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OPERATORS OF HARMONIC ANALYSIS IN GRAND VARIABLE EXPONENT MORREY SPACES

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Abstract. The boundedness statements for the operators of Harmonic Analysis in grand variable exponent Morrey spaces are presented. Operators under consideration involve fractional and Calderón–Zygmund singular integral operators, Hardy–Littleood maximal functions and commutators of singular integrals.

Introduction

In this note, we present the results regarding the boundedness of the operators of Harmonic Analysis in grand variable exponent Morrey spaces ($GVEMS$s, for short) unifying two non-standard Banach function spaces: variable exponent and grand Morrey spaces. The operators under consideration involve fractional and Calderón–Zygmund singular integral operators, Hardy–Littlewood maximal functions, commutators of singular integrals. Among others, we proved Sobolev inequality in these spaces for a variable parameter fractional integral. We studied the problem for operators and spaces defined on quasi–metric measure spaces with doubling measure, however, they are new even for appropriate domains in $\mathbb{R}^n$. We defined $GVEMS$s of various forms. Some variants of $GVEM$s were introduced and studied in [27] from structural properties viewpoint. A variant of $GVEMS$s on quasi-metric measure spaces with non-doubling measure were introduced in [29], where the authors established the Sobolev inequality for appropriate fractional integrals, however, the authors omit necessary norm estimates leading to the desired result.

After Morrey introduced Morrey spaces, it is realized that Morrey spaces are used for various purposes. One of the reasons is that the Morrey spaces describe local regularity more precisely than the Lebesgue spaces. As a result, one can use Morrey spaces widely not only in Harmonic Analysis, but also in PDEs. Grand variable exponent Lebesgue space ($GVELS$, for shor) $L^{p(\cdot),\theta}(\Omega)$, introduced in [17] (see also [22, Ch. 14], [8], [9], for further results) is a special case of $GVEMS$. Grand Lebesgue spaces $L^{p_c)}(\Omega)$ with a constant exponent p_c defined on a bounded domain Ω in $\mathbb{R}^n$ was introduced by T. Iwaniec and C. Sbordone [15] (see also [11] for more general space $L^{p),\theta}(\Omega)$). These spaces are non-reflexive and non-separable (see, e.g., [10]). Further, the grand Morrey spaces with a constant exponent were introduced in [26] (see also [31], for further generalizations). Variable exponent Lebesgue spaces ($VELS$, for short) appeared as a special case of the Musielak–Orlicz spaces introduced by H. Nakano and developed by J. Musielak and W. Orlicz. A large number of various results for non-standard spaces were obtained during the last decade. One of such a non-standard function space is a variable exponent Morrey space which was introduced in [2] in Euclidean spaces and in [16] for quasi-metric measure spaces.

For these and related results, generally speaking, on non-standard function spaces, we refer to the monographs [7], [5], [21], [22] and survey [14].

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Preliminaries

Space of homogeneous type. Let (X, d, μ) be a quasi-metric measure space, i.e., X be a topological space endowed with a locally finite complete measure μ and quasi-metric $d : X \times X \mapsto \mathbb{R}_+$ satisfying the following conditions:

i) $d(x, y) = 0$ if and only if $x = y$;

ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

iii) there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in X$,

$$d(x, y) \leq \kappa[d(x, z) + d(z, y)];$$

(iv) for every neighbourhood V of a point $x \in X$ there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in V.

It is also assumed that all balls $B(x, r) := \{y \in X : d(x, y) < r\}$ in X are measurable and $\mu\{x\} =0$ for all $x \in X$. We assume that $\mu(X) < \infty$.

A measure μ is said to satisfy a doubling condition ($\mu \in DC(X)$) if there is a constant $D_\mu > 0$ such that

$$\mu B(x, 2r) \leq D_\mu \mu B(x, r) \tag{1}$$

for every $x \in X$ and $r > 0$. We will denote by d_X the diameter of X.

A quasi-metric measure space (X, d, μ) with doubling measure μ is called a space of homogeneous type (SHT, for short). An SHT is called normal (see [25]) if there are positive constants N, c_1 and c_2 such that for all r, $0 < r < d_X$,

$$c_1 r^N \leq \mu B(x, r) \leq c_2 r^N. \tag{2}$$

The constant N in (2) is called dimension of an SHT. It can be checked immediately that condition (2) implies doubling condition (1).

