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THE METHOD OF ALMOST SURJECTIVE HOMOMORPHISMS AND THE RELATIVE MEASURABILITY OF FUNCTIONS

ALEKS KIRTADZE

Abstract. In the present paper, an approach to some questions in the theory of invariant (quasi-invariant) measures is discussed. It is useful in certain situations, where the given topological groups or topological vector spaces are equipped with various nonzero σ-finite left invariant (left quasi-invariant) measures.

The study of various problems concerning fundamental concepts of measurability of sets and functions is necessary for further progress of many branches of modern mathematics. In this respect, the notion of measurability may be construed or treated in essentially different ways, according to specific features of the topics that are under consideration (see, for example, [2], [4]).

Let E be a basis set and let μ be a measure on E. If a real-valued function

$$f : E \to \mathbf{R}$$

is such that for every Borel subset Z of the real line $\mathbf{R}$, the set $f^{-1}(Z)$ is measurable with respect to the measure μ, then f is called a measurable function.

Let M be a class of measures on E.

We say that a real valued function is relative measurable with respect to M if there exists at least one measure $\mu \in M$ such that f is measurable with respect to μ.

The notion of a relatively measurable function generalizes the notion of a real-valued measurable function with respect to a concrete measure μ on E. Indeed, if we take the one-element class $M = \{\mu\}$, then the relative measurability of a given function with respect to M is equivalent to the above-mentioned ordinary definition of the μ-measurability of f.

In the present paper, an approach to some questions of the measurability of real-valued functions is discussed and we present the relationship between the problem of measurability of functions and that of the measure extension.

In this connection, there are some well-known classical methods of extending invariant measures: Marczewski's method (see, [10], [11]); the method of Kodaira and Kakutani (see, [9]); the method of Kakutani and Oxtoby (see, [3]). A. Kharazishvili applied a purely algebraic method of direct product and solved W. Sierpinski's problem without any set-theoretical assumption (see, [4]). By using the method of surjective homomorphisms the analogous question was solved for nonzero sigma-finite quasi-invariant (invariant) measures on arbitrary uncountable solvable groups (see, [5]). Note that the method of direct products is an important special case of the method of surjective homomorphisms.

The method of direct products and that of surjective homomorphisms was applied to solve important problems in the invariant (quasi-invariant) measure theory (see, for example, [1], [5], [6], [7], [8]).

A useful method of extending measures is based on the application of those mappings whose graphs are thick from the measure-theoretical point of view. This method was successfully applied by Kodaira and Kakutani in their famous construction of a nonseparable translation-invariant extension of the Lebesgue measure on $\mathbf{R}$ (see, [9]).

Let (G_1, μ_1) and (G_2, μ_2) be any two groups endowed with σ-finite left-invariant measures.

We recall that a subset $\Gamma \subset G_1 \times G_2$ is $(\mu_1 \times \mu_2)$-thick in $G_1 \times G_2$ if, for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset (G_1 \times G_2)$ with $(\mu_1 \times \mu_2)(Z) > 0$, we have $\Gamma \cap Z \neq \emptyset$.

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Let

$$f : G_1 \to G_2$$

be a homomorphism.

We say that f is an almost surjective homomorphism if the graph of f is $(\mu_1 \times \mu_2)$-thick in $G_1 \times G_2$.

The following theorems are true.

Theorem 1. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups and let the group G_2 be equipped with a G_2-left-invariant (left G_2-quasi-invariant) probability measure μ_2 and let*

$$f : G_1 \to G_2$$

be an almost surjective homomorphism of the group G_1 onto the group G_2.

Then there exist two measures and μ_1' such that:

(1) *μ_1 is a non-atomic σ-finite G_1-left-invariant measure on G_1;*
(2) *μ_1' extends μ_1;*
(3) *μ_1' is a G_1-left-invariant measure.*

From Theorem 1, the next statement can be obtained.

Theorem 2. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups and let the group G_2 be equipped with a nonzero σ-finite G_2-left-quasi-invariant measure μ_2 and, moreover, let*

$$f : G_1 \to G_2$$

be an almost surjective homomorphism of the group G_1 onto the group G_2.

Then there exist two measures μ_1 and μ_1' such that:

(1) *μ_1 is a non-atomic σ-finite G_1-left-quasi-invariant measure on G_1;*
(2) *μ_1' extends μ_1;*
(3) *μ_1' is a G_1-left-quasi-invariant measure.*

From Theorem 1, the next statement can be obtained also.

Theorem 3. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups and let the group G_2 be equipped with a G_2-left-invariant (left G_2-quasi-invariant) probability measure μ_2 and, moreover, let*

$$f : G_1 \to G_2$$

be an almost surjective homomorphism of the group G_1 onto the group G_2.

Then the function f is relatively measurable with respect to the class of all extensions of μ_1 on $(G_1, \cdot)$.

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Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0186, Georgia

E-mail address: kirtadze2@yahoo.com