Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 2, 255–266

# NORMAL STRUCTURE IN MODULAR SPACES

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**Abstract.** In this paper, we extend some concepts from geometry of Banach spaces to modular spaces. We prove that the compact convex subsets of any $\rho-$complete modular space, where $\rho$ is convex, continuous and satisfies the $\triangle_2-$condition, and the closed convex subsets of any $\rho-$complete modular space, where $\rho$ is convex and satisfies the (UC1), have normal structure. Specially, we prove that with an essential condition, Chebysheve centers exist.

## 1. Introduction

In the early of 1930, Orlicz and Birnbaum attempted to generalize the Lebesgue function space $L^p$. They studied the function spaces

$$L^{\Phi}=\left\{f:R\longrightarrow R:\ \exists\lambda>0\ \text{ such that } \int\limits_{R}\Phi\big(\lambda|f(x)|\big)dx<\infty\right\},$$

where $\Phi$ acts similarly to a power function $\Phi(t)=t^p$. After that, the convexity assumption on $\Phi$ was omitted. Many applications to differential and integral equations with kernels of nonpower types were good causes for the development of the theory of Orlicz spaces (see, e.g., [11]).

We observe two principal directions of further development. The first one is a theory of Banach function spaces initiated in 1955 by Luxemburg and then developped in a series of joint papers with Zaanen (see, e.g., [12]). The other way, also inspired by the successful theory of Orlicz spaces, is based on replacing the particular integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. This idea was the basis of the theory of modular spaces initiated by Nakano [15] in 1950, in connection with the theory of order spaces and redefined and developed by Orlicz and Musielak in 1959 ( [14]). Even if a metric is not defined, many results in metric fixed point theory can be reformulated in modular spaces, we refer, for instance, to [2, 10].

To manage the pathological behavior of modular in modular spaces, some conditions such as the $\triangle_2-$condition and the Fatou property are usually presumed (see, e.g., [2, 6, 8, 9, 14, 16]). For instance, in [2], Banach's fixed point theorem is given in modular spaces with their modular satisfying both the $\triangle_2-$condition and the Fatou property.

On the other hand, normal structure is one of the basic concepts in the metric fixed point theory. It was introduced by Brodskii and Milman in [3]. In 1980, Bynum [4] introduced the normal structure coefficient $N(X)$ which was applied by Casini and Maluta [5] to obtain a fixed point theorem for a uniformly Lipschitzian mapping. The important application of normal structure is in the fixed point theory and other fields related to the existence of a solution of differential equations and integral equations (see, e.g., [1]).

In this paper, we present and discuss some geometric properties of modular spaces. Namely, we prove that compact convex subsets of any $\rho-$complete modular space where $\rho$ is convex, continuous and satisfies the $\triangle_2-$condition and closed convex subsets of any $\rho-$complete modular space where $\rho$ is convex and satisfies the (UC1), have normal structure and we put an essential condition for the existence of Chebysheve centers.

Also, we give some important properties of the normal structure coeffcients.

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2020 *Mathematics Subject Classification.* 46A80, 46H10.

*Key words and phrases.* Modular spaces; Normal structure; Chebysheve center; Normal structure coefficient.

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## 2. Preliminaries

We begin with recalling some basic facts about modular spaces.

**Definition 2.1.** Let $X$ be a vector space over $K$ ($K = \mathbb{C}$ or $= \mathbb{R}$). A functional $\rho : X \longrightarrow [0, +\infty)$ is called a modular on $X$ if for arbitrary elements $x$ and $y$ of $X$, the following is satisfied:
(1) $\rho(x) = 0$ if and only if $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$ for every $\alpha \in K$ with $|\alpha| = 1$,
(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
It is easy to see that we have $\rho(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \rho(x_1) + \cdots + \rho(x_n)$ for every $\alpha_i \geq 0$ with $\sum\limits_{i=1}^{n} \alpha_i = 1$.

If we replace (3) by

$(3)'$ $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ for every $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with an $s \in (0, 1]$, the modular $\rho$ is called $s-$convex. $1-$convex modular is called convex modular. For a modular $\rho$ on $X$, one can associate a modular space $X_\rho$ defined as

$$X_\rho = \big\{x \in X;\ \lim_{\lambda \to 0} \rho(\lambda x) = 0\big\}.$$

$X_\rho$ is a linear subspace of $X$. Using the modular $\rho$, one can define an $F-$norm [13] on $X_\rho$ by

$$|x|_\rho = \inf \Big\{t > 0;\ \rho\Big(\frac{x}{t}\Big) \leq t\Big\}.$$

If $\rho$ is convex, then

$$\|x\|_\rho = \inf \Big\{t > 0;\ \rho\Big(\frac{x}{t}\Big) \leq 1\Big\}$$

is a norm on $X_\rho$, frequently called the Luxemburg norm [13]. One can also check that $|f_n - f|_\rho \longrightarrow 0$ is equivalent to $\rho\big(\alpha(f_n - f)\big) \longrightarrow 0$ for all $\alpha > 0$.

