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THE INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS AND ENTIRE FUNCTIONS OF FINITE ORDER IN GENERALIZED WEIGHTED GRAND LEBESGUE SPACES

TAMUNA GORGADZE[1], VAKHTANG KOKILASHVILI[2] AND DALI MAKHARADZE[3]

Abstract. We consider generalized weighted grand Lebesgue spaces and establish Bernstein type inequalities for trigonometric polynomials and Nikol'skii type inequalities for entire function of finite order in these spaces.

1. Introduction

Let $1 < p < \infty$ and let φ be a positive, non-decreasing function on $(0, p-1)$ with the condition $\varphi(0+) = 0$. The set of all such functions we denote by Φ_p.

By w we denote a weight function, almost everywhere positive, locally integrable and defined on $\mathbb{R}^1$.

We consider two types of generalized weighted grand Lebesgue spaces. Let X be a bounded set in $\mathbb{R}^n$.

The space $L_w^{p),\varphi}(X)$ is defined as a set of all measurable functions for which the norm

$$\|f\|_{L_w^{p),\varphi}} = \sup_{0<\varepsilon<p-1} \left(\varphi(\varepsilon) \int_X |f(x)|^{p-\varepsilon} w(x) dx \right)^{\frac{1}{p-\varepsilon}}$$

is finite.

The space $\mathcal{L}_w^{p),\varphi}(X)$ is defined as a set of all measurable functions for which the norm

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is finite.

Both these spaces are non-reflexive, non-separable Banach function spaces. Grand Lebesgue spaces $L^{p)}$ were introduced in 1992 by T. Iwaniec and G. Sbordone [2] in their studies related to the integrability properties of the Jacobian in a bounded open set. This is the case where $\varphi(\varepsilon) = \varepsilon$ and $w(x) \equiv 1$. More later, the space $L^{p),\theta}$ for an arbitrary positive θ (the case for $\varphi(\varepsilon) = \varepsilon^\theta$ and $w(x) \equiv 1$) was introduced in their paper by L. Greco, T. Iwaniec and C. Sbordone [1]. Nowadays, the theory of grand Lebesgue spaces is one of intensively developing directions of modern analysis. It turns out that in the theory of partial differential equations, the generalized grand Lebesgue spaces are appropriate to the problems of the existence and uniqueness, and also to the regularity problems for various nonlinear partial differential equations.

Further, we will need the definitions of Muckenhoupt classes of weight functions.

Definition 1.1. A weight function $w(x)$ belongs to the class $A_p (1 < p < \infty)$, if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w^{1-p'}(x) dx \right)^{p-1} < \infty, \ \ p' = \frac{p}{p-1},$$

where I denotes an arbitrary interval on $\mathbb{R}^1$.

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Definition 1.2. A weight function $w(x)$ belongs to the class $A_{p,q}(1<p<q<\infty)$, if

$$\sup_{I\in\mathbb{R}^1}\left(\frac{1}{|I|}\int\limits_I w^q(x)dx\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int\limits_I w^{-p'}(x)dx\right)^{\frac{1}{p'}}<\infty.$$

For the multidimensional case, we need the following

Definition 1.3. A weight function $w(x,y)$ is said to be of class $\mathcal{A}_p$, if

$$\sup\left(\frac{1}{|J|}\int\limits_J w(x,y)dxdy\right)\left(\frac{1}{|J|}\int\limits_J w^{1-p'}(x,y)dxdy\right)^{p-1}<\infty,$$

where the supremum is taken over all rectangles J in $\mathbb{R}^2$ with the edges, parallel to the coordinate axis.

One-weight Bernstein and Nikol'skii type inequalities in the classical weighted grand Lebesgue spaces under the Muckenhoupt weights were established in [3] and [4].

