{\mathrm{r}}})_{g_{\mathrm{r}}} \\
&= (f_{\mathrm{r}}(r) \mid f_{\mathrm{r}}(r'))_{g_{\mathrm{r}}} + g_{\mathrm{r}}((r \mid r')_{f_{\mathrm{r}}}) + (f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r') \mid (r \mid r')_{f_{\mathrm{r}}})_{g_{\mathrm{r}}}.
\end{aligned}$$

But

$$\begin{aligned}
(f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r') \mid (r \mid r')_{f_{\mathrm{r}}})_{g_{\mathrm{r}}} &= g_{\mathrm{c}}(\partial(f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r')), \partial(r \mid r')_{f_{\mathrm{r}}}) \\
&= g_{\mathrm{c}}(\partial(f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r')), (\partial r \mid \partial r')_{\partial f_{\mathrm{r}}}) = g_{\mathrm{c}}(\partial(f_{\mathrm{r}}(r) + f_{\mathrm{r}}(r')), (\partial r \mid \partial r')_{f_{\mathrm{b}}\partial})
\end{aligned}$$

which is zero since $(\partial r \mid \partial r')_{f_{\mathrm{b}}\partial} \in [D_{\mathrm{b}}, D_{\mathrm{b}}]$, f_{b} and hence $f_{\mathrm{b}}\partial$ being a q-map. Thus

$$\begin{aligned}
(r \mid r')_{g_{\mathrm{r}}f_{\mathrm{r}}} = (f_{\mathrm{r}}(r) \mid f_{\mathrm{r}}(r'))_{g_{\mathrm{r}}} + g_{\mathrm{r}}((r \mid r')_{f_{\mathrm{r}}}) &= g_{\mathrm{c}}(\partial f_{\mathrm{r}}(r), \partial f_{\mathrm{r}}(r')) + g_{\mathrm{r}}f_{\mathrm{c}}(\partial r, \partial r') \\
&= g_{\mathrm{c}}(f_{\mathrm{b}}\partial(r), f_{\mathrm{b}}\partial(r')) + g_{\mathrm{r}}f_{\mathrm{c}}(\partial r, \partial r').
\end{aligned}$$

But by the definition this is $(\boldsymbol{gf})_{\mathrm{c}}(\partial r, \partial r')$, so condition iii) indeed holds.

Finally, for condition iv), we have

$$\begin{aligned}
(\boldsymbol{gf})_{\mathrm{r}}(\{c,c'\}) &= g_{\mathrm{r}}(f_{\mathrm{r}}(\{c,c'\})) \\
&= g_{\mathrm{r}}(f_{\mathrm{c}}(c,c')) - g_{\mathrm{r}}(f_{\mathrm{c}}(c',c)) + g_{\mathrm{r}}(\{f_{\mathrm{b}}(c), f_{\mathrm{b}}(c')\}) && \text{(by iv) of 4.1.1)} \\
&= g_{\mathrm{r}}(f_{\mathrm{c}}(c,c')) - g_{\mathrm{r}}(f_{\mathrm{c}}(c',c)) + g_{\mathrm{c}}(f_{\mathrm{b}}(c), f_{\mathrm{b}}(c')) \\
&\quad - g_{\mathrm{c}}(f_{\mathrm{b}}(c'), f_{\mathrm{b}}(c)) + \{g_{\mathrm{b}}f_{\mathrm{b}}(c), g_{\mathrm{b}}f_{\mathrm{b}}(c')\} \\
&= (\boldsymbol{gf})_{\mathrm{c}}(c,c') - (\boldsymbol{gf})_{\mathrm{c}}(c',c) + \{(gf)_{\mathrm{b}}(c), (gf)_{\mathrm{b}}(c')\}. && \square
\end{aligned}$$

Since the composition of 2-linear maps is still 2-linear, sqm's together with 2-linear maps form a category which is denoted by SQM_2. As we have seen, hom's in this category have a group structure. Our immediatle goal is to enrich such hom's and obtain kind of internal hom's in SQM_2.

5. Definition of $\mathbb{Hom}(C,D)$

Now, we wish to introduce a kind of internal Hom for sqm's which we denote by $\mathbb{Hom}(C,D)$. We have already defined $\mathbb{Hom}_{\mathrm{b}}(C,D)$ which is the b-part of $\mathbb{Hom}(C,D)$. Now, we describe an r-part of the $\mathbb{Hom}(C,D)$.

5.1. The group $\mathbb{Hom}_{\mathrm{r}}(C,D)$.

Definition 5.1.1. For sqm's C, D, we put

$$\mathbb{Hom}_{\mathrm{r}}(C,D) := \{f \in \mathsf{Quad}(C_{\mathrm{b}}, D_{\mathrm{r}}) \;\mid\; \mathsf{Im}(- \mid -)_f \subseteq D_{\mathrm{c}}\}.$$

In other words, $\mathbb{Hom}_{\mathrm{r}}(C,D)$ is determined by the pullback square

$$\begin{array}{ccc}
\mathbb{Hom}_{\mathrm{r}}(C,D) & \longrightarrow & \mathsf{Hom}(C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}}, D_{\mathrm{c}}) \\
\downarrow & & \downarrow \\
\mathsf{Quad}(C_{\mathrm{b}}, D_{\mathrm{r}}) & \xrightarrow{f \mapsto (-|-)_f} & \mathsf{Hom}(C_{\mathrm{b}}^{\mathrm{ab}} \otimes C_{\mathrm{b}}^{\mathrm{ab}}, \mathsf{Z}(H)),
\end{array}$$

in particular, $\mathbb{Hom}_{\mathrm{r}}(C,D)$ is a normal subgroup in $\mathsf{Quad}(C_{\mathrm{b}},D_{\mathrm{r}})$. Note also that $\mathsf{Q}(C_{\mathrm{b}},D_{\mathrm{r}})$ is a subset of $\mathbb{Hom}_{\mathrm{r}}(C,D)$.

The next our aim is to define the bundary map $\partial : \mathbb{Hom}_{\mathrm{r}}(C,D) \to \mathbb{Hom}_{\mathrm{b}}(C,D)$.