**Definition 2.2.** Let $X$ be a vector space and $\rho$ be a convex modular defined on $X$.

(a) We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in $X_\rho$ is $\rho-$convergent to $x$ and write $x_n \xrightarrow{\rho} x$ if and only if $\rho(x_n - x) \to 0$ as $n \to \infty$.
(b) A sequence $(x_n)_{n\in\mathbb{N}}$ in $X_\rho$ is called $\rho-$Cauchy whenever $\rho(x_n - x_m) \to 0$ as $m, n \to \infty$.
(c) $X$ is called $\rho-$complete if any $\rho-$Cauchy sequence is $\rho-$convergent.
(d) A subset $B \subset X_\rho$ is called $\rho-$closed if for any sequence $(x_n) \subset B$, $\rho-$convergent to $x \in X_\rho$, we have $x \in B$.
(e) A subset $B \subset X_\rho$ is called $\rho-$bounded if its $\rho-$diameter defined as $\mathrm{diam}(B) = \sup\big\{\rho(x - y) :\ x, y \in B\big\}$ is finite .
(f) $\rho$ is said to satisfy the $\triangle_2-$condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.
(h) $\rho$ is said to satisfy the Fatou property if

$$\rho(x - y) \leq \liminf_{n\to\infty} \rho(x_n - y_n)$$

whenever

$$x_n \xrightarrow{\rho} x \quad \text{and} \quad y_n \xrightarrow{\rho} y \quad \text{as} \quad n \to \infty.$$

### Examples

**Example 2.1** ([10])**.** Let $X = L^p\big([a, b]\big)$ and $\rho(x) = \int\limits_a^b \big|x(t)\big|^p dt$ for $p > 0$, then
(1) $\rho$ is $p-$convex modular for $p < 1$.
(2) $\rho$ is convex modular for $p \geq 1$.
(3) $\rho$ satisfies the $\triangle_2-$condition.

**Example 2.2.** Let $X = \mathbb{R}$. Then $\rho(x) = x^2$ is a modular on $\mathbb{R}$ and satisfies the $\triangle_2-$condition and the Fatou property. It is clear that $(\mathbb{R}, \rho)$ is $\rho-$ *complete* because $\mathbb{R}$ with usual norm defined by an absolute value is a Banach space.

**Remark 2.1.** Note that the family $B = \{B_\rho(0,r), r > 0\}$, where $B_\rho(0,r) = \{x \in X_\rho;\ \rho(x) < r\}$ is a filter base and any element of $B$ is balanced and absorbing. Furthermore, if $\rho$ is convex, then any element of $B$ is convex (see [7]).

**Definition 2.3** ([7])**.** We say that $\rho$ satisfies the property $\mathcal{T}_0$ if for all $\varepsilon > 0$, there exist $L > 0$ and $\delta > 0$ such that $|\rho(x) - \rho(y)| < \varepsilon$ for every $x$, $y$ satisfying $\rho(x) < L$ and $\rho(x-y) < \delta$.

**Remark 2.2.** Note that if the modular $\rho$ satisfies the property $\mathcal{T}_0$, then $X_\rho$ is a separated topological vector space. Also, $\rho$ satisfies the $\triangle_2-$condition if and only if $\rho$ satisfies the property $\mathcal{T}_0$ (see [7]).

The following is another important concept of uniform convexity in normed spaces that generates several different types of uniform convexity in modular spaces which played a major role in the study of normal structure in modular spaces.

**Definition 2.4** ([10])**.** Let $X_\rho$ be a $\rho-$modular space. We define the following uniform convexity type properties of the modular $\rho$:

a) Let $r > 0$, $\varepsilon > 0$. Define
$$D_1(r,\varepsilon) = \{(x,y) :\ x, y \in X_\rho,\ \rho(x) \le r,\ \rho(y) \le r,\ \rho(x-y) \ge \varepsilon r\}.$$
Let $\delta_1(r,\varepsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{x+y}{2}) : (x,y) \in D_1(r,\varepsilon)\}$, if $D_1(r,\varepsilon) \neq \emptyset$, and $\delta_1(r,\varepsilon) = 1$ if $D_1(r,\varepsilon) = \emptyset$. We say that $\rho$ satisfies (UC1) if for every $r > 0$, $\varepsilon > 0$, we have $\delta_1(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

b) We say that $\rho$ satisfies (UUC1) if for every $s \ge 0$, $\varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$ depending on $s$ and $\varepsilon$ such that
$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0 \quad \text{for } r > s.$$

c) Let $r > 0$, $\varepsilon > 0$. Define
$$D_2(r,\varepsilon) = \Big\{(x,y) :\ x, y \in L_\rho,\ \rho(x) \le r,\ \rho(y) \le r,\ \rho\Big(\frac{x-y}{2}\Big) \ge \varepsilon r\Big\}.$$
Let $\delta_2(r,\varepsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{x+y}{2}) : (x,y) \in D_2(r,\varepsilon)\}$, if $D_2(r,\varepsilon) \neq \emptyset$, and $\delta_2(r,\varepsilon) = 1$ if $D(r,\varepsilon) = 1$. We say that $\rho$ satisfies (UC2) if for every $r > 0$, $\varepsilon > 0$, we have $\delta_2(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_2(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

d) We say that $\rho$ satisfies (UUC2) if for every $s \ge 0$, $\varepsilon > 0$ there exists $\eta_2(s,\varepsilon) > 0$ depending on $s$ and $\varepsilon$ such that
$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \quad \text{for } r > s.$$

**Remark 2.3.** Note that the following relationships between the above defined notions exist:

a) (UUCi) implies (UCi) for $i = 1, 2$;
b) $\delta_1(r,\varepsilon) \le \delta_2(r,\varepsilon)$;
c) (UC1) implies (UC2);
d) (UUC1) implies (UUC2) (see [10]).

## 3. Main Results

**Definition 3.1.** Let $C$ be a nonempty $\rho-$bounded subset of a $\rho-$complete modular space $X_\rho$. Then a point $x_0 \in C$ is said to be:

i) a diametral point of $C$ if
$$\sup\{\rho(x_0 - x) :\ x \in C\} = \operatorname{diam} C;$$

ii) a nondiametral point of $C$ if
$$\sup\{\rho(x_0 - x) :\ x \in C\} < \operatorname{diam} C.$$

A nonempty convex subset $C$ of a $\rho-$complete modular space $X_\rho$ is said to have normal structure if each convex $\rho-$bounded subset $D$ of $C$ with at least two points contains a nondiametral point, i.e., there exists $x_0 \in D$ such that
$$\sup\{\rho(x_0 - x) :\ x \in D\} < \operatorname{diam} D.$$

The $\rho-$complete modular space $X_\rho$ is said to have normal structure if every $\rho-$closed convex $\rho-$bounded subset $C$ of $X_\rho$ with $\operatorname{diam} C > 0$ has normal structure.

