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ON THE GENERALIZED WIENER BOUNDED VARIATION SPACES WITH p-VARIABLE

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Abstract. In this paper, the generalized Wiener space of bounded variation with p-variable is investigated. Various results such as uniform convexity and reflexivity are obtained. Characterization of a set of points of discontinuity of functions from this space is also given. For bounded exponents, the existence of right– and left-hand limits at each point is shown. Further, it is proved that there is an unbounded exponent such that in the corresponding space exists a function that does not have the right-hand limit at a point. Also, it is shown that for a wide class of exponents the additivity property of the variation is not fulfilled.

1. Introduction

In 1881, C. Jordan [13], when studying the Fourier series convergence, introduced the notion of a function of bounded variation and established the relationship between those and monotonic functions. Later, motivated by the problems in such areas as the calculus of variations, the convergence of Fourier series, geometric measure theory, mathematical physics, etc. mathematicians generalized the idea to generalize the concept of bounded variation in various directions. Because of a great number of authors who have generalized the notion of bounded variation, we do not list them here.

The topic of function spaces with a variable exponent is an important area of research at present, mainly, due to its wide applications in the modeling of electrorheological fluids, in the study of image processing and differential equations with a non-standard growth. Different aspects concerning these spaces with variable exponents can be found in [3, 4] and references therein.

To shorten the expressions that will appear in our further reasoning, we introduce the following notation. Denote by $P := \{Q_k\}_{k=1}^n$ the partition of $[a;b]$, where $Q_k = [t_{k-1};t_k]$ and $a = t_0 < \cdots < t_n = b$. Besides, let $f(Q_k) := f(t_k) - f(t_{k-1})$ and $\Pi[a;b]$ denote the set of all finite partitions of $[a;b]$.

The first attempt to investigate the bounded variation spaces with a variable exponent was made by H. Herda [12]. Following H. Nakano, he generalized the Wiener p-th variation in the same way as the $L^{p(\cdot)}$ space generalizes classical L^p space[1]. H. Herda established various properties of such spaces, namely, modular completeness, uniform convexity, and reflexivity.

Recently, the authors of [2], introduced the space of functions of bounded variation with p-variable $BV^{p(\cdot)}$. They considered the exponent $p : [a;b] \to (1;+\infty)$ such that $\sup p(x) < +\infty$ and for such exponent defined the functional by

$$V_{[a;b]}^{p(\cdot)}(f) := \sup_{\Pi^*[a;b]} \sum_{k=1}^{n} |f(Q_k)|^{p(x_k)},$$

where $\Pi^*[a;b]$ is a set of tagged partitions P^* of $[a;b]$, i.e., a partition of the segment $[a;b]$ together with a finite sequence of numbers $x_1, \ldots, x_n$ subject to the conditions that for each k, $t_{k-1} \le x_k \le t_k$.

In [2], the authors defined the space of functions of $p(\cdot)$-bounded variation by

$$BV^{p(\cdot)}[a;b] = \{f : [a;b] \to \mathbb{R} \mid f(a) = 0,\ ||f||_{BV^{p(\cdot)}} < +\infty\},$$

where

$$||f||_{BV^{p(\cdot)}} = \inf\left\{\lambda > 0 \mid V_{[a;b]}^{p(\cdot)}(f/\lambda) \le 1\right\}$$

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[1]The so-called variable exponent Lebesgue space $L^{p(\cdot)}$ first was introduced by W. Orlicz [19].

is the Luxemburg norm. The authors of paper [2] investigated properties of the $BV^{p(\cdot)}[a;b]$ space. Namely, they proved that $BV^{p(\cdot)}[a;b]$ is the Banach space, if $q(x) \geq p(x)$ for all $x \in [a;b]$, then $BV^{p(\cdot)}[a;b] \hookrightarrow BV^{q(\cdot)}[a;b]$, and also, they proved Helly's principle of choice type result in the $BV^{p(\cdot)}[a;b]$. Moreover, the analogue of absolutely p-continuous functions is defined in the framework of a variable space and the fact that this space of absolutely continuous functions is a closed and separable subspace of $BV^{p(\cdot)}[a;b]$ is proved. (For different results concerning the space $BV^{p(\cdot)}[a;b]$ and other spaces of bounded variation with p-variable, see [5, 14–16]).

A more general form of the definition of functions of bounded variation was introduced in a series of papers by S. Gnilka [6–11] (see also Ch.II, §10, Definition 10.4 and the following results in [17]). Although Gnilka's definition of bounded variation is more general than that in [2] as certain restrictions are imposed on the variational function, some results of paper [2] can't be obtained from the previous results.

In the present paper, we are going to introduce the notion of bounded variation with p-variable differently and investigate the obtained $WBV_{p(\cdot)}$ space of functions.