Lemma 5.1.2. *Let C, D be sqm's and let $q : C_{\mathrm{b}} \to D_{\mathrm{r}}$ be a quadratic map. Define*

$$\boldsymbol{\partial}(q)_{\mathrm{b}} = \partial q, \quad \boldsymbol{\partial}(q)_{\mathrm{r}} = q\partial, \quad \boldsymbol{\partial}(q)_{\mathrm{c}} = (- \mid -)_q.$$

Then $\boldsymbol{\partial}(q) : C \to D$ is a quadratic map. Moreover, $\boldsymbol{\partial}(q)$ is 2-linear if and only if $q \in \mathbb{Hom}_{\mathrm{r}}(C,D)$.

Proof. Property i) of Definition 4.1.1 follows from the associativity of the composition. Since ∂ is a homomorphism, it follows that $\partial(c \mid c')_q = (c \mid c')_{\partial q}$ and $(\partial r \mid \partial r')_q = (r \mid r')_{q\partial}$, which proves conditions ii) and iii) of *loc. cit.* Property iv) is a consequence of Lemma 2.2.1.

Finally, $\mathsf{Im}(- \mid -)_q \subseteq D_{\mathrm{c}}$ is the same as $\mathsf{Im}((\boldsymbol{\partial} q)_{\mathrm{c}}) \subseteq D_{\mathrm{c}}$, so $\boldsymbol{\partial} q$ is 2-linear iff $q \in \mathbb{Hom}_{\mathrm{r}}(C,D)$. □

Lemma 5.1.3. *Let C, D be sqm's. Then the map $\boldsymbol{\partial} : \mathsf{Quad}(C_{\mathrm{b}},D_{\mathrm{r}}) \to \mathbb{Quad}(C,D)$ is a homomorphism.*

Proof. Indeed, assume $q_1, q_2 \in \mathsf{Quad}(C_{\mathrm{b}},D_{\mathrm{r}})$; then we have

$$\begin{aligned}(\boldsymbol{\partial}(q_1+q_2))_{\mathrm{b}} &= \partial(q_1+q_2)\\ &= \partial q_1 + \partial q_2\\ &= (\boldsymbol{\partial} q_1)_{\mathrm{b}} + (\boldsymbol{\partial} q_2)_{\mathrm{b}}\\ &= (\boldsymbol{\partial} q_1 + \boldsymbol{\partial} q_2)_{\mathrm{b}}.\end{aligned}$$

Similarly, $(\boldsymbol{\partial}(q_1+q_2))_{\mathrm{r}} = (\boldsymbol{\partial} q_1 + \boldsymbol{\partial} q_2)_{\mathrm{r}}$. We also have

$$(\boldsymbol{\partial}(q_1+q_2))_{\mathrm{c}}(c,c') = (c \mid c')_{q_1+q_2} = (c \mid c')_{q_1} + (c \mid c')_{q_2} + [q_1(c'), q_2(c)].$$

On the other hand,

$$\begin{aligned}(\boldsymbol{\partial} q_1 + \boldsymbol{\partial} q_2)_{\mathrm{c}}(c,c') &= (\boldsymbol{\partial} q_1)_{\mathrm{c}}(c,c') + (\boldsymbol{\partial} q_2)_{\mathrm{c}}(c,c') + \{(\boldsymbol{\partial} q_1)_{\mathrm{b}}(c'), (\boldsymbol{\partial} q_2)_{\mathrm{b}}(c)\}\\ &= (c \mid c')_{q_1} + (c \mid c')_{q_2} + \{\partial q_1(c'), \partial q_2(c)\}\\ &= (c \mid c')_{q_1} + (c \mid c')_{q_2} + [q_1(c'), q_2(c)]\end{aligned}$$

Thus $(\boldsymbol{\partial}(q_1+q_2))_{\mathrm{c}} = (\boldsymbol{\partial} q_1 + \boldsymbol{\partial} q_2)_{\mathrm{c}}$. □

Thus there also is a pullback square of the form

$$\begin{array}{ccc}\mathbb{Hom}_{\mathrm{r}}(C,D) & \xrightarrow{\boldsymbol{\partial}} & \mathbb{Hom}_{\mathrm{b}}(C,D)\\ \downarrow & \lrcorner & \downarrow\\ \mathsf{Quad}(C_{\mathrm{b}},D_{\mathrm{r}}) & \xrightarrow{\boldsymbol{\partial}} & \mathbb{Quad}(C,D).\end{array}$$

5.2. **Definition of the bracket.**

Lemma 5.2.1. *For any $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{Hom}_{\mathrm{b}}(C,D)$, the formula*

$$\{\boldsymbol{f}, \boldsymbol{g}\}(c) := \{f_{\mathrm{b}}(c), g_{\mathrm{b}}(c)\}$$

defines an element $\{\boldsymbol{f}, \boldsymbol{g}\} \in \mathbb{Hom}_{\mathrm{r}}(C,D)$ with

$$(c \mid c')_{\{\boldsymbol{f},\boldsymbol{g}\}} = \{f_{\mathrm{b}}(c), g_{\mathrm{b}}(c')\} + \{f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)\}.$$

Moreover, one has the following equalities:

$$\begin{aligned}\{\boldsymbol{f}_1 + \boldsymbol{f}_2, \boldsymbol{g}\} &= \{\boldsymbol{f}_1, \boldsymbol{g}\} + \{\boldsymbol{f}_2, \boldsymbol{g}\},\\ \{\boldsymbol{f}, \boldsymbol{g}_1 + \boldsymbol{g}_2\} &= \{\boldsymbol{f}, \boldsymbol{g}_1\} + \{\boldsymbol{f}, \boldsymbol{g}_2\}.\end{aligned}$$

Therefore this yields a bilinear map $\{-,-\} : \mathbb{Hom}_{\mathrm{b}}(C,D) \times \mathbb{Hom}_{\mathrm{b}}(C,D) \to \mathbb{Hom}_{\mathrm{r}}(C,D)$.

