Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 3, 345–348

MORE ON DECOMPOSITION OF BIOPERATION-CONTINUITY

CARLOS CARPINTERO[1], NEELAMEGARAJAN RAJESH[2] AND ENNIS ROSAS[3]

Abstract. In this paper, we define the notions of weakly (γ,γ')-(β,β')-continuity and weakly* (γ,γ')-(β,β')-continuity in bioperation-topological spaces and investigate some of their properties and relationships with (γ,γ')-(β,β')-continuity.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed, a significant theme in General Topology and Real Analysis concerns various modified forms of continuity, separation axioms etc. utilizing generalized open sets. Kasahara [3] defined the concept of an operation on topological spaces. Ogata and Maki [4] introduced the notion of $\tau_{(\gamma,\gamma')}$ which is the collection of all (γ,γ')-open sets in a topological space (X,τ). Carpintero et al. [2] introduced some new types of sets via bioperation and obtained a new decomposition of bioperation-continuity. In this paper, we define the concepts of weakly (γ,γ')-(β,β')-continuity and weakly* (γ,γ')-(β,β')-continuity in bioperation-topological spaces and investigate some of their properties and relationships.

2. Preliminaries

In this section, we give some fundamental notions, by citing several previous definitions which will be used in the proof of the main results of this paper.

The closure and the interior of a subset A of (X,τ) are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1 ([3])**.** Let (X,τ) be a topological space. An operation γ on the topology τ is a function from τ into the power set $\mathcal{P}(X)$ of X such that $V\subset V^\gamma$ for each $V\in\tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma:\tau\to\mathcal{P}(X)$.

Definition 2.2 ([4])**.** A topological space (X,τ) equipped with two operations, say, γ and γ', defined on τ is called a bioperation-topological space, it is denoted by (X,τ,γ,γ').

Definition 2.3. A subset A of a bioperation topological space (X,τ,γ,γ') is said to be a (γ,γ')-open set [4] if for each $x\in A$ there exist open neighborhoods U and V of x such that $U^\gamma\cup V^{\gamma'}\subset A$. The complement of a (γ,γ')-open set is called a (γ,γ')-closed set. $\tau_{(\gamma,\gamma')}$ denotes a set of all (γ,γ')-open sets in (X,τ,γ,γ').

Definition 2.4 ([4])**.** For a subset A of a bioperation-topological space (X,τ,γ,γ'), (γ,γ')-$\mathrm{Cl}(A)$ denotes the intersection of all (γ,γ')-closed sets containing A, that is, (γ,γ')-$\mathrm{Cl}(A)=\cap\{F: A\subset F, X\backslash F\in\tau_{(\gamma,\gamma')}\}$.

Definition 2.5. Let A be any subset of X. The (γ,γ')-$\mathrm{Int}(A)$ is defined as (γ,γ')-$\mathrm{Int}(A)=\cup\{U: U,$ is a (γ,γ')-open set and $U\subset A\}$.

Remark 2.6. A subset A of a bioperation topological space (X,τ,γ,γ') is:
(1) (γ,γ')-open if and only if $A=(\gamma,\gamma')$-$\mathrm{Int}(A)$;
(2) (γ,γ')-closed if and only if $A=(\gamma,\gamma')$-$\mathrm{Cl}(A)$.

2020 *Mathematics Subject Classification.* 54C05, 54C08, 54E55.
Key words and phrases. (γ,γ')-open set; Weakly (γ,γ')-(β,β')-continuous function; Weakly* (γ,γ')-(β,β')-continuous function.

Definition 2.7 ([1])**.** Let (X, τ) be a topological space, A be a subset of X and γ and γ' be the operations on τ. Then A is said to be (γ, γ')-preopen if $A \subset (\gamma, \gamma')\text{-Int}((\gamma, \gamma')\text{-Cl}(A))$.

Definition 2.8 ([1])**.** A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is said to be (γ, γ')-(β, β')-continuous if $f^{-1}(G)$ is (γ, γ')-open in X for every $G \in \sigma_{(\beta, \beta')}$.

Definition 2.9 ([1])**.** A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is said to be (γ, γ')-(β, β')-precontinuous if $f^{-1}(G)$ is (γ, γ')-preopen in X for every $G \in \sigma_{(\beta, \beta')}$.