For example, rectifiable curves in $\mathbb{C}$ with Euclidean distance and arc-length measure satisfying Carleson (regularity) condition is a normal SHT with $N = 1$, the nilpotent Lie groups with Haar measure, is a normal SHT with $N = Q$, where Q is a homogeneous dimension; let a domain $\Omega \subset \mathbb{R}^n$ satisfy the condition: there is a positive constant c such that for all $x \in \Omega$, and $r \in (0, d_X)$, $|B(x, r)| \geq cr^n$. Then Ω with Lebesgue measure is an example of a normal SHT with $N = n$. For more examples and properties of SHT we refer, e.g, to [4].

Variable exponent Lebesgue and Morrey spaces. We denote by $P(X)$ the family of all real-valued μ- measurable functions p on X such that

$$1 < p_- \leq p_+ < \infty,$$

where $p_- := p_-(X) := \inf_X p(x);\quad p_+ := p_+(X) := \sup_X p(x)$.

We say that a function $p(\cdot) \in P(X)$ belongs to the class $\mathcal{P}^{\log}(X)$ if there is a positive constant ℓ such that for all $x, y \in X$ with $d(x, y) \leq 1/2$,

$$|p(x) - p(y)| \leq \frac{\ell}{-\ln(d(x, y))}.$$

Let $p(\cdot) \in P(X)$. A variable exponent Lebesgue space ($VELS$, for short) denoted by $L^{p(\cdot)}(X)$ (or by $L^{p(x)}(X)$) is the class of all measurable μ- functions f on X for which

$$S_p(f) := \int_X |f(x)|^{p(x)} d\mu(x) < \infty.$$

The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf \big\{\lambda > 0 : S_p\big(f/\lambda\big) \leq 1\big\}.$$

It is known that $L^{p(\cdot)}(X)$ is a Banach space (for this and other properties of $VELS$s we refer, e.g., to [24], [7]).

Let $p(\cdot)$ and $q(\cdot)$ be variable exponents defined on an SHT such that $1 < q_- \le q(\cdot) \le p(\cdot) \le p_+ < \infty$. A measurable locally integrable function f on X belongs to the class $M^{p(\cdot)}_{q(\cdot)}(X)$ (see [16]) if

$$\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)} = \sup_{x\in X, 0<r<d_X} (\mu B(x,r))^{\frac{1}{p(x)}-\frac{1}{q(x)}} \|f\|_{L^{q(\cdot)}(B(x,r))} < \infty.$$

If $p(\cdot) = q(\cdot)$, then $M^{p(\cdot)}_{q(\cdot)}(X) = L^{p(\cdot)}(X)$ is a $VELS$.

For the spaces $M^{p(\cdot)}_{q(\cdot)}(X)$, the following relation:

$$M^{p(\cdot)}_{r(\cdot)}(X) \hookrightarrow M^{p(\cdot)}_{q(\cdot)}(X), \quad 1 < q_- \le q(x) \le r(x) \le p(x) \le p_+ < \infty$$

holds.

Let $\lambda(x)$ be a measurable function on X with values in $[0,1]$. Another type of extension of the classical Morrey space to the variable exponent setting, denoted by $L^{p(\cdot),\lambda(\cdot)}(X)$ (see [2] for Euclidean spaces) is the set of measurable functions on X such that

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} = \sup_{x\in X,\ r\in(0,d_X)} \left\| r^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{B(x,r)} \right\|_{L^{p(\cdot)}} < \infty.$$

The following embedding holds for the space $L^{p(\cdot),\lambda(\cdot)}(X)$: If $p(\cdot)$ and $r(\cdot)$ belong to $\mathcal{P}^{\log}(X)$, $p(x) \le r(x)$ and $\frac{n-\lambda(x)}{p(x)} \ge \frac{n-\mu(x)}{r(x)}$, then (see [2] for Euclidean spaces, but the proof for an SHT is the same) $L^{r(\cdot),\mu(\cdot)}(X) \hookrightarrow L^{p(\cdot),\lambda(\cdot)}(X)$.