Proof. We have

$$\begin{aligned}\{f_{\mathrm{b}}(c+c'), g_{\mathrm{b}}(c+c')\} &= \{f_{\mathrm{b}}(c)+f_{\mathrm{b}}(c')+(c \mid c')_{f_{\mathrm{b}}}, g_{\mathrm{b}}(c)+g_{\mathrm{b}}(c')+(c \mid c')_{g_{\mathrm{b}}}\}\\ &= \{f_{\mathrm{b}}(c)+f_{\mathrm{b}}(c'), g_{\mathrm{b}}(c)+g_{\mathrm{b}}(c')\}\,.\end{aligned}$$

Here, we have used the fact that f_{b} and g_{b} are q-maps and therefore their cross-effects lie in the commutator subgroup. Thus the expected expansion for the cross-effect of $\{\boldsymbol{f},\boldsymbol{g}\}$ follows. The rest is straightforward. □

Lemma 5.2.2. *For any sqm's C and D, the maps*

$$\mathbb{Hom}_{\mathrm{b}}(C,D)\times\mathbb{Hom}_{\mathrm{b}}(C,D)\xrightarrow{\{-,-\}}\mathbb{Hom}_{\mathrm{r}}(C,D)\xrightarrow{\boldsymbol{\partial}}\mathbb{Hom}_{\mathrm{b}}(C,D)$$

define a sqm $\mathbb{Hom}(C,D)$ with

$$\begin{aligned}\mathbb{Hom}(C,D)_{\mathrm{b}} &:= \mathbb{Hom}_{\mathrm{b}}(C,D),\\ \mathbb{Hom}(C,D)_{\mathrm{r}} &:= \mathbb{Hom}_{\mathrm{r}}(C,D).\end{aligned}$$

Proof. We have already proved that $\boldsymbol{\partial}$ and $\{-,-\}:\mathbb{Hom}_{\mathrm{b}}(C,D)\times\mathbb{Hom}_{\mathrm{b}}(C,D)\to\mathbb{Hom}_{\mathrm{r}}(C,D)$ are well-defined and the bracket is bilinear. Let us compute $\boldsymbol{\partial}\{\boldsymbol{f},\boldsymbol{g}\}:C\to D$, where $\boldsymbol{f}$, $\boldsymbol{g}$ are 2-linear maps $C\to D$. We have

$$(\boldsymbol{\partial}\{\boldsymbol{f},\boldsymbol{g}\})_{\mathrm{b}}(c)=\partial\{f_{\mathrm{b}}(c),g_{\mathrm{b}}(c)\}=[f_{\mathrm{b}}(c),g_{\mathrm{b}}(c)]=([\boldsymbol{f},\boldsymbol{g}]_{\mathrm{b}})(c).$$

We have also

$$(\boldsymbol{\partial}\{\boldsymbol{f},\boldsymbol{g}\})_{\mathrm{r}}(r)=\{\boldsymbol{f},\boldsymbol{g}\}\,\partial(r)=\{f_{\mathrm{b}}(\partial r),g_{\mathrm{b}}(\partial r)\}=\{\partial f_{\mathrm{b}}(r),\partial g_{\mathrm{b}}(r)\}=[f_{\mathrm{r}}(r),g_{\mathrm{r}}(r)]=([\boldsymbol{f},\boldsymbol{g}]_{\mathrm{r}})(r).$$

We can write

$$\begin{aligned}(\boldsymbol{\partial}\{\boldsymbol{f},\boldsymbol{g}\})_{\mathrm{c}}(c,c') &= (c \mid c')_{\{\boldsymbol{f},\boldsymbol{g}\}}\\ &= \{f_{\mathrm{b}}(c),g_{\mathrm{b}}(c')\}+\{f_{\mathrm{b}}(c'),g_{\mathrm{b}}(c)\} && \text{(by 5.2.1).}\end{aligned}$$

Comparing with Lemma 4.2.3, we conclude that $\boldsymbol{\partial}(\{\boldsymbol{f},\boldsymbol{g}\})=[\boldsymbol{f},\boldsymbol{g}]$. Therefore condition i) of Definition 3.1.1 holds. The other conditions are easier to check. For $h,h'\in\mathbb{Hom}_{\mathrm{r}}(C,D)$, we can write

$$\{\boldsymbol{\partial}(h),\boldsymbol{\partial}(h')\}\,(c)=\{(\partial h)_{\mathrm{b}}(c),(\partial h')_{\mathrm{b}}(c)\}=\{\partial h(c),\partial h'(c)\}=[h(c),h(c')]=[h,h'](c)$$

which shows that condition ii) of Definition 3.1.1 indeed holds. Finally, for $\boldsymbol{f},\boldsymbol{f}'\in\mathbb{Hom}_{\mathrm{b}}(C,D)$ one has

$$\{\boldsymbol{f},\boldsymbol{f}'\}\,(c)+\{\boldsymbol{f}',\boldsymbol{f}\}\,(c)=\{f_{\mathrm{b}}(c),{f'}_{\mathrm{b}}(c)\}+\{{f'}_{\mathrm{b}}(c),f_{\mathrm{b}}(c)\}=0.$$

which shows that $\mathbb{Hom}(C,D)$ is really a sqm. □

Remark 5.2.3. By the adjunction $(-)_{\mathrm{b}}\dashv\mathbb{L}$ of 3.1.5, every sqm C is equipped with a canonical morphism $C\to\mathbb{L}(C_{\mathrm{b}})$. Comparing with 4.2.4 one sees that for any sqm D, the induced homomorphism

$$\mathbb{Hom}_{\mathrm{b}}(\mathbb{L}(C_{\mathrm{b}}),D)\to\mathbb{Hom}_{\mathrm{b}}(C,D)$$

can be identified with

$$\mathbb{Hom}(C,D)_{\mathrm{r}}\xrightarrow{\boldsymbol{\partial}}\mathbb{Hom}(C,D)_{\mathrm{b}}.$$

Indeed, as mentioned in 4.2.4, one has $\mathbb{Hom}_{\mathrm{b}}(\mathbb{L}(C_{\mathrm{b}}),D)\cong\mathbb{Hom}_{\mathrm{r}}(C,D)$ and the induced homomorphism is easily seen to match with $\boldsymbol{\partial}$ in $\mathbb{Hom}(D,C)$.

Thus the category of sqm's and 2-linear maps are of a kind of "internal hom objects" $\mathbb{Hom}(-,-)$.