3. Weakly (γ, γ')-(β, β')-continuous Functions

In this section, we introduce the notions of weakly (γ, γ')-(β, β')-continuity and weakly* (γ, γ')-(β, β')-continuity in bioperation-topological spaces and obtain some of their characterizations and its relationships with the notion of (γ, γ')-(β, β')-continuity.

Definition 3.1. A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is said to be weakly (γ, γ')-(β, β')-continuous if for each $x \in X$ and each $G \in \sigma_{(\beta, \beta')}$ containing $f(x)$, there exists $F \in \tau_{(\gamma, \gamma')}$ containing x such that $f(F) \subset (\beta, \beta')\text{-Cl}(G)$.

Now, we give an illustrative example of a weakly (γ, γ')-(β, β')-continuous function.

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. We define the operations $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ as follows:

$$A^{\gamma} = \begin{cases} A & \text{if } b \notin A, \\ \text{Cl}(A) & \text{if } b \in A, \end{cases} \quad \text{and} \quad A^{\gamma'} = \begin{cases} \text{Cl}(A) & \text{if } b \notin A, \\ A & \text{if } b \in A. \end{cases}$$

Let $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. We define the operations $\beta, \beta' : \sigma \to \mathcal{P}(X)$ as follows:

$$A^{\beta} = \begin{cases} \{b\} & \text{if } A = \{b\}, \\ Y & \text{if } A \neq \{b\}, \end{cases} \quad \text{and} \quad A^{\beta'} = \begin{cases} \{a, b\} & \text{if } A = \{b\}, \\ Y & \text{if } A \neq \{b\}. \end{cases}$$

Define $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ as $f(a) = b$, $f(b) = a$ and $f(c) = c$. It is easy to see that f is weakly (γ, γ')-(β, β')-continuous.

Theorem 3.3. Let $(X, \tau, \gamma, \gamma')$ and $(Y, \sigma, \beta, \beta')$ be bioperation-topological spaces. A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is weakly (γ, γ')-(β, β')-continuous if and only if for each $G \in \sigma_{(\beta, \beta')}$, $f^{-1}(G) \subset (\gamma, \gamma')\text{-Int}(f^{-1}((\beta, \beta')\text{-Cl}(G)))$.

Proof. Let $G \in \sigma_{(\beta, \beta')}$ and $x \in f^{-1}(G)$. Since f is weakly (γ, γ')-(β, β')-continuous, there exists $F \in \tau_{(\gamma, \gamma')}$ such that $x \in F$ and $f(F) \subset (\beta, \beta')\text{-Cl}(G)$. Hence $x \in F \subset f^{-1}((\beta, \beta')\text{-Cl}(G))$ and $x \in (\gamma, \gamma')\text{-Int}(f^{-1}((\beta, \beta')\text{-Cl}(G)))$. Therefore $f^{-1}(G) \subset (\gamma, \gamma')\text{-Int}(f^{-1}((\beta, \beta')\text{-Cl}(G)))$. Conversely, let $x \in X$ and $G \in \sigma_{(\beta, \beta')}$ containing $f(x)$. Then $x \in f^{-1}(G) \subset (\gamma, \gamma')\text{-Int}(f^{-1}((\beta, \beta')\text{-Cl}(G)))$. Let $F = (\gamma, \gamma')\text{-Int}(f^{-1}((\beta, \beta')\text{-Cl}(G)))$. Then $f(F) = f((\gamma, \gamma')\text{-Int}(f^{-1}((\beta, \beta')\text{-Cl}(G)))) \subset f(f^{-1}((\beta, \beta')\text{-Cl}(G))) \subset (\beta, \beta')\text{-Cl}(G)$. This shows that f is a weakly (γ, γ')-(β, β')-continuous function. □

Theorem 3.4. If $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is a (γ, γ')-(β, β')-continuous function, then it is weakly (γ, γ')-(β, β')-continuous.

Proof. The proof is similar to that of Theorem 3.3. □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. As in Example 3.2, the function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is weakly (γ, γ')-(β, β')-continuous, but is not (γ, γ')-(β, β')-continuous.

At this point, it is possible to find an additional condition in order to show an equivalence between a (γ, γ')-(β, β')-continuous function and a weakly (γ, γ')-(β, β')-continuous one. First, we define the notion of a (γ, γ')-regular space.