2. Bernstein Type Inequalities in $\mathbf{L}_{\mathrm{w}}^{\mathbf{p}),\varphi}$ and $\mathcal{L}_{\mathrm{w}}^{\mathbf{p}),\varphi}$ for Trigonometric Polynomials

Theorem 2.1. *Let $1<p<\infty$, $\varphi\in\Phi_p$. Suppose $\alpha(\alpha>0)$ denotes the order of Weyl fractional derivatives.*

i) *If $w\in A_p$, then the inequality*

$$\left\|T_n^{(\alpha)}\right\|_{L_w^{p),\varphi}(-\pi,\pi)}\le cn^a\left\|T_n\right\|_{L_w^{p),\varphi}(-\pi,\pi)}$$

holds for arbitrary polynomials T_n with a constant c, independent of T_n.

ii) *If $w^p\in A_p$, then for arbitrary trigonometric polynomials T_n, we have*

$$\left\|T_n^{(\alpha)}\right\|_{\mathcal{L}_w^{p),\varphi}(-\pi,\pi)}\le cn^a\left\|T_n\right\|_{\mathcal{L}_w^{p),\varphi}(-\pi,\pi)}.$$

In the sequel, by $T_{mn}(x,y)$ we denote two-dimensional trigonometric polynomial of order m with respect to the variable x and of order n with respect to y. We set $Q:=(-\pi,\pi)\times(-\pi,\pi)$.

Theorem 2.2. *Let $1<p<\infty,\varphi\in\Phi_p$. Let $\alpha>0$ and $\beta>0$ be the orders of Weyl fractional derivatives. The following statements are true:*

i) *If $w\in\mathcal{A}_p$, then the inequality*

$$\left\|\frac{\partial^{\alpha+\beta}T_{mn}(x,y)}{\partial x^\alpha\partial y^\beta}\right\|_{L_w^{p),\varphi}(Q)}\le c_1m^\alpha n^\beta\left\|T_{mn}\right\|_{L_w^{p),\varphi}(Q)}$$

holds for arbitrary trigonometric polynomials $T_{mn}(x,y)$.

ii) *If $w^p\in\mathcal{A}_p$, then*

$$\left\|\frac{\partial^{\alpha+\beta}T_{mn}(x,y)}{\partial x^\alpha\partial y^\beta}\right\|_{\mathcal{L}_w^{p),\varphi}(Q)}\le c_1m^\alpha n^\beta\left\|T_{mn}\right\|_{\mathcal{L}_w^{p),\varphi}(Q)}$$

with the constant, independent of m,n and T_{mn}.

3. Nikol'skii Type Inequalities

Theorem 3.1. *Let $1<p<q<\infty$ and $\varphi\in\Phi_p$. We set $\varphi_1=\varphi^{\frac{q}{p}}$. Suppose that $w\in A_{p,q}$. Then for arbitrary trigonometric polynomials T_n, we have*

$$\left\|T_n\right\|_{\mathcal{L}_w^{q),\varphi_1}}\le cn^{\frac{1}{p}-\frac{1}{q}}\left\|T_n\right\|_{\mathcal{L}_w^{p),\varphi}}.$$

Theorem 3.2. *Let $1<p<q<\infty$ and $\varphi\in\Phi_p$. We set $\varphi_1=\varphi^{\frac{q}{p}}$. Assume that $w\in A_{1+\frac{q}{p'}}$. Then for arbitrary trigonometric polynomials T_n, we have*

$$\left\|T_nw^\gamma\right\|_{L_w^{q),\varphi_1}}\le cn^{\frac{1}{p}-\frac{1}{q}}\left\|T_n\right\|_{L_w^{p),\varphi}},$$

where

$$\gamma = \frac{1}{p} - \frac{1}{q}.$$

4. The Inequalities for Entire Functions of Finite Order

First of all, we define the space suited to the real line. The generalized weighted grand Lebesgue space on the real line is defined via two weights w and v. This space is defined as a set of all measurable functions on $\mathbb{R}^1$ for which

$$\|f\|_{L_v^{p),\varphi}(\mathbb{R}^1,w)} = \sup_{0<\varepsilon<p-1} \left\{ \varphi(\varepsilon) \int\limits_{\mathbb{R}^1} |f(x)|^{p-\varepsilon} w(x) v^{\varepsilon}(x) dx \right\}^{\frac{1}{p-\varepsilon}} < \infty$$

where $wv^{\varepsilon} \in L_w^1\left(\mathbb{R}^1\right)$ for all $0 < \varepsilon < p-1$.