The following theorems state that compact convex subsets of any $\rho-$complete modular space where $\rho$ is convex, continuous and satisfies the $\triangle_2-$condition and closed convex subsets of any $\rho-$complete modular space where $\rho$ is convex and satisfies the (UC1), have this geometric property.

**Theorem 3.1.** *Let $X_\rho$ be a $\rho-$complete modular space where $\rho$ is convex, continuous and satisfies the $\triangle_2-$condition. Then every compact convex subset $C$ of $X_\rho$ has normal structure.*

*Proof.* Suppose, for contradiction, that $C$ does not have normal structure. Then there exists a convex $\rho-$bounded subset of $C$ as $D$ with at least two points such that all points of $D$ are diametral. Now we construct a sequence $\{x_i\}_{i=1}^{\infty}$ in $D$ such that

$$\rho(x_i - x_j) = \operatorname{diam} D \quad \text{for all } i, j \in \mathbb{N},\ i \neq j.$$

For this, let $x_1$ be an arbitrary point in $D$. Since all points of $D$ are diametral, therefore $\operatorname{diam} D = \sup\limits_{x\in D} \rho(x_1 - x)$. Then there exists a point $x_2 \in D$ such that $\operatorname{diam} D = \rho(x_1 - x_2)$. As $D$ is convex, then $\frac{x_1+x_2}{2} \in D$. Next, we choose a point $x_3 \in D$ such that $\operatorname{diam} D = \rho\big(x_3 - \frac{x_1+x_2}{2}\big)$. Proceeding in the same manner, we obtain a sequence $\{x_n\}$ in $D$ such that $\operatorname{diam} D = \rho\big(x_{n+1} - \sum\limits_{i=1}^{n} \frac{x_i}{n}\big)$, $n \geq 2$. As $\rho$ is convex, so,

$$\begin{aligned}
\operatorname{diam} D =& \rho\Big(x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n}\Big)\\
=& \rho\Big(\frac{(x_{n+1} - x_1) + (x_{n+1} - x_2) + \cdots + (x_{n+1} - x_n)}{n}\Big)\\
\leq& \frac{1}{n}\Big(\sum_{i=1}^{n} \rho(x_{n+1} - x_i)\Big)\\
\leq& \frac{1}{n}\sum_{i=1}^{n} \operatorname{diam} D\\
=& \operatorname{diam} D,
\end{aligned}$$

and then $\operatorname{diam}(D) = \rho(x_{n+1} - x_i)$, $1 \leq i \leq n$.

Since $\{x_n\} \subset D \subset C$ and $C$ is compact subset of $X_\rho$, there exists a subsequence $\{x_{n_k} = x'_k\}$ and an $a \in C$ such that $\rho(x_{n_k} - x) \to 0$. Therefore

$$\begin{aligned}
\operatorname{diam} D =& \rho(x'_{k+1} - x'_i) = \rho(x'_{k+1} - a + a - x'_i)\\
=& \rho\Big(\frac{1}{2}\big(2(x'_{k+1} - a)\big) + \frac{1}{2}\big(2(a - x'_i)\big)\Big)\\
\leq& \rho\big(2(x'_{k+1} - a)\big) + \rho\big(2(a - x'_i)\big).
\end{aligned}$$

Since $\rho$ satisfies the $\triangle_2-$condition, therefore $\rho\big(2(x'_{k+1} - a)\big) \xrightarrow{k\to\infty} 0$. Thus $\operatorname{diam} D \leq 0$. This implies that the sequence $\{x_n\}$ has no convergent subsequences. This contradicts the compactness of $C$. $\square$

**Theorem 3.2.** *Let $X_\rho$ be a $\rho-$complete modular space where $\rho$ is convex and satisfies* (UC1). *Then every convex $\rho-$closed, $\rho-$bounded subset $C$of $X_\rho$ has normal structure.*

*Proof.* Let $D$ be a $\rho-$bounded convex subset of $C$ with $\operatorname{diam} D = d > 0$. Let $x_1$ be an arbitrary point in $D$. Choose a point $x_2 \in D$ such that $\rho(x_1 - x_2) \geq d - \frac{d}{2} = \frac{d}{2}$. Because $D$ is convex, $\frac{x_1+x_2}{2} \in D$. Set $x_0 = \frac{x_1+x_2}{2}$. Since $\rho$ satisfies $(UC1)$, there is $\delta_1(r, \varepsilon) > 0$ for every $r > 0$, $\varepsilon > 0$. Therefore, if $\rho(x) \leq r$, $\rho(y) \leq r$, $\rho(x - y) \geq \varepsilon r$, then

$$\rho\Big(\frac{x + y}{2}\Big) \leq r\big(1 - \delta_1(r, \varepsilon)\big). \tag{1}$$

Put $r = d$, $\varepsilon = \frac{1}{2}$. Since $x_1, x_2 \in D$, $\rho(x_1) \leq d = r$, $\rho(x_2) \leq d = r$ and $\rho(x_1 - x_2) \geq \frac{d}{2} = d \times \frac{1}{2} = r\varepsilon$, hence from (1), there exists $\delta_1(r,\varepsilon) = \delta_1(d, \frac{1}{2}) > 0$ such that $\rho\big(\frac{x_1+x_2}{2}\big) \leq r\big(1 - \delta_1(r,\varepsilon)\big) = d\big(1 - \delta_1\big(d, \frac{1}{2}\big)\big)$. Hence for $x \in D$, we have

$$\begin{aligned}\rho(x - x_0) =&\rho\Big(x - \frac{x_1 + x_2}{2}\Big)\\ =&\rho\Big(\frac{(x - x_1) + (x - x_2)}{2}\Big)\\ \leq& d\Big(1 - \delta_1\big(d, \frac{1}{2}\big)\Big) < d, \qquad \Big(\delta_1\big(d, \frac{1}{2}\big) > 0\Big).\end{aligned}$$