Definition 3.6. A bioperation-topological space $(X, \tau, \gamma, \gamma')$ is called a (γ, γ')-regular space, if for each $x \in X$ and each $F \in \tau_{(\gamma,\gamma')}$ containing x, there exists $G \in \tau_{(\gamma,\gamma')}$ containing x such that $G \subset (\gamma, \gamma')\text{-Cl}(G) \subset F$.

Theorem 3.7. Let $(X, \tau, \gamma, \gamma')$ and $(Y, \sigma, \beta, \beta')$ be bioperation-topological spaces, where $(Y, \sigma, \beta, \beta')$ is a (β, β')-regular space. A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is weakly (γ, γ')-(β, β')-continuous if and only if it is (γ, γ')-(β, β')-continuous.

Proof. Let $x \in X$ and $G \in \sigma_{(\beta,\beta')}$ containing $f(x)$. Since (Y, σ, σ') is a (γ, γ')-regular space, there exists $H \in \tau_{(\gamma,\gamma')}$ containing $f(x)$ such that $H \subset (\gamma, \gamma')\text{-Cl}(H) \subset G$. Since f is weakly (γ, γ')-(β, β')-continuous, there exists $F \in \tau_{(\gamma,\gamma')}$ containing x such that $f(F) \subset (\beta, \beta')\text{-Cl}(H) \subset G$. Thus f is (γ, γ')-(β, β')-continuous. The converse is clear. $\square$

Example 3.8. As in Example 3.2, $(Y, \sigma, \beta, \beta')$ is not a (β, β')-regular space.

Definition 3.9. The (γ, γ')-frontier of the subset A of a bioperation-topological space $(X, \tau, \gamma, \gamma')$ is defined by $(\gamma, \gamma')\text{-Cl}(F) - (\gamma, \gamma')\text{-Int}(F)$ and denoted by $(\gamma, \gamma')\text{-}Fr(F)$.

Definition 3.10. A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is said to be weakly* (γ, γ')-(β, β')-continuous if for each $G \in \sigma_{(\beta,\beta')}$, $f^{-1}((\beta, \beta')\text{-}Fr(G))$ is (γ, γ')-closed in X.

Example 3.11. The function f defined in Example 3.5, is weakly* (γ, γ')-(β, β')-continuous.

At this point, there naturally arises the question whether there exists any relationship between weakly (γ, γ')-(β, β')-continuous functions and weakly* (γ, γ')-(β, β')-continuous functions. The following examples show that the notions of weakly (γ, γ')-(β, β')-continuous functions and weakly* (γ, γ')-(β, β')-continuous functions are independent of each other.

Example 3.12. Let $X = \{a, b\}$, $\tau = \{\emptyset, X\}$, $\sigma = \mathcal{P}(X)$ and γ, γ' be identity operations defined on τ and let β, β' be also identity operations defined on σ. Then the identity function $f : (X, \tau, \gamma, \gamma') \to (X, \sigma, \beta, \beta')$ is weakly (γ, γ')-(β, β')-continuous, but not weakly* (γ, γ')-(β, β')-continuous.

Example 3.13. The function f defined in Example 3.11 is weakly* (γ, γ')-(β, β')-continuous, but is not weakly (γ, γ')-(β, β')-continuous.

Theorem 3.14. If $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is a (γ, γ')-(β, β')-continuous function, then it is weakly* (γ, γ')-(β, β')-continuous.

Proof. It is straightforward. $\square$

Remark 3.15. The converse of Theorem 3.14 is not true, as we can see in the following Example.

Example 3.16. The function $f : (X, \tau, \gamma, \gamma') \to (X, \sigma, \beta, \beta')$ defined in Example 3.12 is weakly* (γ, γ')-(β, β')-continuous, but not (γ, γ')-(β, β')-continuous.

We have the following implication diagram

$$\begin{array}{c} \text{weakly } (\gamma, \gamma')\text{-}(\beta, \beta')\text{-continuous} \\ \uparrow \\ (\gamma, \gamma')\text{-}(\beta, \beta')\text{-continuous} \\ \downarrow \\ \text{weakly}^* \ (\gamma, \gamma')\text{-}(\beta, \beta')\text{-continuous.} \end{array}$$

Theorem 3.17. A function $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is (γ, γ')-(β, β')-continuous if and only if it is both weakly (γ, γ')-(β, β')-continuous and weakly* (γ, γ')-(β, β')-continuous.