6. Extending Functoriality of $\mathbb{Hom}(C,D)$

For any sqm C, D, E and a 2-linear map $\boldsymbol{g}:D\to E$, we construct a natural 2-linear map

$$\boldsymbol{g}_{\odot}:\mathbb{Hom}(C,D)\to\mathbb{Hom}(C,E).$$

Quite similarly, for any 2-linear map $\boldsymbol{f}:C\to D$, we construct a natural 2-linear map

$$\cdot\boldsymbol{f}:\mathbb{Hom}(D,E)\to\mathbb{Hom}(C,E).$$

6.1. **Construction of the map $\boldsymbol{g}_\odot$.** We start with the following

Definition 6.1.1. For 2-linear maps $\boldsymbol{f},\boldsymbol{f}':C\to D$ and $\boldsymbol{g}:D\to E$, the 2-linear map

$$(\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}:C\to E$$

is the value on the pair $\boldsymbol{f},\boldsymbol{f}'$ of the cross-effect $(-\mid-)_{\boldsymbol{g}\cdot}$ of the map

$$\boldsymbol{g}\cdot:\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)\to\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)$$

given by a precomposition with $\boldsymbol{g}$.

Lemma 6.1.2. *For any 2-linear maps $\boldsymbol{f},\boldsymbol{f}':C\to D$ and $\boldsymbol{g}:D\to E$, the 2-linear map $(\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}:C\to E$ is given by*

$$\left((\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{b}}(c)=(f_{\mathrm{b}}(c)\mid f'_{\mathrm{b}}(c))_{g_{\mathrm{b}}},\quad\left((\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{r}}(c)=(f_{\mathrm{r}}(c)\mid f'_{\mathrm{r}}(c))_{g_{\mathrm{r}}}$$

and

$$\left((\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{c}}(c,c')=g_{\mathrm{c}}(f_{\mathrm{b}}(c'),f'_{\mathrm{b}}(c))-g_{\mathrm{c}}(f'_{\mathrm{b}}(c),f_{\mathrm{b}}(c')).$$

Proof. The equalities for the b- and r-levels are straightforward; for the c-level, we compute

$$\begin{aligned}\left((\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{c}}(c,c')&=\left(-(\boldsymbol{gf}+\boldsymbol{gf}')\right)_{\mathrm{c}}(c,c')+\left(\boldsymbol{g}(\boldsymbol{f}+\boldsymbol{f}')\right)_{\mathrm{c}}(c,c')\\&\quad+\left\{\left(-(\boldsymbol{gf}+\boldsymbol{gf}')\right)_{\mathrm{b}}(c'),\left(\boldsymbol{g}(\boldsymbol{f}+\boldsymbol{f}')\right)_{\mathrm{b}}(c)\right\}\\&=-\left(\left(\boldsymbol{gf}+\boldsymbol{gf}'\right)_{\mathrm{c}}(c,c')\right)+\left\{\left(\boldsymbol{gf}+\boldsymbol{gf}'\right)_{\mathrm{b}}(c'),\left(\boldsymbol{gf}+\boldsymbol{gf}'\right)_{\mathrm{b}}(c)\right\}\\&\quad+g_{\mathrm{c}}(f_{\mathrm{b}}(c)+f'_{\mathrm{b}}(c),f_{\mathrm{b}}(c')+f'_{\mathrm{b}}(c'))+g_{\mathrm{r}}(\left(\boldsymbol{f}+\boldsymbol{f}'\right)_{\mathrm{c}}(c,c'))\\&\quad+\left\{\left(-(\boldsymbol{gf}+\boldsymbol{gf}')\right)_{\mathrm{b}}(c'),\left(\boldsymbol{g}(\boldsymbol{f}+\boldsymbol{f}')\right)_{\mathrm{b}}(c)\right\}\\&=-\left\{g_{\mathrm{b}}f_{\mathrm{b}}(c'),g_{\mathrm{b}}f'_{\mathrm{b}}(c)\right\}-\left(\boldsymbol{gf}'\right)_{\mathrm{c}}(c,c')-\left(\boldsymbol{gf}\right)_{\mathrm{c}}(c,c')\\&\quad+g_{\mathrm{c}}(f_{\mathrm{b}}(c),f_{\mathrm{b}}(c'))+g_{\mathrm{c}}(f_{\mathrm{b}}(c),f'_{\mathrm{b}}(c'))\\&\quad+g_{\mathrm{c}}(f'_{\mathrm{b}}(c),f_{\mathrm{b}}(c'))+g_{\mathrm{c}}(f'_{\mathrm{b}}(c),f'_{\mathrm{b}}(c'))\\&\quad+g_{\mathrm{r}}(f_{\mathrm{c}}(c,c')+f'_{\mathrm{c}}(c,c')+\{f_{\mathrm{b}}(c'),f'_{\mathrm{b}}(c)\}),\end{aligned}$$

since $\{-,(g_{\mathrm{b}}(f_{\mathrm{b}}+f'_{\mathrm{b}}))(c)\}=\{-,(g_{\mathrm{b}}f_{\mathrm{b}}+g_{\mathrm{b}}f'_{\mathrm{b}})(c)\}$ as g_{b} is a q-map and $\{-,-\}$ vanishes on commutators; then substituting

$$(\boldsymbol{gf})_{\mathrm{c}}(c,c')=g_{\mathrm{r}}f_{\mathrm{c}}(c,c')+g_{\mathrm{c}}(f_{\mathrm{b}}(c),f_{\mathrm{b}}(c'))$$

and the similar expression for $\boldsymbol{gf}'$ and taking into account that g_{r} is linear on values of f_{c} and f'_{c}, one obtains

$$\begin{aligned}\left((\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{c}}(c,c')&=-\left\{g_{\mathrm{b}}f_{\mathrm{b}}(c'),g_{\mathrm{b}}f'_{\mathrm{b}}(c)\right\}+g_{\mathrm{r}}\left(\{f_{\mathrm{b}}(c'),f'_{\mathrm{b}}(c)\}\right)\\&=g_{\mathrm{c}}(f_{\mathrm{b}}(c'),f'_{\mathrm{b}}(c))-g_{\mathrm{c}}(f'_{\mathrm{b}}(c),f_{\mathrm{b}}(c'))\end{aligned}$$

by property iv) of 4.1.1. □

Corollary 6.1.3. *For any 2-linear map $\boldsymbol{g}:D\to E$, the map*

$$\boldsymbol{g}\cdot:\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)\to\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)$$

is quadratic.