The first motivation to introduce bounded variation with p-variable in somewhat different way is that the space $BV^{p(\cdot)}[a;b]$ is not "stable" with respect to the changes of the exponent function even on the countable set. In other words, if we change the exponent function on the countable set, then we change the corresponding space. Therefore, in our definition, we use the mean values of the exponent at the partition intervals. Such a definition of the bounded variation allows us to get a more "stable" space with respect to the changes in the exponent on a set of measure zero. Besides, below we will show that $BV^{p(\cdot)}[a;b] \subset WBV_{p(\cdot)}[a;b]$ for all exponents p, and also, we will give an example of the exponent for which $WBV_{p(\cdot)}[a;b] \backslash BV^{p(\cdot)}[a;b] \neq \emptyset$; this effect is achieved due to the fact that the constant exponent changes on a set of measure zero (see Theorem 2.6).

Another motivation to consider the mean values of the exponent is that in 2016, independently, there appeared two papers [1] and [14] in which the Riesz bounded variation was introduced with a variable exponent. Besides, the main result in these papers was a generalization of the Riesz result (about representation as to the indefinite integral of a function from the L^p space) to a variable exponent Lebesgue spaces. In the [1], the authors considered the so-called tagged partition as above, in the definition of the space $BV^{p(\cdot)}[a;b]$, and proved the corresponding result by using the log-Hölder continuity of the exponent. On the other hand, in [14], the authors introduced a Riesz bounded variation with p-variable by using mean values of the exponent which allowed them to prove the above-mentioned result for a much wider set of exponents such that the Hardy–Littlewood maximal operator is bounded on the $L^{p(\cdot)}$ space.

These reasons motivated us to consider the $WBV_{p(\cdot)}[a;b]$ space and investigate various properties of this space. Besides, we have obtained some new results that were not considered in the papers published earlier.

2. The Space $WBV_{p(\cdot)}$ of Functions of Bounded Variation

Let $p : [a;b] \to [1;+\infty)$ be a Lebesgue function, integrable on $[a;b]$, and for the interval $Q \subset [a;b]$ define

$$\bar{p}(Q) := \left(\frac{1}{|Q|} \int\limits_Q \frac{1}{p(x)} dx \right)^{-1}.$$

Here and throughout the whole paper, for the set A, the symbol $|A|$ denotes Lebesgue measure of the set A and χ_A is the characteristic function of the set A.

Definition 2.1. Let us consider the functional

$$V_a^b(p,f) := \sup_{\Pi[a;b]} \sum |f(Q_k)|^{\bar{p}(Q_k)} < +\infty. \tag{2.1}$$

We define the generalized Wiener bounded variation spaces with p-variable by

$$WBV_{p(\cdot)}[a;b] = \{f : [a;b] \to \mathbb{R} \mid f(a) = 0,\ ||f||_{WBV_{p(\cdot)}} < +\infty\},$$

where

$$||f||_{WBV_{p(\cdot)}} = \inf\left\{\lambda > 0 \mid V_a^b(p, f/\lambda) \le 1\right\}$$

is the Luxemburg norm.

Remark 2.1. Note that in this definition, the exponent p may be unbounded.

Now, we formulate the results concerning the functional properties of the space $WBV_{p(\cdot)}$.

Theorem 2.1. *The space $WBV_{p(\cdot)}$ is a Banach space.*

Theorem 2.2. *If the exponent $p : [a;b] \to [1;+\infty)$ is such that $1 < \operatorname{essinf} p(x) \le \operatorname{esssup} p(x) < +\infty$, then the space $WBV_{p(\cdot)}$ is uniformly convex and reflexive.*

Theorem 2.3. *The space $WBV_{p(\cdot)}$ is not separable.*

Theorem 2.4. *Given the functions $p_1, p_2 : [a;b] \to [1;+\infty)$ and $p_1(x) \le p_2(x)$ a.e. $x \in [a;b]$, then $WBV_{p_1(\cdot)} \hookrightarrow WBV_{p_2(\cdot)}$.*

Next, we investigate the relationship between the $BV^{p(\cdot)}$ and $WBV_{p(\cdot)}$ spaces. Let us start with the following

Theorem 2.5. *Suppose $p : [a;b] \to [1;+\infty)$ is bounded, then $BV^{p(\cdot)} \subset WBV_{p(\cdot)}$.*

Corollary 2.1. *Let $p : [a;b] \to [1;+\infty)$ be a bounded function. Then $WBV_{p(\cdot)}$ has a sub-space, isomorphic to c_0.*

It is clear that if for some $p \ge 1$, the exponent function is such that $p(x) = p$ for all x, then $WBV_{p(\cdot)} = BV^{p(\cdot)}$. So, there arises a question: Does the inclusion $WBV_{p(\cdot)} \subset BV^{p(\cdot)}$ hold for all bounded exponents? The answer to this question is negative.

Theorem 2.6. *There exists a bounded exponent for which $WBV_{p(\cdot)} \backslash BV^{p(\cdot)} \ne \emptyset$.*

Further, we show the properties of the functions from the $WBV_{p(\cdot)}$ space.

Theorem 2.7. *If $f \in WBV_{p(\cdot)}$, then f is bounded.*

The next results concern the existence of one-sided limits at each point of a function from $WBV_{p(\cdot)}$ and how the existence of one-sided limits depends on the exponent p.