Consequently,

$$\sup\big\{\rho(x - x_0) : \ x \in D\big\} < d = \operatorname{diam} D.$$

□

**Theorem 3.3.** *Let $X_\rho$ be a $\rho-$complete modular space where $\rho$ is convex and satisfies $(UC1)$. Then $X_\rho$ has normal structure.*

*Proof.* This follows from Theorem 3.2. □

The following notion plays an important role in the study of normal structure.

**Definition 3.2.** A $\rho-$bounded $(x_n)$ in $\rho-$complete modular space is said to be a diametral sequence if

$$\lim_{n\to\infty} \rho\big(x_{n+1}, \operatorname{co}(\{x_1, x_2, \ldots, x_n\})\big) = \operatorname{diam}(\{x_n\}),$$

where $\rho(x, A) = \inf\limits_{y\in A} \rho(x - y)$.

The following result gives an important fact relating to normal structure and nondiametral sequence.

**Proposition 3.1.** *A convex $\rho-$bounded subset $C$ of a $\rho-$complete modular space $X_\rho$ has normal structure if and only if it does not contain a diametral sequence.*

*Proof.* Suppose that $C$ contains a diametral sequence $\{x_n\}$. Then the set $C^{'} = \operatorname{co}(\{x_n\})$ is a convex subset of $C$ and each point of $C^{'}$ is a diametral point. Thus $C$ fails to have normal structure. Let $x_0$ be an arbitrary point in $C^{'}$. Therefore there exists $N_0$ such that for every $n \geq N_0$, we have $x_0 \in \operatorname{co}(x_1, x_2, \ldots, x_n)$. Hence

$$\begin{aligned}\operatorname{diam}(x_n) = \operatorname{diam} C^{'} = \sup_{u,v\in C'} \rho(u - v) \geq& \sup_{u\in C'} \rho(u - x_0)\\ \geq&\rho(x_{n+1} - x_0)\\ \geq&\rho\big(x_{n+1}, \operatorname{co}(x_1, x_2, \ldots, x_n)\big).\end{aligned}$$

Since the sequence $\{x_n\}$ is diametral sequence, we have

$$\begin{aligned}\operatorname{diam}(x_n) = \operatorname{diam} C^{'} = \sup_{u\in \operatorname{co}} \rho(u - x_0) =& \lim_{n\to\infty} \rho(x_{n+1} - x_0)\\ =& \lim_{n\to\infty} \rho\big(x_{n+1}, \operatorname{co}(x_1, x_2, \ldots, x_n)\big).\end{aligned}$$

Hence for every $x_0 \in C$, we have

$$\sup_{u\in \operatorname{co}} \rho(u - x_0) = \operatorname{diam} \operatorname{co}.$$

Conversely, suppose that $C$ contains a convex $\rho-$bounded subset $D$ with $d = \operatorname{diam}(D) > 0$ and each point of $D$ is a diametral point. By induction, we construct a sequence $\{x_n\}$ in $D$ such that

$$y_0 = x_1, \qquad y_{n-1} = \sum_{i=1}^{n} \frac{x_i}{n}.$$

Because $y_{n-1}$ is a diametral point in $D$, then for $0 < \varepsilon < d$, there exists an $x_{n+1} \in D$ such that

$$\rho(x_{n+1} - y_{n-1}) > d - \frac{\varepsilon}{n^2}.$$

Suppose $x \in \operatorname{co}\big(\{x_1, x_2, \dots, x_n\}\big)$, say $x = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Set $0 < \lambda := \max\big\{\lambda_1, \lambda_2, \dots, \lambda_n\big\}$. Then

$$\frac{1}{n}\Big(1 - \frac{\lambda_i}{\lambda}\Big) \geq 0 \quad \text{and} \quad \frac{1}{n\lambda} + \frac{1}{n}\sum_{i=1}^{n}\Big(1 - \frac{\lambda_i}{\lambda}\Big) = 1.$$

Hence

$$\frac{1}{n\lambda}x + \frac{1}{n}\sum_{i=1}^{n}\Big(1 - \frac{\lambda_i}{\lambda}\Big)x_i = \frac{1}{n\lambda}x + \frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n\lambda}\sum_{i=1}^{n}\lambda_i x_i = y_{n-1}.$$

Observe that

$$\begin{aligned}
d - \frac{\varepsilon}{n^2} <& \rho(x_{n+1} - y_{n-1}) \\
=& \rho\Big(\frac{1}{n\lambda}(x_{n+1} - x) + \frac{1}{n}\sum_{i=1}^{n}\Big(1 - \frac{\lambda_i}{\lambda}\Big)(x_{n+1} - x_i)\Big) \\
\leq& \frac{1}{n\lambda}\rho(x_{n+1} - x) + \frac{1}{n}\sum_{i=1}^{n}\Big(1 - \frac{\lambda_i}{\lambda}\Big)\rho(x_{n+1} - x_i) \\
\leq& \frac{1}{n\lambda}\rho(x_{n+1} - x) + \frac{1}{n}\sum_{i=1}^{n}\Big(1 - \frac{\lambda_i}{\lambda}\Big)d \\
=& \frac{1}{n\lambda}\rho(x_{n+1} - x) + \Big(1 - \frac{1}{n\lambda}\Big)d.
\end{aligned}$$