Proof. The b-component of $(\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}$ lands in the commutator subgroup of E_{b} and its r- and c-components land in E_{c}, hence $(\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}$ is a central element of $\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)$. The linearity identity

$$(\boldsymbol{f}\mid\boldsymbol{f}'+\boldsymbol{f}'')_{\boldsymbol{g}}=(\boldsymbol{f}\mid\boldsymbol{f}')_{\boldsymbol{g}}+(\boldsymbol{f}\mid\boldsymbol{f}'')_{\boldsymbol{g}}$$

and another one for $(\boldsymbol{f}'+\boldsymbol{f}''\mid\boldsymbol{f})_{\boldsymbol{g}}$ follows from the fact that the maps $(-\mid-)_{g_{\mathrm{b}}}$, $(-\mid-)_{g_{\mathrm{r}}}$ and $g_{\mathrm{c}}(-,-)$ are bilinear and vanish on commutators. □

We use the above quadratic map $\boldsymbol{g}\cdot$ as the b-part of a quadratic map

$$\boldsymbol{g}_{\odot} : \mathbb{Hom}(C, D) \to \mathbb{Hom}(C, E),$$

with the r-part $\mathbb{Hom}(C, D)_{\mathrm{r}} \to \mathbb{Hom}(C, E)_{\mathrm{r}}$ equal to $g_{\mathrm{r}}\cdot : \mathbb{Hom}_{\mathrm{r}}(C, D) \to \mathbb{Hom}_{\mathrm{r}}(C, E)$. For this to make sense, one must have that for any quadratic map $q : C_{\mathrm{b}} \to D_{\mathrm{r}}$ with $\mathsf{Im}(- \mid -)_q \subseteq D_{\mathrm{c}}$, the map $g_{\mathrm{r}}q$ is also quadratic with $\mathsf{Im}(- \mid -)_{g_{\mathrm{r}}q} \subseteq E_{\mathrm{c}}$. Indeed,

$$\begin{aligned}(r \mid r')_{g_{\mathrm{r}}q} &= -(g_{\mathrm{r}}q(r) + g_{\mathrm{r}}q(r')) + g_{\mathrm{r}}q(r + r') = -(g_{\mathrm{r}}q(r) + g_{\mathrm{r}}q(r')) + g_{\mathrm{r}}(q(r) + q(r') + (r \mid r')_q) \\ &= -(g_{\mathrm{r}}q(r) + g_{\mathrm{r}}q(r')) + g_{\mathrm{r}}(q(r) + q(r')) + g_{\mathrm{r}}((r \mid r')_q) + (q(r) + q(r') \mid (r \mid r')_q)_{g_{\mathrm{r}}} \\ &= (q(r) \mid q(r'))_{g_{\mathrm{r}}} + g_{\mathrm{r}}((r \mid r')_q),\end{aligned}$$

arguing exactly as in the proof of 4.3.2.

For the c-part of $\boldsymbol{g}_{\odot}$, we also need a homomorphism

$$(\boldsymbol{g}_{\odot})_{\mathrm{c}} : \mathbb{Hom}_{\mathrm{b}}(C, D)^{\mathrm{ab}} \otimes \mathbb{Hom}_{\mathrm{b}}(C, D)^{\mathrm{ab}} \to \mathbb{Hom}_{\mathrm{r}}(C, E).$$

To this end, we define

Definition 6.1.4. For 2-linear maps $\boldsymbol{f}, \boldsymbol{f}' : C \to D$ and $\boldsymbol{g} : D \to E$, the map

$$(\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}} : C_{\mathrm{b}} \to E_{\mathrm{r}}$$

is given by the equality

$$(\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}(c) = g_{\mathrm{c}}(f_{\mathrm{b}}(c), f'_{\mathrm{b}}(c)).$$

We then have

Lemma 6.1.5. *For any $\boldsymbol{f}$, $\boldsymbol{f}'$ and $\boldsymbol{g}$ as above, $(\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}$ is an element of $\mathbb{Hom}_{\mathrm{r}}(C, E)$ with*

$$\boldsymbol{\partial}((\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}) = (\boldsymbol{f} \mid \boldsymbol{f}')_{\boldsymbol{g}}.$$

Proof. One has

$$\begin{aligned}(c \mid c')_{(\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}} &= -\left(g_{\mathrm{c}}(f_{\mathrm{b}}(c), f'_{\mathrm{b}}(c)) + g_{\mathrm{c}}(f_{\mathrm{b}}(c'), f'_{\mathrm{b}}(c'))\right) + g_{\mathrm{c}}(f_{\mathrm{b}}(c + c'), f'_{\mathrm{b}}(c + c')) \\ &= g_{\mathrm{c}}(f_{\mathrm{b}}(c), f'_{\mathrm{b}}(c')) + g_{\mathrm{c}}(f_{\mathrm{b}}(c'), f'_{\mathrm{b}}(c)),\end{aligned}$$

since f_{b}, f'_{b} are q-maps and g_{c} vanishes on commutators. By the same reason, the latter expression is bilinear in c and c'. Moreover, its values belong obviously to E_{c}, so $(\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}$ indeed belongs to $\mathbb{Hom}_{\mathrm{r}}(C, E)$.

Next, we calculate

$$\begin{aligned}\boldsymbol{\partial}((\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}})_{\mathrm{b}}(c) =& \partial g_{\mathrm{c}}(f_{\mathrm{b}}(c), f'_{\mathrm{b}}(c)) = (f_{\mathrm{b}}(c) \mid f'_{\mathrm{b}}(c))_{g_{\mathrm{b}}} = \left((\boldsymbol{f} \mid \boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{b}}(c), \\ \boldsymbol{\partial}((\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}})_{\mathrm{r}}(c) =& g_{\mathrm{c}}(f_{\mathrm{b}}(\partial c), f'_{\mathrm{b}}(\partial c)) = g_{\mathrm{c}}(\partial f_{\mathrm{r}}(c), \partial f'_{\mathrm{r}}(c)) = (f_{\mathrm{r}}(c) \mid f'_{\mathrm{r}}(c))_{g_{\mathrm{r}}} \\ =& \left((\boldsymbol{f} \mid \boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{r}}(c), \\ \boldsymbol{\partial}((\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}})_{\mathrm{c}}(c, c') =& (c \mid c')_{(\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}} = g_{\mathrm{c}}(f_{\mathrm{b}}(c), f'_{\mathrm{b}}(c')) + g_{\mathrm{c}}(f_{\mathrm{b}}(c'), f'_{\mathrm{b}}(c)) \\ =& \left((\boldsymbol{f} \mid \boldsymbol{f}')_{\boldsymbol{g}}\right)_{\mathrm{c}}(c, c').\end{aligned}$$