Theorem 2.8. *Let $p : [a;b] \to [1;+\infty)$ be an essentially bounded function and $f \in WBV_{p(\cdot)}$, then at each point of the domain of function f there exist left-hand and right-hand limits (at the end points of the interval, we consider only $f(a+)$ and $f(b-)$).*

One of the direct consequences of Theorem 2.8 is the following

Corollary 2.2. *Let $p : [a;b] \to [1;+\infty)$ be an essentially bounded function. Then if the function $f \in WBV_{p(\cdot)}$ has a point of discontinuity, it should be only of the first kind. The same is true for the functions of the space $BV^{p(\cdot)}$.*

Corollary 2.3. *Let $p : [a;b] \to [1;+\infty)$ be an essentially bounded function, then the set of discontinuous points of the function $f \in WBV_{p(\cdot)}$ is at most countable. The same is true for the functions of the space $BV^{p(\cdot)}$.*

Here, there arise the questions: What role does the boundedness of the exponent in the previous results play? Do the same results maintain if the exponent is not essentially bounded? As it turned out, the answers to these questions are negative.

Theorem 2.9. *There exists an essentially unbounded exponent $p : [a;b] \to [1;+\infty)$ and the function $f \in WBV_{p(\cdot)}$ such that $f(a+)$ does not exist.*

Theorem 2.10. *There is an essentially unbounded exponent $p : [a;b] \to [1;+\infty)$ and $f \in WBV_{p(\cdot)}$ such that the set of discontinuity points of f has continuum power.*

Now, we present the results concerning the additivity of the variation $V_a^b(p, \cdot)$.

Theorem 2.11. *For any point $c \in (a;b)$, we have*

$$V_a^c(p,f) + V_c^b(p,f) \le V_a^b(p,f).$$

The following result shows even a more contrasting property of the $WBV_{p(\cdot)}$ space compared to the Jordan, Wiener and Gnilka's bounded variation classes. In general, if the exponent p is not equivalent to the constant, then there might not exist a number $C > 0$ such that the following inequality

$$V_a^b(p,f) \le C \cdot \big(V_a^c(p,f) + V_c^b(p,f)\big)$$

holds for all $c \in (a;b)$ and for all $f \in WBV_{p(\cdot)}$.

Introduce the notation

$$\bar{p}_-^x(a;b) := \inf\{\bar{p}([c;d]) \mid x \in [c;d] \subset [a;b]\}.$$

Theorem 2.12. *Let there be given the function $p : [a;b] \to [1;+\infty)$. If there exists the point $x \in (a;b)$ such that*

$$\bar{p}_-^x(a;b) < \max\{\bar{p}_-^x(a;x), \bar{p}_-^x(x;b)\}, \tag{2.2}$$

then for any number $C \ge 1$ there exists the function $f := f_C$ such that $f \in WBV_{p(\cdot)}$ and

$$V_a^b(p,f) > C \cdot \big(V_a^x(p,f) + V_x^b(p,f)\big).$$

Condition (2.2) seems to be non-transparent and we are trying to fix this. Consider $\bar{p}_-^x(a;b)$. By its definition, we get

$$\begin{aligned}
\bar{p}_-^x(a;b) &= \inf\{\bar{p}([c;d]) \mid x \in [c;d] \subset [a;b]\} \\
&= \inf\left\{\left(\frac{1}{d-c}\int_c^d \frac{1}{p(t)}dt\right)^{-1} \mid x \in [c;d] \subset [a;b]\right\} \\
&= \left(\sup\left\{\frac{1}{d-c}\int_c^d \frac{1}{p(t)}dt \mid x \in [c;d] \subset [a;b]\right\}\right)^{-1} = \left(M\Big(\frac{1}{p}\Big)(x)\right)^{-1}.
\end{aligned}$$

Here, the symbol M denotes the Hardy–Littlewood maximal operator. By the analogous reasoning, we obtain

$$\bar{p}_-^x(a;x) = \left(M^-\Big(\frac{1}{p}\Big)(x)\right)^{-1} \quad \text{and} \quad \bar{p}_-^x(x;b) = \left(M^+\Big(\frac{1}{p}\Big)(x)\right)^{-1},$$

where M^- and M^+ denote the one-sided Hardy–Littlewood maximal operators. So, in terms of the Hardy–Littlewood maximal operator, we can rewrite condition (2.2) in the following form:

$$M\Big(\frac{1}{p}\Big)(x) > \min\left\{M^-\Big(\frac{1}{p}\Big)(x), M^+\Big(\frac{1}{p}\Big)(x)\right\}.$$

The investigation of the class of such exponents for which there exists a point $x \in (a;b)$ such that the above inequality holds is beyond of the scope of our paper and let it remain an open problem.

Now, we formulate Helly's principle of choice type result for the space $WBV_{p(\cdot)}$.

Theorem 2.13. *Let $p : [a;b] \to [1;+\infty)$ be an essentially bounded function and $\mathcal{F}$ be a uniformly bounded infinite family of functions of the space $WBV_{p(\cdot)}$. If the variations of functions from $\mathcal{F}$ are bounded by the same number, then from the family $\mathcal{F}$, we can choose the sequence which is pointwise convergent to the function $f \in WBV_{p(\cdot)}$.*

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