Hence

$$\begin{aligned}
\rho(x_{n+1} - x) \geq& \Big(d - \frac{\varepsilon}{n^2} - \big(1 - \frac{1}{n\lambda}\big)d\Big)n\lambda \\
=& n\lambda\Big(\frac{d}{n\lambda} - \frac{\varepsilon}{n^2}\Big) \\
=& d - \frac{\varepsilon\lambda}{n} \geq d - \frac{\varepsilon}{n}, \qquad (2)
\end{aligned}$$

therefore

$$d \geq \rho(x_{n+1} - x) \geq d - \frac{\varepsilon}{n} \qquad (\, x_{n+1}, x \in D).$$

This implies

$$\lim_{n\to\infty} \rho(x_{n+1} - x) = d \quad \text{for every} \quad x \in \operatorname{co}\big(\{x_1, x_2, \dots, x_n\}\big).$$

Now, we show that

$$\lim_{n\to\infty} \rho\big(x_{n+1}, \operatorname{co}(x_1, x_2, \dots, x_n)\big) = d = \operatorname{diam} D = d(x_n) = \operatorname{diam}\big(\{x_n\}\big).$$

Since $\rho(x_n - x_m) \leq d = \operatorname{diam} D$ for every $x_n, x_m \in \{x_n\} \subset D$ and $x \in \operatorname{co}\big(\{x_1, x_2, \dots, x_n\}\big)$ is an arbitrary point in (2), we put $x = x_1$. Therefore

$$\rho(x_2 - x_1) \geq d - \frac{\varepsilon}{1} = d - \varepsilon, \qquad (\, x = x_1 \in \operatorname{co}(x_1) = \{x_1\}).$$

Thus

$$d - \varepsilon \leq \rho(x_2 - x_1) \leq \sup \rho(x_n - x_m) \leq d, \ \text{for any } \varepsilon > 0.$$

So,

$$d(x_n) = \sup \rho(x_n - x_m) = d. \qquad \square$$

**Lemma 3.1.** *Let $X_\rho$ be a $\rho-$complete modular space where $\rho$ is convex and $A \subset X_\rho$. Then* $\operatorname{diam} \operatorname{co}(A) = \operatorname{diam} A$.

*Proof.* Set $\operatorname{diam} A = \delta$. We show that for every $a, b \in \operatorname{co}(A)$, $\rho(a-b) < \delta$.

Let $z$ be an arbitrary point in $A$. Then $A \subset \overline{S(z,\delta)} = \{x : \ \rho(x-z) \le \delta\} = S_1(z)$. We show that $S_1(z)$ is convex. For the given $x_1, x_2 \in S_1(z)$, $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$, we have

$$\begin{aligned}\rho(\lambda_1 x_1 + \lambda_2 x_2 - z) =& \rho(\lambda_1 x_1 + \lambda_2 x_2 - \lambda_1 z - \lambda_2 z)\\ =& \rho\big(\lambda_1(x_1 - z) + \lambda_2(x_2 - z)\big)\\ \le& \lambda_1 \rho(x_1 - z) + \lambda_2 \rho(x_2 - z)\\ \le& \lambda_1 \delta + \lambda_2 \delta = \delta.\end{aligned}$$

Therefore

$$\operatorname{co}(A) \subset S_1(z).$$

Since $a \in \operatorname{co}(A) \subset S_1(z)$, we have $\rho(a-z) = \rho(z-a) \le \delta$. Then $z \in \overline{S(a,\delta)}$. Therefore

$$A \subset \overline{S(a,\delta)} = S_1(a).$$

Since $S_1(a)$ is convex, we have $\operatorname{co}(A) \subset S_1(A)$. Also, since $a, b \in \operatorname{co}(A)$ are arbitrary points, therefore $\operatorname{diam} \operatorname{co}(A) \le \delta = \operatorname{diam}(A)$. Conversely, since $A \subset \operatorname{co}(A)$, we have $\operatorname{diam} A \le \operatorname{diam} \operatorname{co}(A)$. In conclusion, $\operatorname{diam} A = \operatorname{diam} \operatorname{co}(A)$. □

Many spaces satisfy a property, stronger than a normal structure.

**Definition 3.3.** A nonempty convex subset $C$ of a $\rho-$complete modular space is said to have uniformly normal structure if there exists a constant $\alpha \in (0,1)$, independent of $C$, such that each $\rho-$closed convex $\rho-$bounded subset $D$ of $C$ with $\operatorname{diam}(D) > 0$ contains a point $x_0 \in C$ such that

$$\sup \{\rho(x_0 - x) : \ x \in D\} \le \alpha \operatorname{diam} D.$$

**Theorem 3.4.** *Let $X_\rho$ be a $\rho-$complete modular space where $\rho$ is convex and satisfies* (UC1). *Then $X_\rho$ has uniformly normal structure.*

*Proof.* For a $\rho - closed$ convex $\rho-$bounded subset $C$ of $X_\rho$ with $d = \operatorname{diam} C > 0$, from Theorem 3.2, there exists a point $x_0 \in C$ such that $\rho(x - x_0) \le d\left(1 - \delta_1\left(d, \frac{1}{2}\right)\right)$. This implies that

$$\sup \{\rho(x - x_0); \ x \in C\} \le \alpha \operatorname{diam} C,$$

where $\alpha = \left(1 - \delta_1\left(d, \frac{1}{2}\right)\right) < 1$. Therefore $X_\rho$ has uniformly normal structure. □

**Definition 3.4.** Let $C$ be a nonempty $\rho-$bounded subset of a $\rho-$complete modular space $X_\rho$. We adopt the following notations:

$$\begin{aligned} r_x(C) =& \sup\{\rho(x-y) : \ y \in C\}, \ x \in C;\\ r(C) =& \inf\{r_x(C) : \ x \in C\} = \inf\{\sup_{y\in C} \rho(x-y) : \ x \in C\};\\ Z(C) =& \{x \in C : \ r_x(C) = r(C)\};\\ r_X(C) =& \inf\{r_x(C) : \ x \in X\}.\end{aligned}$$

The number $r(C)$ is called the Chebyshev radius of $C$ and the set $Z(C)$ is called the Chebyshev center of $C$. Note that for any $x \in C$,

$$r(C) \le r_x(C) \le \operatorname{diam}(C).$$

The following result gives an essential condition for the existence of Chebyshev centers .