Grand variable exponent Lebesgue and Morrey spaces. We are interested in $GVEMS$s denoted by $M^{p(\cdot)}_{q(\cdot),\theta}(X)$, $L^{p(\cdot),\lambda(\cdot),\theta}(X)$ and $\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X)$, respectively, where $\theta > 0$ and $\lambda(\cdot) \in [0,1]$. These spaces are defined with respect to the following norms:

$$\|f\|_{M^{p(\cdot)}_{q(\cdot),\theta}(X)} = \sup_{0<\varepsilon<q_--1} \varepsilon^{\frac{\theta}{q_--\varepsilon}} \cdot \|f\|_{L^{p(\cdot)}_{q(\cdot)-\varepsilon}} < \infty,$$

where $1 < q_- \le q(\cdot) \le p(\cdot) \le p_+ < \infty$,

$$\|f\|_{L^{p(\cdot),\lambda(\cdot),\theta}} = \sup_{0<\varepsilon<p_--1} \varepsilon^{\frac{\theta}{p_--\varepsilon}} \|f\|_{L^{p(x)-\eta(x),\lambda(x)}(X)},$$

and

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}} = \sup\left\{ \eta_+^{\frac{\theta}{p_--\eta_+}} \|f\|_{L^{p(x)-\eta(x),\lambda(\cdot)}(X)} : 0 < \eta_- \le \eta_+ < p_- - 1, \eta(\cdot)\in\mathcal{P}^{\log}(X) \right\} < \infty,$$

respectively. Observe that if $p(\cdot) \in \mathcal{P}^{\log}(X)$, then

(i)

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}} \approx \sup \eta_+^{\frac{\theta}{p_--\eta_+}} \mu B(x,r)^{-\frac{\lambda(x)}{p(x)-\eta(x)}} \|f\|_{L^{p(x)-\eta(x)}(B(x,r))},$$

where the supremum is taken over all $x \in X$, $r \in (0, d_X)$, and $\eta(\cdot)$ satisfying the condition $0 < \eta_- \le \eta_+ < p_- - 1$, $\eta(\cdot) \in \mathcal{P}^{\log}(X)$;

(ii)

$$\|f\|_{L^{p(\cdot),\lambda(\cdot),\theta}} \approx \sup_{0<\varepsilon<p_--1} \varepsilon^{\frac{\theta}{p_--\varepsilon}} \Big(\sup_{x\in X, 0<r<d_X} \mu B(x,r)^{-\frac{\lambda(x)}{p(x)-\varepsilon}} \|f\|_{L^{p(x)-\varepsilon}(B(x,r))} \Big).$$

The following embeddings:

$$M^{p(\cdot)}_{q(\cdot)}(X) \hookrightarrow M^{p(\cdot)}_{q(\cdot),\theta}(X) \hookrightarrow M^{p(\cdot)}_{q(\cdot)-\varepsilon}(X), \ \ 0 < \varepsilon < p_- - 1, \ \ q(\cdot) \in \mathcal{P}^{\log}(X),$$

$$L^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow L^{p(\cdot),\lambda(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot)-\varepsilon,\lambda(\cdot)}(X), \ \ 0 < \varepsilon < p_- - 1, \ \ p(\cdot) \in \mathcal{P}^{\log}(X),$$

$$L^{p(\cdot),\lambda(\cdot)}(X) \hookrightarrow \mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot)-\eta(\cdot),\lambda(\cdot)}(X),$$

hold, where $0 < \eta_- \le \eta_+ < p_- - 1$, $p(\cdot) \in \mathcal{P}^{\log}(X)$.

For constant exponents, the grand Morrey spaces were introduced and studied in [26], [31], [23], [20], [19], [18] (see also [22] and references therein) from the operators boundedness and applications to PDEs viewpoint.