□

Proposition 6.1.6. *For any sqm C and any 2-linear map $\boldsymbol{g} : D \to E$, the equalities*

$$(\boldsymbol{g}_{\odot})_{\mathrm{b}}(\boldsymbol{f}) = \boldsymbol{g}\boldsymbol{f}, \quad (\boldsymbol{g}_{\odot})_{\mathrm{r}}(q) = g_{\mathrm{r}}q, \ \text{ and } \ (\boldsymbol{g}_{\odot})_{\mathrm{c}}(\boldsymbol{f}, \boldsymbol{f}') = (\boldsymbol{f} \| \boldsymbol{f}')_{\boldsymbol{g}}$$

for $\boldsymbol{f}, \boldsymbol{f}' \in \mathbb{Hom}_{\mathrm{b}}(C, D)$ and $q \in \mathbb{Hom}_{\mathrm{r}}(C, D)$ define a quadratic map $\boldsymbol{g}_{\odot} : \mathbb{Hom}(C, D) \to \mathbb{Hom}(C, E)$.

Proof. We know by 6.1.3 that $(\boldsymbol{g}_{\odot})_{\mathrm{b}}$ is quadratic and we have just checked that $(\boldsymbol{g}_{\odot})_{\mathrm{r}}$ is. Next, for $q : C_{\mathrm{b}} \to D_{\mathrm{r}}$ in $\mathbb{Hom}_{\mathrm{r}}(C, D)$, we have

$$\boldsymbol{\partial}\,(\boldsymbol{g}_{\odot})_{\mathrm{r}}(q) = \boldsymbol{\partial}(g_{\mathrm{r}}q) = (\partial g_{\mathrm{r}}q, g_{\mathrm{r}}q\partial, (- \mid -)_{g_{\mathrm{r}}q})$$

and

$$(\boldsymbol{g}_{\odot})_{\mathrm{b}}(\boldsymbol{\partial}q) = \boldsymbol{g} \cdot (\partial q, q\partial, (- \mid -)_q) = \left(g_{\mathrm{b}}\partial q, g_{\mathrm{r}}q\partial, g_{\mathrm{c}}(\partial q(-), \partial q(-)) + g_{\mathrm{r}}(- \mid -)_q\right),$$

which is the same because one has $g_{\mathrm{b}}\partial=\partial g_{\mathrm{r}}$, $g_{\mathrm{c}}(\partial(-),\partial(-))=(-\mid -)_{g_{\mathrm{r}}}$ and $(-\mid -)_{g_{\mathrm{r}}q}=g_{\mathrm{r}}(-\mid -)_q+(q(-)\mid q(-))_{g_{\mathrm{r}}}$. Thus the diagram

$$\begin{array}{ccc}\mathbb{H}\mathrm{om}_{\mathrm{r}}(C,D) & \longrightarrow & \mathbb{H}\mathrm{om}_{\mathrm{r}}(C,E)\\ \downarrow{\scriptstyle\boldsymbol{\partial}} & & \downarrow{\scriptstyle\boldsymbol{\partial}}\\ \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D) & \longrightarrow & \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)\end{array}$$

commutes. Next, to check the commutativity of the upper triangle in the diagram

$$\begin{array}{ccc}\mathbb{H}\mathrm{om}_{\mathrm{r}}(C,D)^{\mathrm{ab}}\otimes\mathbb{H}\mathrm{om}_{\mathrm{r}}(C,D)^{\mathrm{ab}} & \xrightarrow{(-|-)_{(\boldsymbol{g}\odot)_{\mathrm{r}}}} & \mathbb{H}\mathrm{om}_{\mathrm{r}}(C,E)\\ \downarrow{\scriptstyle\boldsymbol{\partial}\otimes\boldsymbol{\partial}} & \nearrow{\scriptstyle(-\|-)_{\boldsymbol{g}}} & \downarrow{\scriptstyle\boldsymbol{\partial}}\\ \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)^{\mathrm{ab}}\otimes\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)^{\mathrm{ab}} & \xrightarrow{(-|-)_{(\boldsymbol{g}\odot)_{\mathrm{b}}}} & \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)\end{array}$$

we take any $q,q'\in\mathsf{Q}(C_{\mathrm{b}},D_{\mathrm{r}})$, then for any $c\in C_{\mathrm{b}}$, one has

$$(\boldsymbol{\partial}q\|\boldsymbol{\partial}q')_{\boldsymbol{g}}(c)=g_{\mathrm{c}}(\partial q(c),\partial q'(c))=(q(c)\mid q'(c))_{g_{\mathrm{r}}}=(q\mid q')_{(\boldsymbol{g}\odot)_{\mathrm{r}}}(c);$$

as for the lower triangle, its commutativity follows directly from 6.1.5. □

We next have

Lemma 6.1.7. *For any $\boldsymbol{f},\boldsymbol{f}'\in\mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)$ and $\boldsymbol{g},\boldsymbol{g}'\in\mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E)$, one has*

$$(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}+\boldsymbol{g}'}=(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}}+(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}'}+\left\{\boldsymbol{g}\boldsymbol{f}',\boldsymbol{g}'\boldsymbol{f}\right\}.$$

