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ON SOME SHARP ESTIMATES OF TOEPLITZ OPERATOR IN SOME SPACES OF HARDY–LIZORKIN TYPE OF ANALYTIC FUNCTIONS IN THE POLYDISC

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Abstract. We provide some new sharp assertions on the action of the Toeplitz operator $T\varphi$ in new $F^{p,q}_\alpha$ type spaces of analytic functions of several complex variables extending previously known assertions proved by various authors.

1. Introduction

Let U^n be the unit polydisc in $\mathbb{C}^n$, i.e., $U^n = \{z \in \mathbb{C}^n : \ |z| < 1, \ j = 1, \dots n\}$. Suppose that $H(U^n)$ is the space of all analytic functions in U^n. Let

$$F^{p,q}_\alpha(U^n) = \left\{ f \in H(U^n) : \|f\|^p_{F^{p,q}_\alpha} = \int\limits_{T^n} \left(\int\limits_{I^n} |f(r\xi)|^q (1-r)^{\alpha q - 1} dr \right)^{\frac{p}{q}} d\xi < \infty \right\}$$

be the holomorphic Lizorkin-Triebel space, where $0 < p, \ q < \infty$, $\alpha > 0$, $T^n = \{|z_j| = 1, j = 1, \dots n\}$, $I^n = (0,1] \times \cdots \times (0,1]$, $dr = dr_1 \cdots dr_n$, $d\xi = d\xi_1 \cdots d\xi_n$, $(1-r)^\alpha = \prod_{k=1}^n (1-r_k)^\alpha$, $r_k \in (0,1)$ (see, for example, [2, 3, 6, 9, 10]). Note that for particular case $p = q$ we have the classical Bergman class, while for $q = 2$ we have the so-called Hardy–Lizorkin space H^p_β for some β, that is, $H^p_\beta = \{f \in H(U^n) : D^\beta f \in H^p\}$, $0 < p \le \infty$, $\beta > 0$, where $D^\beta f$ is the fractional derivative of an analytic function f in U^n. Note also that (see definitions bellow) for this particular cases, mapping properties of the classical Toeplitz operator $T\varphi$ are well-studied in unit disk, unit ball and unit polydisc. We study operators T_φ in more general type spaces $F^{p,q}_\alpha$ defined in the polydisc. Our main sharp result provides some criteria for the symbol of φ to obtain the boundednes of T_φ in the above-mentioned spaces.

We define the classical Hardy space $H^p(U^n)$, $0 < p \le \infty$, as follows (see also, for example, [1,2,7,8]): let

$$H^p(U^n) = \{f \in H(U^n) : \ \|f\|_{H^p} = \sup_{r \in I^n} M_p(f,r) < \infty\},$$

and let

$$M_p(f,r) = \left(\int\limits_{T^n} |f(r\xi)|^p dm_n(\xi) \right)^{\frac{1}{p}},$$

where $r\xi = (r_1\xi_1, \dots, r_n\xi_n)$, and dm_n is a normalized Lebesgues measure on T^n, $r_j \in (0,1)$, $j = 1, \dots, n$. Note that $M_p(f,r)$ is increasing for each r_j; for $p = \infty$, we obtain the classical and well-studied class $H^\infty(U^n)$ of all bounded analytic functions in U^n (see for example [7] for this class of functions). Various sharp results about mapping properties of Toeplitz operators can bee found in other papers about various function spaces defined on the unit ball or polydisc. It is worth mentioning, for instance, [4, 5], where such sharp-type results are obtained in particular cases of $F^{p,q}_\alpha$, namely, in Bergman and Hardy type spaces defined on the unit ball or the unit polydisc. Similar results for particular values of parameters are well-known (see, for example, [1, 2, 4, 5, 9]).

Such sharp type results regarding the boundedness of Toeplitz operators have various applications (see, for example, [1, 2, 5, 9]).

We remind the reader the standard definition of the Toeplitz operators T_h in the unit polydisc. Let $h \in L^1(T^n)$. Then we define the Toeplitz operator T_h as follows:

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$$(T_h f)(z) = \frac{1}{(2\pi)^n} \int\limits_{T^n} \frac{f(\xi_1, \dots, \xi_n) h(\xi_1, \dots, \xi_n)}{\prod_{k=1}^n (1 - \bar{\xi}_k z_k)} d\xi_1 \cdots d\xi_n,$$

$z_k \in U$, $k = 1, \dots, n$.

Note that if $\min(p, q) > 1$, then the general mixed norm analytic function spaces $F_\alpha^{p,q}$ in the unit polydisc are Banach spaces; further, these spaces are complete metric spaces for all other values of p and q.

We emphasize that the behavior of operators defined on the unit polydisc is substantially different from the action of T_h in the unit ball in $\mathbb{C}^n$ (see [1, 4, 5, 9] for example).

Our aim is to set criteria for the action of Toeplitz operators T_φ from $F_{\alpha,k}^{p,q}(U^n)$ into Bergman–Sobolev and Hardy–Lizorkin type spaces in the unit polydisc, under the assumption that φ is holomorphic, $\varphi \in H(D^n)$ (under some restrictions on symbol of Toeplitz operator).

For the formulation of our main result in the polydisc, we define some new function spaces. Let dm_{2n} be the normalized Lebesgues measure in U^n, $D^s f$, $0 < s \leq \infty$, be the fractional derivative of a holomorphic function f:

$$(D^s f)(z) = \sum_{|k| \geq 0} |\frac{\Gamma(s + k + 1)\Gamma(s + 1)}{\Gamma(k + 1)} a_k z^k,$$

$a_k z^k = a_{k_1 \cdots k_m} z_1^{k_1} \cdots z_m^{k_m}$, $f(z) = \sum_{|k| \geq 0} a_k z^k$, $z \in U^n$, $\Gamma(\alpha + 1) = \prod_{j=1}^m \Gamma(\alpha_j + 1)$, $\alpha_j > -1$, $j = 1, \dots, m$.