Maximal, calderón–Zygmund singular and fractional integral operators on SHT. Let (X,d,μ) be an SHT. Denote by Mf the centered Hardy–Littleood maximal function defined on X:

$$(Mf)(x)=\sup_{x\in X,0<r<d_X}(\mu B(x,r))^{-1}\int\limits_{B(x,r)}|f(y)|d\mu(y),\quad x\in X.$$

Let K be the Calderón–Zygmund operator

$$(Kf)(x)=\lim_{\varepsilon\to 0}\int\limits_{X\setminus B(x,\varepsilon)}k(x,y)f(y)d\mu(y),\quad x\in X, \tag{3}$$

where $k(x,y)$ is the Calderón-Zygmund kernel on $X\times X$, i.e., $k: X\times X\setminus\{(x,x):\ x\in X\}\to\mathbb{R}$ is a measurable function such that

$$|k(x,y)|\le \frac{c}{\mu B(x,d(x,y))}\,,\quad x,y\in X,\quad x\ne y;$$

$$\big|k(x_1,y)-k(x_2,y)\big|+\big|k(y,x_1)-k(y,x_2)\big|\le c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big)\,\frac{1}{\mu B(x_2,d(x_2,y))}$$

for all x_1, x_2 and y satisfying the condition $d(x_2,y)>d(x,x_2)$, where ω − is a positive nondecreasing function on $(0,\infty)$, satisfying the Δ_2 condition ($\omega(2t)\le c\omega(t)$, $t>0$) and the Dini condition $\int_0^1\frac{\omega(t)}{t}\,dt<\infty$. In addition, we assume that for some p_0, $1<p_0<\infty$, and all $f\in L^{p_0}(X,\mu)$, the limit in (3) exists a.e. on X, and the operator T is bounded in $L^{p_0}(X,\mu)$. For the Calderón–Zygmund theory on SHT, we refer also to [4], [1].

In the case an SHT is normal with dimension N, we define the fractional integral operator

$$\big(T_{\alpha(\cdot)}f\big)(x):=\int\limits_X\frac{f(y)}{d(x,y)^{N-\alpha(x)}}\,d\mu(y),\quad 0<\alpha_-\le\alpha(x)\le\alpha_+<N,\ \ x\in X.$$

Commutators of singular integrals. We say that a function b defined on X belongs to BMO if

$$\|b\|_{BMO}=\frac{1}{\mu(B)}\int\limits_B|b(x)-b_B|d\mu(x)<\infty,$$

where $b_B=\frac{1}{\mu(B)}b(y)d\mu(y)$.

A well-known result of Coifman, Rochberg and Weiss [3] states that the commutator operator $K_bf=K(bf)-bKf$, where K is the Calderón–Zygmund operator on $\mathbb{R}^n$, bounded on $L^p(\mathbb{R}^n)$ for $b\in BMO(\mathbb{R}^n$ and $1<p<\infty$. The boundedness of K_b on SHT in the classical Lebesgue spaces was studied in [30]. The same problems in variable exponent Morrey spaces defined on Euclidean spaces were explored in [12].

Main Results

Some properties of $GVEMSs$. Denote by the symbol $[X]_Y$ the closure of a space X in Y. The following statement holds (see [27] for $L^{p(\cdot),\lambda(\cdot),\theta}(X)$):

Proposition 1. *Let (X,d,μ) be an SHT. Suppose that $\theta>0$, $0\le\lambda(x)<1$, $p(\cdot),q(\cdot)\in P(X)$, $\lambda(\cdot)\in\mathcal{P}^{\log}(X)$. Then the spaces $M^{q(\cdot),\theta}(X)$, $L^{p(\cdot),\lambda(\cdot),\theta}(X)$, $\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}_{p(\cdot)}(X)$ are Banach spaces. Further,*

(i) *if $q(\cdot)\in P(X)\cap\mathcal{P}^{\log}(X)$, then*

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{q_--\varepsilon}}\|f\|_{M^{p(\cdot)}_{q(\cdot)-\varepsilon}(X)}=0,$$

for all $f\in\Big[M^{p(\cdot)}_{q(\cdot)}(X)\Big]_{M^{p(\cdot)}_{q(\cdot),\theta}(X)}$;