Proof. One calculates

$$\begin{aligned}(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}+\boldsymbol{g}'}(c)&=(\boldsymbol{g}+\boldsymbol{g}')_{\mathrm{c}}\left(f_{\mathrm{b}}(c),{f'}_{\mathrm{b}}(c)\right)\\&=g_{\mathrm{c}}(f_{\mathrm{b}}(c),{f'}_{\mathrm{b}}(c))+{g'}_{\mathrm{c}}(f_{\mathrm{b}}(c),{f'}_{\mathrm{b}}(c))+\{g_{\mathrm{b}}{f'}_{\mathrm{b}}(c),{g'}_{\mathrm{b}}f_{\mathrm{b}}(c)\}\\&=(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}}(c)+(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}'}(c)+\left\{\left(\boldsymbol{g}\boldsymbol{f}'\right)_{\mathrm{b}}(c),(\boldsymbol{g}'\boldsymbol{f})_{\mathrm{b}}(c)\right\}\\&=(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}}(c)+(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}'}(c)+\left\{\boldsymbol{g}\boldsymbol{f}',\boldsymbol{g}'\boldsymbol{f}\right\}(c).\end{aligned}$$ □

Corollary 6.1.8. *The map*

$$\mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E)\to\mathbb{H}\mathrm{om}_{\mathrm{b}}(\mathbb{H}\mathrm{om}(C,D),\mathbb{H}\mathrm{om}(C,E))$$

given by the assignment $\boldsymbol{g}\mapsto(\boldsymbol{g}_\odot)$ is a homomorphism.

Proof. For $\boldsymbol{g},\boldsymbol{g}'\in\mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E)$, we calculate

$$\begin{aligned}\left((\boldsymbol{g}+\boldsymbol{g}')\odot\right)_{\mathrm{b}}(\boldsymbol{f})&=(\boldsymbol{g}+\boldsymbol{g}')\boldsymbol{f}=\boldsymbol{g}\boldsymbol{f}+\boldsymbol{g}'\boldsymbol{f}=\left((\boldsymbol{g}\odot)_{\mathrm{b}}+(\boldsymbol{g}\odot)_{\mathrm{b}}\right)(\boldsymbol{f}),\\ \left((\boldsymbol{g}+\boldsymbol{g}')\odot\right)_{\mathrm{r}}(q)&=(g_{\mathrm{r}}+{g'}_{\mathrm{r}})q=g_{\mathrm{r}}q+{g'}_{\mathrm{r}}q=\left((\boldsymbol{g}\odot)_{\mathrm{r}}+(\boldsymbol{g}\odot)_{\mathrm{r}}\right)(q),\\ \left(\left((\boldsymbol{g}+\boldsymbol{g}')\odot\right)_{\mathrm{c}}(\boldsymbol{f},\boldsymbol{f}')\right)(c)&=(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}+\boldsymbol{g}'}(c)=(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}}(c)+(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}'}(c)+\left\{\boldsymbol{g}\boldsymbol{f}',\boldsymbol{g}'\boldsymbol{f}\right\}(c)\\&=(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}}(c)+(\boldsymbol{f}\|\boldsymbol{f}')_{\boldsymbol{g}'}(c)+\{g_{\mathrm{b}}{f'}_{\mathrm{b}}(c),{g'}_{\mathrm{b}}f_{\mathrm{b}}(c)\}\\&=\left((\boldsymbol{g}\odot)_{\mathrm{c}}(\boldsymbol{f},\boldsymbol{f}')\right)(c)+\left((\boldsymbol{g}'\odot)_{\mathrm{c}}(\boldsymbol{f},\boldsymbol{f}')\right)(c)\\&\quad+\left\{(\boldsymbol{g}\odot)_{\mathrm{b}}(\boldsymbol{f}'),(\boldsymbol{g}'\odot)_{\mathrm{b}}(\boldsymbol{f})\right\}(c)\\&=\left((\boldsymbol{g}\odot(-)+\boldsymbol{g}'\odot(-))_{\mathrm{c}}(\boldsymbol{f},\boldsymbol{f}')\right)(c).\end{aligned}$$ □

6.2. **Construction of the map $\cdot\boldsymbol{f}$.** We will use the following observation.

Lemma 6.2.1. *Composition of 2-linear maps is linear on the right. That is, for any 2-linear maps $\boldsymbol{f}:C\to D$ and $\boldsymbol{g},\boldsymbol{g}':D\to E$ one has*

$$(\boldsymbol{g}+\boldsymbol{g}')\boldsymbol{f}=\boldsymbol{g}\boldsymbol{f}+\boldsymbol{g}'\boldsymbol{f}.$$

Proof. On the b-level,

$$((\boldsymbol{g}+\boldsymbol{g}')\boldsymbol{f})_{\mathrm{b}} = (g_{\mathrm{b}}+g'_{\mathrm{b}})f_{\mathrm{b}} = g_{\mathrm{b}}f_{\mathrm{b}} + g'_{\mathrm{b}}f_{\mathrm{b}} = (\boldsymbol{g}\boldsymbol{f}+\boldsymbol{g}'\boldsymbol{f})_{\mathrm{b}}\,,$$

and, similarly, on the r-level; on the c-level, one has

$$\begin{aligned}((\boldsymbol{g}+\boldsymbol{g}')\boldsymbol{f})_{\mathrm{c}}(a,b) =& (\boldsymbol{g}+\boldsymbol{g}')_{\mathrm{c}}(f_{\mathrm{b}}(a),f_{\mathrm{b}}(b)) + (\boldsymbol{g}+\boldsymbol{g}')_{\mathrm{r}}(f_{\mathrm{c}}(a,b))\\ =& g_{\mathrm{c}}(f_{\mathrm{b}}(a),f_{\mathrm{b}}(b)) + g'_{\mathrm{c}}(f_{\mathrm{b}}(a),f_{\mathrm{b}}(b)) + \{g_{\mathrm{b}}f_{\mathrm{b}}(b), g'_{\mathrm{b}}f_{\mathrm{b}}(a)\}\\ &+ g_{\mathrm{r}}f_{\mathrm{c}}(a,b) + g'_{\mathrm{r}}f_{\mathrm{c}}(a,b)\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{g}\boldsymbol{f}+\boldsymbol{g}'\boldsymbol{f})_{\mathrm{c}}(a,b) =& (\boldsymbol{g}\boldsymbol{f})_{\mathrm{c}}(a,b) + (\boldsymbol{g}'\boldsymbol{f})_{\mathrm{c}}(a,b) + \{(\boldsymbol{g}\boldsymbol{f})_{\mathrm{b}}(b), (\boldsymbol{g}'\boldsymbol{f})_{\mathrm{b}}(a)\}\\ =& g_{\mathrm{c}}(f_{\mathrm{b}}(a),f_{\mathrm{b}}(b)) + g_{\mathrm{r}}f_{\mathrm{c}}(a,b) + g'_{\mathrm{c}}(f_{\mathrm{b}}(a),f_{\mathrm{b}}(b)) + g'_{\mathrm{r}}f_{\mathrm{c}}(a,b)\\ &+ \{g_{\mathrm{b}}f_{\mathrm{b}}(b), g'_{\mathrm{b}}f_{\mathrm{b}}(a)\}\,.\end{aligned}$$