**Proposition 3.2.** *Let $X_\rho$ be a $\rho-$complete modular space, where $\rho$ is convex and satisfies* (UUC2), *$\triangle_2-$condition and Fatou property, and let $C$ be a convex and $\rho-$closed subset of $X_\rho$. Then $Z(C)$ is a nonempty $\rho-$closed convex subset of $C$.*

*Proof.* For $x \in C$, we set

$$C_n(x) := \left\{y \in C, \ \rho(x-y) \le r(C) + \frac{1}{n}\right\}, \quad n \in \mathbb{N}.$$

Then $C_n(x)$ is a nonempty $\rho-$closed convex subset of $C$.

For any $n \geq 1$, $r(C) + \frac{1}{n}$ is not infimum. Then there exists $y_n \in C$ such that $r_{y_n}(C) \leq r(C) + \frac{1}{n}$.
Therefore, for every $z \in C$,

$$\rho(y_n - z) \leq r_{y_n}(C) \leq r(C) + \frac{1}{n}, \quad \forall n \geq 1. \tag{3}$$

Then for $z = x$, we have $\rho(y_n - x) \leq r(C) + \frac{1}{n}$. So, $y_n \in C_n(x)$. Thus $C_n(x) \neq \emptyset$, for any $n \geq 1$.

Let $y \in \overline{C_n(x)}$. Therefore there exists $(y_m) \subset C_n(x)$ such that $y_m \to y$. Then for any $m$, we have

$$\rho(x - y_m) \leq r(C) + \frac{1}{n}. \tag{4}$$

Now, since $\rho$ satisfies the Fatou property, from (4), we have

$$\rho(x - y) \leq \liminf_{m \to \infty} \rho(x - y_m) \leq r(C) + \frac{1}{n}.$$

Then $y \in C_n(x)$. Thus $C_n(x)$ is $\rho-$closed.

For $y_1, y_2 \in C_n(x)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, we have

$$\begin{aligned}
\rho\big(x - (\alpha y_1 + \beta y_2)\big) =& \rho(\alpha x + \beta x - \alpha y_1 - \beta y_2) \\
=& \rho\big(\alpha(x - y_1) + \beta(x - y_2)\big) \\
\leq& \alpha\rho(x - y_1) + \beta\rho(x - y_2) \\
\leq& \alpha\Big(r(C) + \frac{1}{n}\Big) + \beta\Big(r(C) + \frac{1}{n}\Big) \\
=& r(C) + \frac{1}{n}.
\end{aligned}$$

Hence $C_n(x)$ is convex.

For $n \geq 1$, put $C_n = \bigcap_{x \in C} C_n(x)$. Now, we show that $C_n$ is a nonempty, $\rho-$closed, convex subset of $X_\rho$ and $C_{n+1} \subset C_n$.

From (3), there exists $y_n \in C$ such that for every $x \in C$, we have $\rho(x - y_n) \leq r(C) + \frac{1}{n}$. Therefore $y_n \in C_n(x)$ for every $x \in C$. Then $y_n \in C_n = \bigcap_{x \in C} C_n(x)$. Thus $C_n$ is nonempty.

Let $y \in \overline{C_n}$. Therefore there exists $(y_m) \subset C_n$ such that $y_m \xrightarrow{\rho} y$. Then for every $x \in C$, we have $(y_m) \subset C_n(x)$, and hence $\rho(x - y_m) \leq r(C) + \frac{1}{n}$.

Now, since $\rho$ satisfies Fatou property, we have

$$\rho(x - y) \leq \liminf_{m \to \infty} \rho(x - y_m) \leq r(C) + \frac{1}{n}$$

for every $x \in C$. Therefore for every $x \in C$, we have $y \in C_n(x)$. Then $y \in \bigcap_{x \in C} C_n(x) = C_n$.

Also, we prove that $C_{n+1} \subset C_n$.

Let $y \in C_{n+1} = \bigcap_{x \in C} C_{n+1}(x)$. Then for every $x \in C$, we have $y \in C_{n+1}(x)$. Thus

$$\rho(x - y) \leq r(c) + \frac{1}{n+1} \leq r(c) + \frac{1}{n}.$$

Hence for every $x \in C$, $y \in C_n(x)$. Then $y \in C_n = \bigcap_{x \in C} C_n(x)$.

Now, since $\rho$ satisfies (UUC2), therefore $X_\rho$ has property (R) ( [10, Theorem 4.2]). Thus $\bigcap C_n \neq \emptyset$.
We claim that $Z(C) = \bigcap_{n \in \mathbb{N}} C_n$, and so, $Z(C) \neq \emptyset$.

Let $x \in Z(C)$.

By definition of the $Z(C)$ , we have $r_x(C) = r(C)$. Therefore for every $y \in C$ and $n \geq 1$ ,

$$\rho(x - y) \leq r_x(C) = \sup_{y \in C} \rho(x - y) \leq r_x(C) + \frac{1}{n} = r(C) + \frac{1}{n}.$$

Then we have $x \in C_n(y)$. Therefore $x \in \bigcap_{y \in C} C_n(y) = C_n$.