The grand Lebesgue spaces over a set of infinite measure were introduced in [5].

Theorem 4.1. *Let $1 < p < \infty, \varphi \in \Phi_p$ and $w \in A_p, v^{\gamma} \in A_p$ for some $\gamma > 0$. Then for an arbitrary entire function f of order τ from $L_v^{p),\varphi}\left(\mathbb{R}^1, w\right)$, we have*

$$\|f'\|_{L_v^{p),\varphi}(\mathbb{R}^1,w)} \le c\tau \|f\|_{L_v^{p),\varphi}(\mathbb{R}^1,w)}$$

with a constant c, independent of f.

For the Nikol'skii type inequalities, it is more convenient to address to the other definition of $L_w^{p),\varphi}, \mathcal{L}_w^{p),\varphi}$ on the real line.

$$L_w^{p),\varphi}(\mathbb{R}^1) = \left\{ f : \left[\sup_{0<\varepsilon<\sigma} \varphi(\varepsilon) \int\limits_{\mathbb{R}^1} |f(x)|^{p-\varepsilon} w(x) dx \right]^{\frac{1}{p-\varepsilon}} < \infty \right\},$$

$$\mathcal{L}_w^{p),\varphi}(\mathbb{R}^1) = \left\{ f : \left[\sup_{0<\varepsilon<\sigma} \varphi(\varepsilon) \int\limits_{\mathbb{R}^1} |f(x) w(x)|^{p-\varepsilon} dx \right]^{\frac{1}{p-\varepsilon}} < \infty \right\},$$

where σ is a small positive constant.

Theorem 4.2. *Let $1 < p < q < \infty$ and $\varphi \in \Phi_p$. We set $\varphi_1 = \varphi^{\frac{q}{p}}$. Suppose that $w \in A_{1+\frac{q}{p'}}$. Then for an arbitrary function f of order τ, we have*

$$\left\| f w^{\frac{1}{p}-\frac{1}{q}} \right\|_{L_w^{q),\varphi_1}} \le c\tau^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_w^{p),\varphi}}$$

with a constant c, independent of f.

Theorem 4.3. *Let $1 < p < q < \infty$, $\varphi \in \Phi_p$. We set $\varphi_1 = \varphi^{\frac{q}{p}}$. Suppose that $w \in A_{p,q}$. Then the inequality*

$$\|f\|_{\mathcal{L}_w^{q),\varphi_1}} \le c\tau^{\frac{1}{p}-\frac{1}{q}} \|f\|_{\mathcal{L}_w^{p),\varphi}}$$

holds for an arbitrary function f of order τ, where the constant c is independent of f.

References

1. L. Greco, T. Iwaniec, C. Sbordone, Inverting the p-harmonic operator. *Manuscripta Math.* **92** (1997), no. 2, 249–258.
2. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses. *Arch. Rational Mech. Anal.* **119** (1992), no. 2, 129–143.
3. V. Kokilashvili, A. Meskhi, Some fundamental inequalities for trigonometric polynomials and imbeddings of grand Besov spaces. *Proc. A. Razmadze Math. Inst.* **165** (2014), 105–116.
4. V. Kokilashvili, A. Meskhi, Weighted extrapolation in Iwaniec-Sbordone spaces. Applications to integral operators and approximation theory. (Russian) *Tr. Mat. Inst. Steklova* **293** (2016), Funktsional. Prostranstva, Teor. Priblizh., Smezhnye Razdely Mat. Anal., 167–192; *translation in Proc. Steklov Inst. Math.* **293** (2016), no. 1, 161–185.
5. S. Samko, S. M. Umarkhadzhiev, On Iwaniec-Sbordone spaces on sets which may have infinite measure. *Azerb. J. Math.* **1** (2011), no. 1, 67–84.

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[1]Batumi Shota Rustaveli State University, 35/32 Ninoshvili/Rustaveli Str., Batumi 6010, Georgia

[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

[3]Batumi Shota Rustaveli State University, 35/32 Ninoshvili/Rustaveli Str., Batumi 6010, Georgia
Email address: 61009029720@bsu.ge
Email address: dali.makharadze@bsu.edu.ge