If $f \in H(U^n)$, then for any $s \in \mathbb{N}$, $D^s f \in H(U^n)$. Let $F_{\alpha,k}^{p,q}(U^n) = \{f \in H(U^n) : \|D^k f\|_{F_\alpha^{p,q}} < \infty\}$, $0 < p, q, \alpha < \infty$, $k \in \mathbb{N}$;

$$A_{\alpha,m}^s(U^n) = F_{\alpha/s,m}^{s,s} = \left\{ f \in H(U^n) : \|f\|_{A_{\alpha,m}}^s = \int\limits_{U^n} |(D^m f)(z)|^s (1 - |z|)^{\alpha - 1} dm_{2n}(z) < \infty \right\},$$

$m \in \mathbb{N}$, $0 < s, \alpha < \infty$ (Bergman–Sobolev space). Suppose that

$$H_m^s(U^n) = \{f \in H(U^n) : \|D^m f\|_{H^s} < \infty, \ m \in \mathbb{N}, \ 0 < s < \infty\}$$

is analytic Hardy–Lizorkin space in the unit polydisc U^n.

It can easily shown that these both scales of analytic function spaces in the unit polydisc are Banach spaces for all values of s, $s \geq 1$, and they are complete metric spaces for other values of s, $s > 0$.

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of functions or variables being discussed.

2. Main Results

We now formulate the main results of this note.

Theorem 1. *Let* $0 < \max(p, q) \leq s$, $1 < s < \infty$, $k = \alpha + \frac{1}{p} - \frac{1}{s} + m\Big(1 - \frac{1}{s}\Big)$, $m, k \in \mathbb{N}$. *Then the operator* $T_{\bar{\varphi}}$ *is bounded from* $F_{\alpha,k}^{p,q}(U^n)$ *into* $A_{m,m}^s(U^n)$ *if and only if* $\varphi \in H^\infty(U^n)$ *and* $\|\varphi\|_\infty \leq \|T_{\bar{\varphi}}\|$.

Theorem 2. *Let* $0 < \max(p, q) \leq 1$, $1 < s < \infty$, $k = \frac{1}{p} - \frac{1}{s} + \alpha + m$, $k, m \in \mathbb{N}$. *Then the operator* $T_{\bar{\varphi}}$ *is bounded from* $F_{\alpha,k}^{p,q}(U^n)$ *into* $H_m^s(U^n)$ *if and only if* $\varphi \in H^\infty(U^n)$ *and* $\|\varphi\|_\infty \leq \|T_{\bar{\varphi}}\|$.

For the proof of the main results we need some lemmas.

Lemma 1. *Let* $f \in F^{p,q}_{\alpha,k}(U^n)$, $s > 1$, $k \in \mathbb{N}$, $0 < p$, $q \leq s$ *and* $k = \dfrac{(\alpha + \frac{1}{p})s - 1}{s} + m\big(1 - \dfrac{1}{s}\big)$. *Then the following inequality holds:*

$$\Big(\int\limits_{U^n} |D^k f(w)|^s (1 - |w|)^{s(k-1)-(m-1)(1-\frac{1}{s})} dm_{2n}(w)\Big)^{\frac{1}{s}}$$
$$\leq c\Big(\int\limits_{T^n}\Big(\int\limits_{I^n} |D^k f(w)|^q (1 - |w|)^{\alpha q - 1} d|w|\Big)^{\frac{p}{q}} dm_n(\xi)\Big)^{\frac{1}{p}}.$$

Remark 1. Similar proof can be provided for the case when the standard weights $(1 - |w|)^\alpha$ are replaced by $w(r)$, $r \in (0, 1)$ (see, for example, [4,9]).

Lemma 2. *Let* $G \in H(U^n)$, $1 < s < \infty$. *Then the following estimates are valid:*

$$M_s(D^m G, R^2) \leq c(1 - R)^{-m} M_s(G, R), \ R \in I^n, \ m \in \mathbb{N}$$
$$M_s(G, R^2) \leq c_1(1 - R)^{\frac{1}{s}-1} M_1(G, R), \ R \in I^n.$$

Remark 2. It is well-known that those estimates are valid for $n = 1$ (see [2,5,6,9]).

Lemma 3 ([11,12])**.** *Let* f *be analytic in* $0 \leq r_j < |z_j| < R_j$, $1 \leq j \leq m$, *and* f *be* C^s *continuous in the closure of this domain. Then for* $0 \leq p \leq q \leq \infty$, $\rho \in (r, R)$,

$$\|f_\rho\|_{H^q} \leq c(m, p, q) \prod_{j=1}^{m} ((\rho_j - r_j), (R_j - \rho_j))^{\frac{1}{q} - \frac{1}{p}} \max_{\substack{V_j = r_j, R_j \\ j = 1, \dots, m}} \|f_V\|_{H^p}.$$

Lemma 4 ([9,10])**.** *Let* $0 < \max(p, q) \leq s < \infty$, $\alpha > 0$. *Then*

$$\Big(\int\limits_{U^n} |f(w)|^s (1 - |w|)^{s(\alpha + \frac{1}{p}) - 2} dm_{2n}(w)\Big)^{\frac{1}{s}} \leq c\|f\|_{F^{p,q}_\alpha}.$$

Remark 3. The same approach can be used to get criteria on the symbol φ, for which $(T_{\bar{\varphi}})$ is bounded from $F^{p,q}_{k,\alpha}$ into X, where X is different from A^m_s, and H^s is a quasi-norm subspace of $H(U^n)$.

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