Because $n$ is arbitrary, we have $x \in \bigcap\limits_{n\in\mathbb{N}} C_n$.

Conversely, suppose that $x \in \bigcap\limits_{n\in N} C_n$. We claim that $x \in Z(C)$, therefore it suffices to show that $r_x(C) = r(C)$.

By definition, we have

$$\inf_{x\in C} r_x(C) = r(C) \leq r_x(C). \tag{5}$$

Therefore it suffices to show that $r_x(C) \leq r(C)$.

Since $x \in \bigcap_{n\in\mathbb{N}} C_n$, for any $n \in \mathbb{N}$, we have $x \in C_n = \bigcap\limits_{y\in C} C_n(y)$ and so, $x \in C_n(y)$ for any $n \in \mathbb{N}$ and $y \in C$.

Therefore by definition, we have $\rho(x-y) \leq r(C) + \frac{1}{n}$ for any $y \in C$ and $n \in \mathbb{N}$. Then $r_x(C) \leq r(C) + \frac{1}{n}$ for any $n \in \mathbb{N}$. Thus

$$r_x(C) \leq r(C). \tag{6}$$

From (5) and (6), we conclude that $r_x(C) = r(C)$. Then $x \in Z(C)$. □

**Proposition 3.3.** *Let $X_\rho$ be a $\rho-$complete modular space where $\rho$ is convex and satisfies* (UUC2), *$\triangle_2-$condition and Fatou property, and let $C$ be a convex and $\rho-$closed subset of $X_\rho$ with* $\operatorname{diam} C > 0$. *Suppose that $C$ has normal structure. Then* $\operatorname{diam} Z(C) < \operatorname{diam} C$.

*Proof.* Note that $Z(C) \neq \emptyset$ by Proposition 3.2. Since $C$ has normal structure, there exists at least one nondiametral point $x_0 \in C$, i.e.,

$$r_{x_0}(C) = \sup\left\{\rho(x_0 - x),\ x \in C\right\} < \operatorname{diam} C.$$

Let $u$ and $v$ be any two points of $Z(C)$. Then $r_u(C) = r_v(C) = r(C)$. Because

$$\begin{aligned}\rho(u-v) &\leq \sup\left\{\rho(u-x), x \in Z(C)\right\}\\ &\leq \sup\left\{\rho(u-x), x \in C\right\} = r(C) = \inf_{x\in C} r_x(C)\\ &\leq r_{x_0}(C) < \operatorname{diam} C,\end{aligned}$$

it follows that

$$\operatorname{diam} Z(C) < \operatorname{diam} C. \qquad \square$$

## 4. Normal Structure Coefficient

**Definition 4.1.** Let $X$ be a $\rho-$complete modular space. Then

$$N(X) = \inf\left\{\frac{\operatorname{diam} C}{r(C)}\right\}$$

is said to be a normal structure coefficient, where the infimum is taken over all $\rho-$closed convex $\rho-$bounded subsets $C$ of $X$ with $\operatorname{diam} C > 0$.

**Lemma 4.1.** *Let $X$ be a $\rho-$complete modular space. Then $X$ has uniformly normal structure if and only if $N(X) > 1$.*

*Proof.* Let $X$ have a uniformly normal structure. Then there exists a constant $\alpha \in (0,1)$, independent of $X$, such that each $\rho-$closed convex $\rho-$bounded subset $D$ of $X$ with $\operatorname{diam} D > 0$ contains a point $x_0 \in X$ such that

$$\sup\left\{\rho(x_0 - x); x \in D\right\} = r_{x_0}(D) \leq \alpha \operatorname{diam} D.$$

Then $r(D) \leq \alpha \operatorname{diam} D$. Therefore $\frac{1}{\alpha} \leq \frac{\operatorname{diam} D}{r(D)}$. Hence

$$\frac{1}{\alpha} \leq \inf\left\{\frac{\operatorname{diam} D}{r(D)}\right\} = N(X).$$

Then $N(X) > 1$.

Inversely, let $N(X) > 1$. Therefore we have $1 < N(X) \leq \frac{\operatorname{diam} D}{r(D)}$. Then $r(D) \leq \frac{1}{N(X)} \operatorname{diam} D$.

Now, we set $\alpha = \frac{1}{N(X)}$ and note that we have $N(X) > 1$. Therefore $\alpha = \frac{1}{N(X)} \in (0,1)$. Then there exists $x_0 \in D$ such that $r_{x_0}(D) \leq \alpha \operatorname{diam} D$. $\square$

We now give an important property of the normal structure coefficient.

**Theorem 4.1.** *Let $X_\rho$ be a $\rho-$complete modular space, where $\rho$ is convex and satisfies* (UC1). *Then for every $\rho-$closed convex $\rho-$bounded subset $C$ of $X_\rho$,*

$$N(X) \geq \frac{1}{1-\alpha},$$

*where $\alpha = \inf_C \delta_1(d_C, 1)$.*

*Proof.* Let $C$ be a $\rho-$closed convex $\rho-$bounded subset of $X_\rho$ with $d_C = \operatorname{diam} C > 0$ and let $0 < \varepsilon < d_C$. Choose $x$ and $y$ in $C$ such that $\rho(x-y) \geq d_C - \varepsilon$.