By the centrality of the images of g_{c} and $\{-,-\}$, the lemma follows. □

Corollary 6.2.2. *There are the trinatural maps*

$$\circ_E^{C,D} : \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D) \to \mathsf{Hom}_{\mathsf{SQM}}(\mathbb{H}\mathrm{om}(D,E), \mathbb{H}\mathrm{om}(C,E))$$

which on the b-level are given by composition, i. e., send an $\boldsymbol{f} \in \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,D)$ *to the strict map* $\cdot\boldsymbol{f} : \mathbb{H}\mathrm{om}(D,E) \to \mathbb{H}\mathrm{om}(C,E)$ *with*

$$(\cdot\boldsymbol{f})_{\mathrm{b}}(\boldsymbol{g}) = \boldsymbol{g}\boldsymbol{f}.$$

Proof. We have just seen that $(\cdot\boldsymbol{f})_{\mathrm{b}} : \mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E) \to \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)$ is a homomorphism. We next define $(\cdot\boldsymbol{f})_{\mathrm{r}} : \mathsf{Q}(D_{\mathrm{b}},E_{\mathrm{r}}) \to \mathsf{Q}(C_{\mathrm{b}},E_{\mathrm{r}})$ by the formula

$$(\cdot\boldsymbol{f})_{\mathrm{r}}(q) = qf_{\mathrm{b}}.$$

We then have to check that the diagram

$$\begin{array}{ccccc}\mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E)^{\mathrm{ab}} \otimes \mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E)^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & \mathsf{Q}(D_{\mathrm{b}},E_{\mathrm{r}}) & \xrightarrow{\boldsymbol{\partial}} & \mathbb{H}\mathrm{om}_{\mathrm{b}}(D,E)\\ \downarrow{\scriptstyle (\cdot\boldsymbol{f})_{\mathrm{b}}^{\mathrm{ab}}\otimes(\cdot\boldsymbol{f})_{\mathrm{b}}^{\mathrm{ab}}} & & \downarrow{\scriptstyle (\cdot\boldsymbol{f})_{\mathrm{r}}} & & \downarrow{\scriptstyle (\cdot\boldsymbol{f})_{\mathrm{b}}}\\ \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)^{\mathrm{ab}} \otimes \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)^{\mathrm{ab}} & \xrightarrow{\{-,-\}} & \mathsf{Q}(C_{\mathrm{b}},E_{\mathrm{r}}) & \xrightarrow{\boldsymbol{\partial}} & \mathbb{H}\mathrm{om}_{\mathrm{b}}(C,E)\end{array}$$

commutes. Indeed, for a q-map $q : D_{\mathrm{b}} \to E_{\mathrm{r}}$ one has

$$\begin{aligned}((\cdot\boldsymbol{f})_{\mathrm{b}}\boldsymbol{\partial}q)_{\mathrm{b}} &= ((\boldsymbol{\partial}q)\boldsymbol{f})_{\mathrm{b}} = (\boldsymbol{\partial}q)_{\mathrm{b}}f_{\mathrm{b}} = \partial qf_{\mathrm{b}} = \boldsymbol{\partial}(qf_{\mathrm{b}}) = (\boldsymbol{\partial}(\cdot\boldsymbol{f})_{\mathrm{r}}(q))_{\mathrm{b}}\,,\\ ((\cdot\boldsymbol{f})_{\mathrm{b}}\boldsymbol{\partial}q)_{\mathrm{r}} &= ((\boldsymbol{\partial}q)\boldsymbol{f})_{\mathrm{r}} = (\boldsymbol{\partial}q)_{\mathrm{r}}f_{\mathrm{r}} = q\partial f_{\mathrm{r}} = qf_{\mathrm{b}}\partial = \boldsymbol{\partial}(qf_{\mathrm{b}}) = (\boldsymbol{\partial}(\cdot\boldsymbol{f})_{\mathrm{r}}(q))_{\mathrm{r}}\,,\\ ((\cdot\boldsymbol{f})_{\mathrm{b}}\boldsymbol{\partial}q)_{\mathrm{c}}(a,b) &= ((\boldsymbol{\partial}q)\boldsymbol{f})_{\mathrm{c}}(a,b) = (\boldsymbol{\partial}q)_{\mathrm{c}}(f_{\mathrm{b}}(a),f_{\mathrm{b}}(b)) + q\partial f_{\mathrm{c}}(a,b)\\ &= (f_{\mathrm{b}}(a) \mid f_{\mathrm{b}}(b))_q + q\left((a \mid b)_{f_{\mathrm{b}}}\right) = (a \mid b)_{qf_{\mathrm{b}}} \qquad \text{(by 2.3.4)}\\ &= (\boldsymbol{\partial}(\cdot\boldsymbol{f})_{\mathrm{r}}(q))_{\mathrm{c}}\,,\end{aligned}$$

which proves the commutativity of the right-hand square. For the left-hand one, given any 2-linear maps $\boldsymbol{g}, \boldsymbol{g}' : D \to E$, one has

$$\begin{aligned}(\cdot\boldsymbol{f})_{\mathrm{r}}(\{\boldsymbol{g},\boldsymbol{g}'\})(c) &= \{\boldsymbol{g},\boldsymbol{g}'\}f_{\mathrm{b}}(c) = \{g_{\mathrm{b}}f_{\mathrm{b}}(c), g'_{\mathrm{b}}f_{\mathrm{b}}(c)\} = \{\boldsymbol{g}\boldsymbol{f},\boldsymbol{g}'\boldsymbol{f}\}(c)\\ &= \{(\cdot\boldsymbol{f})_{\mathrm{b}}(\boldsymbol{g}), (\cdot\boldsymbol{f})_{\mathrm{b}}(\boldsymbol{g}')\}(c).\end{aligned}$$

□

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