(ii) *if $p(\cdot)\in\mathcal{P}^{\log}(X)$, then*

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{\theta}{p_--\varepsilon}}\|f\|_{L^{p(\cdot)-\varepsilon,\lambda(\cdot)}}=0$$

for all $f \in \left[L^{p(\cdot),\lambda(\cdot)}(X)\right]_{L^{p(\cdot),\lambda(\cdot),\theta}(X)}$;

(iii) *if* $p(\cdot) \in \mathcal{P}^{\log}(X)$, *then for* $f \in \left[\mathcal{L}^{p(\cdot),\lambda(\cdot)}(X)\right]_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta)}(X)}$, *we find that for any sequence* $\eta^{(n)}(\cdot) \in \mathcal{P}^{\log}(X)$ *such that* $\eta_+^{(n)} \to 0$, $n \to \infty$,

$$(\eta_+^{(n)})^{\frac{\theta}{p_- - \eta_+^{(n)}}} \|f(\cdot)\|_{\mathcal{L}^{p(\cdot)-\eta^{(n)}(\cdot),\lambda(\cdot)}(X)} \to 0, \quad n \to \infty.$$

Boundedness. For the maximal operator, we have the next statements.

Theorem 1. *Let* (X, d, μ) *be an SHT. Suppose that* $\theta > 0$ *and* $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$. *Suppose also that* $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(X)$. *Then the maximal operator* M *is bounded in* $M^{p(\cdot)}_{q(\cdot),\theta}(X)$.

Theorem 2. *Let* (X, d, μ) *be an SHT. Let* $\theta > 0$ *and that* $0 \le \lambda(x) \le \lambda_+ < 1$. *Suppose that* $p(\cdot)$ *is an exponent on* X *such that* $p(\cdot) \in P(X) \cap \mathcal{P}^{\log}(X)$. *Then the maximal operator* M *is bounded* (a) *in* $L^{p(\cdot),\lambda(\cdot),\theta}(X)$, (b) *in* $\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X)$.

Let us denote by $\mathcal{D}(X)$ the set of all bounded functions on X.

Our results regarding the Calderón–Zygmund operators read as follows.

Theorem 3. *Let the conditions of Theorem* 1 *be satisfied. Then there is a positive constant* C *such that*

$$\|Kf\|_{L^{p(\cdot),\lambda,\theta}(X)} \le C\|f\|_{L^{p(\cdot),\lambda,\theta}(X)}$$

for all $f \in \mathcal{D}(X)$.

Theorem 4. *Let the conditions of Theorem* 2 *be satisfied. Then there is a positive constant* C *such that for all* $f \in \mathcal{D}(X)$, *the inequalities*

(i) $\|Kf\|_{L^{p(\cdot),\lambda(\cdot),\theta}(X)} \le C\|f\|_{L^{p(\cdot),\lambda(\cdot),\theta}(X)}$;

(ii) $\|Kf\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X)} \le C\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X)}$

hold.

Now we discuss fractional integral operators. The following theorem holds (cf. [29]).

Theorem 5. *Let* (X, d, μ) *be a normal SHT with dimension* N. *Suppose that* $p \in P(X) \cap \mathcal{P}^{\log}(X)$, $\alpha(\cdot) \in \mathcal{P}^{\log}(X)$, *where* $0 < \alpha_- \le \alpha_+ < N$, $\sup\{N\lambda(x) + \alpha(x)p(x)\} < N$. *We set* $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha(x)}{N(1-\lambda(x))}$. *Let* $\theta > 0$. *Then* $T_{\alpha(\cdot)}$ *is bounded from* $\mathcal{L}^{p(\cdot),\lambda,\theta}(X)$ *to* $\mathcal{L}^{q(\cdot),\lambda(\cdot),q_-\theta/p_-}(X)$.

Regarding the operator K_b, we have

Theorem 6. *Let the conditions of Theorem* 4 *be satisfied and let* $b \in BMO$. *Then there is a positive constant* C *such that for all* $f \in \mathcal{D}(X)$, *the following inequalities:*

(i) $\|K_b f\|_{L^{p(\cdot),\lambda(\cdot),\theta}(X)} \le C\|f\|_{L^{p(\cdot),\lambda(\cdot),\theta}(X)}$;

(ii) $\|K_b f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X)} \le C\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),\theta}(X)}$

hold.

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