Since $C$ is convex, we have $v = \frac{x+y}{2} \in C$. Therefore $r_v(C) = \sup_{x \in C} \rho(v-x)$. Then there exists $u \in C$ such that

$$\rho(u-v) \geq r_v(C) - \varepsilon. \tag{7}$$

We have $\rho(u-x) \leq d_C$, $\rho(u-v) \leq d_C$ and $\rho\big((u-x)-(u-y)\big) = \rho(x-y) \geq d_C - \varepsilon$. Now, set $\varepsilon' = \frac{d_C - \varepsilon}{d_C}$ and $r = d_C$. Then

$$\rho\big((u-x)-(u-y)\big) = \rho(y-x) \geq d_C - \varepsilon = \frac{d_C - \varepsilon}{d_C} \times d_C = \varepsilon' \cdot r.$$

Since $\rho$ satisfies (UC1), we have

$$\begin{aligned}
\delta_1(r, \varepsilon') =& \delta_1\Big(d_C, \frac{d_C - \varepsilon}{d_C}\Big) \\
\leq& 1 - \frac{1}{d_C}\rho\Big(\frac{(u-x)+(u-y)}{2}\Big) \\
=& 1 - \frac{1}{d_C}\rho\Big(u - \frac{x+y}{2}\Big) \\
=& 1 - \frac{1}{d_C}\rho(u-v).
\end{aligned}$$

Hence

$$\rho(u-v) \leq d_C\Big(1 - \delta_1\Big(d_C, \frac{d_C - \varepsilon}{d_C}\Big)\Big). \tag{8}$$

From (7) and (8), we have $r_v(C) - \varepsilon \leq \rho(u-v) \leq d_C\big(1 - \delta_1\big(d_C, \frac{d_C-\varepsilon}{d_C}\big)\big)$ and by definition of $r_v(C)$, $r(C) \leq r_v(C) \leq \rho(u-v) + \varepsilon$. Thus

$$r(C) \leq \Big(1 - \delta_1\Big(d_C, \frac{d_C - \varepsilon}{d_C}\Big)\Big) d_C + \varepsilon.$$

Hence by the continuity of $\delta_1$, we have

$$r(C) \leq d_C\big(1 - \delta_1(d_C, 1)\big).$$

Then $\frac{\operatorname{diam} C}{r(C)} \geq \frac{1}{1 - \delta_1(d_C,1)}$. Hence

$$\begin{aligned}
N(X) \geq& \inf_C \frac{1}{1 - \delta_1(d_C, 1)} \\
=& \frac{1}{\sup_C \big(1 - \delta_1(d_C, 1)\big)} \\
=& \frac{1}{1 - \inf_C \delta_1(d_C, 1)} \\
=& \frac{1}{1-\alpha}.
\end{aligned}$$

Therefore we get the desired result. □

The following theorem states that the property "uniformly normal structure" is stable under modular perturbations.

**Theorem 4.2.** *Let $X$ be a complete modular space and let $X_1 = (X, \rho_1)$ and $X_2 = (X, \rho_2)$, where $\rho_1$ and $\rho_2$ are two equivalent modulars on $X$ satisfying*

$$\alpha\rho_1(x) \leq \rho_2(x) \leq \beta\rho_1(x), \quad x \in X$$

*for $\alpha, \beta > 0$. If $k = \frac{\beta}{\alpha}$, then*

$$k^{-1}N(X_1) \leq N(X_2) \leq kN(X_1).$$

*Proof.* Note that for a nonempty $\rho-$bounded convex $\rho-$closed subset $C$ of $X$,

$$\begin{aligned}\alpha \sup\big\{\rho_1(x-y),\ x, y \in C\big\} \leq& \sup\big\{\rho_2(x-y),\ x, y \in C\big\}\\ \leq& \beta \sup\big\{\rho_1(x-y),\ x, y \in C\big\}.\end{aligned}$$

Therefore

$$\alpha \operatorname{diam}_{\rho_1}(C) \leq \operatorname{diam}_{\rho_2}(C) \leq \beta \operatorname{diam}_{\rho_1}(C). \tag{9}$$

Let $x$ be an arbitrary point in $C$. Then for every $y \in C$, we have

$$\alpha\rho_1(x-y) \leq \rho_2(x-y) \leq \beta\rho_1(x-y).$$

Therefore

$$\alpha r_x^{\rho_1}(C) \leq r_x^{\rho_2}(C) \leq \beta r_x^{\rho_1}(C).$$

Hence

$$\alpha r_{\rho_1}(C) \leq r_{\rho_2}(C) \leq \beta r_{\rho_1}(C).$$

Then

$$\frac{1}{\beta r_{\rho_1}(C)} \leq \frac{1}{r_{\rho_2}(C)} \leq \frac{1}{\alpha r_{\rho_1}(C)}. \tag{10}$$

From (9) and (10), we have

$$\frac{\alpha \operatorname{diam}_{\rho_1}(C)}{\beta r_{\rho_1}(C)} \leq \frac{\operatorname{diam}_{\rho_2}(C)}{\beta r_{\rho_1}(C)} \leq \frac{\operatorname{diam}_{\rho_2}(C)}{r_{\rho_2}(C)} \leq \frac{\operatorname{diam}_{\rho_2}(C)}{\alpha r_{\rho_1}(C)} \leq \frac{\beta \operatorname{diam}_{\rho_1}(C)}{\alpha r_{\rho_1}(C)}.$$

Hence

$$\frac{\alpha}{\beta}\frac{\operatorname{diam}_{\rho_1}(C)}{r_{\rho_1}(C)} \leq \frac{\operatorname{diam}_{\rho_2}(C)}{r_{\rho_2}(C)} \leq \frac{\beta}{\alpha}\frac{\operatorname{diam}_{\rho_1}(C)}{r_{\rho_1}(C)}.$$

Therefore

$$k^{-1}\frac{\operatorname{diam}_{\rho_1}(C)}{r_{\rho_1}(C)} \leq \frac{\operatorname{diam}_{\rho_2}(C)}{r_{\rho_2}(C)} \leq k\frac{\operatorname{diam}_{\rho_1}(C)}{r_{\rho_1}(C)}.$$

Then

$$k^{-1}N(X_1) \leq N(X_2) \leq kN(X_1).$$ □

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(Received 01.07.2020)

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