Proof. Let $x \in X$ and $f(x) \in G \in \sigma_{(\beta,\beta')}$. Since f is weakly (γ, γ')-(β, β')-continuous, there exists $F \in \tau_{(\gamma,\gamma')}$ containing x such that $f(F) \subset (\beta, \beta')\text{-Cl}(G)$. Thus $f(x) \notin (\beta, \beta')\text{-}Fr(G)$. Hence $x \notin f^{-1}((\beta, \beta')\text{-}Fr(G))$ and $F - f^{-1}((\beta, \beta')\text{-}Fr(G)) \in \tau_{(\gamma,\gamma')}$ containing x since f is weakly* (γ, γ')-(β, β')-continuous. Let $y \in F \subset f^{-1}((\beta, \beta')\text{-}Fr(G))$. Then $y \in F$ and so $f(y) \in (\beta, \beta')\text{-Cl}(G)$. But $y \notin f^{-1}((\beta, \beta')\text{-}Fr(G))$ and hence $f(y) \notin (\beta, \beta')\text{-}Fr(G)$. Thus $f(y) \in (\beta, \beta')\text{-Int}(G) \subset G$. Finally, we obtain f is a (γ, γ')-(β, β')-continuous function. The converse is obvious. $\square$

Proposition 3.18. If $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is a (γ, γ')-(β, β')-precontinuous function and satisfies (γ, γ')-$\mathrm{Cl}(f^{-1}(G)) \subset f^{-1}((\beta, \beta')$-$\mathrm{Cl}(G))$ for each $G \in \sigma_{(\beta,\beta')}$, then f is weakly (γ, γ')-(β, β')-continuous.

Proof. Let $x \in X$ and $G \in \sigma_{(\beta,\beta')}$ containing $f(x)$. By the hypothesis, we have (γ, γ')-$\mathrm{Cl}(f^{-1}(G)) \subset f^{-1}((\beta, \beta')$-$\mathrm{Cl}(G))$. Since f is (γ, γ')-precontinuous, $x \in f^{-1}(G) \subset (\gamma, \gamma')$-$\mathrm{Int}((\gamma, \gamma')$-$\mathrm{Cl}(f^{-1}(G)))$ and so there exists $F \in \tau_{(\gamma,\gamma')}$ containing x such that $F \subset (\gamma, \gamma')$-$\mathrm{Cl}(f^{-1}(G))$. Thus we have $f(F) \subset (\beta, \beta')$-$\mathrm{Cl}(G)$. Hence f is a weakly (γ, γ')-(β, β')-continuous function. □

Conclusions

(1) The notions of weakly (γ, γ')-(β, β')-continuous functions and weakly* (γ, γ')-(β, β')-continuous functions are independent of each other.
(2) All $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$, that is, (γ, γ')-(β, β')-continuous function, is weakly (γ, γ')-(β, β')-continuous.
(3) All $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$, that is, (γ, γ')-(β, β')-continuous function, is weakly* (γ, γ')-(β, β')-continuous.
(4) $f : (X, \tau, \gamma, \gamma') \to (Y, \sigma, \beta, \beta')$ is (γ, γ')-(β, β')-continuous function if and only if it is both weakly (γ, γ')-(β, β')-continuous function and weakly* (γ, γ')-(β, β')-continuous function.

References

1. C. Carpintero, N. Rajesh, E. Rosas, On a class of (γ, γ')-preopen sets in a topological space. *Fasc. Math.* **46** (2011), 25–35.
2. C. Carpintero, N. Rajesh, E. Rosas, On decomposition of bioperation-continuity. *C. R. Acad. Bulgare Sci.* **73** (2020), no. 4, 443–450.
3. S. Kasahara, Operation-compact spaces. *Math. Japon.* **24** (1979/80), no. 1, 97–105.
4. J. Umehara, H. Maki, T. Noiri, Bioperations on topological spaces and some separation axioms. *Mem. Fac. Sci. Kochi Univ. Ser. A Math.* **13** (1992), 45–59.

(Received 13.09.2021)

[1]Facultad de Ciencias Básicas Corporación Universitaria del Caribe-CECAR Sincelejo, Colombia

[2]Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005 Tamilnadu, India

[3]Departamento de Ciencias Naturales y Exactas, Universidad de la Costa, Barranquilla-Colombia
Email address: carpintero.carlos@gmail.com
Email address: nrajesh_topology@yahoo.co.in
Email address: ennisrafael@gmail.com