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THE PUNCH PROBLEMS OF THE PLANE THEORY OF VISCOELASTICITY FOR THE HALF PLANE

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Abstract. The paper considers the concrete problems of the punch for a viscoelastic half-plane by the Kelvin–Voigt model. It is known that many buildings and composite materials exhibit viscoelastic properties which are reflected in Hooke's law in which the stresses are proportional both to the deformations and to their derivatives in time.

The purpose of the present paper is to study the some concrete problems of the punch for a viscoelastic half-plane by means of Kolosov–Muskhelishvili's method for the Kelvin–Voigt model and get formulas for the distribution of the tangential and normal stresses under the punch.

Introduction

For viscoelastic bodies, following the Kelvin–Voigt model, Hook's law in the plane theory of elasticity has the form [3]

$$\begin{aligned} X_x &= \lambda\vartheta + 2\mu e_{xx} + \lambda^*\dot{\vartheta} + 2\mu^*\dot{e}_{xx},\\ Y_y &= \lambda\vartheta + 2\mu e_{yy} + \lambda^*\dot{\vartheta} + 2\mu^*\dot{e}_{yy},\\ X_y &= \mu\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) + \mu^*\Big(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y}\Big), \end{aligned} \tag{1}$$

where $\vartheta = e_{xx} + e_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, $X_x,\ Y_y,\dots,e_{xy}$ are the functions of the variables $x,\ y,\ t$ (Under t we will always mean the time parameter and the dots in the expressions $\dot{\vartheta},\dots,\dot{u}$ will denote time derivatives). The constants λ and μ are elastic, while λ^* and μ^* are viscoelastic.

We cite here certain well-known Kolosov–Muskhelishvili's formulas which may, as is known, be attributed to any solid bodies [2]

$$\begin{aligned} X_x + Y_y &= 4\,\mathrm{Re}[\Phi(z,t)] = 4\,\mathrm{Re}\big[\varphi'(z,t)\big],\\ Y_y - X_x + 2iX_y &= 2\big[\bar{z}\Phi'(z,t) + \Psi(z,t)\big],\\ Y_y - iX_y &= \Phi(z,t) + \overline{\Phi(z,t)} + z\overline{\Phi'(z,t)} + \overline{\Psi(z,t)}. \end{aligned} \tag{2}$$

We assume that the resultant vector $(X;Y)$ of the external forces, acting on the boundary, is finite and the stresses and rotation vanish at infinity, thus for large $|z|$, we have

$$\Phi(z,t) = -\frac{X+iY}{2\pi z} + o\Big(\frac{1}{z}\Big),\ \ \Psi(z,t) = \frac{X-iY}{2\pi z} + o\Big(\frac{1}{z}\Big).$$

From the correlations (1) and (2) we obtain the formula (see [1,4]) (we assume that at time $t=0$, the displacements $u_0(x,y,0) = v_0(x,y,0) = 0$):

$$2\mu^*(u+iv) = \int\limits_0^t \Big[\varkappa^*\varphi(z,\tau)e^{k(\tau-t)} + \Big(\varphi(z,\tau) - z\overline{\varphi'(z,\tau)} - \overline{\psi(z,\tau)}\Big)e^{m(\tau-t)}\Big]d\tau, \tag{3}$$

where

$$\varkappa^* = \frac{2\mu^*}{\lambda^*+\mu^*},\ \ k = \frac{\lambda+\mu}{\lambda^*+\mu^*},\ \ m = \frac{\mu}{\mu^*}.$$

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Formula (3) is an analogue of Kolosov–Muskhelishvili's formula for the second basic problem of the plane theory of elasticity (see [2]) for a viscoelastic isotropic body. This formula implies that

$$2\mu^* v'(z,t) = \operatorname{Im} \int\limits_0^t \Big[e^{k(\tau-t)}\Phi(z,\tau)d\tau$$
$$+ \int\limits_0^t e^{m(\tau-t)}\Big(\Phi(z,\tau) - \overline{\Phi(z,\tau)} - z\overline{\Phi'(z,\tau)} - \overline{\Psi(z,\tau)}\Big)\Big] d\tau.$$

Statement of the Problem

Let a viscoelastic body occupy a lower half-plane S^- and by L we denote the boundary of that domain (i.e., the Ox-axis). Assume that at each point of $L' = [-1,1]$ there is a normal displacement $v^-(x,0,t) = f(x)$ at time t, for which the segment L' undergoes a normal pressure $N(x,0,t)$. In the resulting "trench", established, the punch with the given form of base $y = f(x)$ and maintained balance is pressed by the vertical force $N_0 = \int_{-1}^{1} N(x,0,t)dx$ (in what follows, by N_0 we mean the value $N_0 H(t)$, where $H(t)$ is the Heaviside step function. In the same sense, we understand all other quantities that depend on time t). The law of distribution of normal pressure $N(x,0,t)$ is unknown in advance, it is the main unknown value of the problem and must be determined so that the equilibrium of the punch is maintained under the conditions of constant deformation of the boundary, i.e., we have the problem of the stress relaxation.

Usually, the linear theory of elasticity considers only a small displacement and a shallow punch; in this connection the boundary conditions are set on the segment L' (i.e., on the undeformed part $L' = [-1,1]$ of the boundary of the half-plane).

So, the boundary conditions can be written in the form

$$\begin{aligned} X_y^-(x,t) &= \alpha N(x,t), \quad \alpha = \text{const} > 0, \ x \in L', \\ X_y^-(x,t) &= Y_y^-(x,t) = 0, \quad x \in L'' = L - L', \\ v^-(x,t) &= f(x) + c, \quad x \in L', (c = \text{const}), \end{aligned} \tag{4}$$

where $y = f(x)$ is the given function defining the base shape of the punch before pressing into the half-plane. In (4), by $X_y^-(x,t), \dots, v^-(x,t)$ we have denoted the expressions $X_y^-(x,0,t), \dots, v^-(x,0,t)$. The total tangential stress in the case under consideration has the form $T_0 = \alpha N_0$ and hence, the resultant vector of outer forces acting onto the punch (which are assumed to be prescribed) is of the kind $(X,Y) = (\alpha N_0, -N_0)$.

Solution of the Problem

The considered problem, in the general form, when the displacement changes in time, i.e., $v(x,0,t) = f(x,t)$, where $f(x,0) = f(x)$, has been studied in [1], where formulas for the distribution of the tangential $p(x,t) = Y_y^-(x,t)$ and normal $T(x,t) = X_y^-(x,t)$ stresses under the punch are obtained and in our case, we have

$$\begin{aligned} P(x,t) = Y_y(x,t) &= -2 \operatorname{Im}\Big[\frac{i}{1+i\alpha}\Phi^-(x,t)\Big], \\ T(x,t) = X_y^-(x,t) &= -2\alpha \operatorname{Im}\Big[\frac{i}{1+i\alpha}\Phi^-(x,t)\Big], \end{aligned} \tag{5}$$

where

$$\Phi^-(x,t) = e^{(\lambda_1+i\lambda_2)t}\Big[\Phi^-(x,0) + \frac{\Omega^-(x)}{\lambda_1+i\lambda_2}\Big(e^{(\lambda_1+i\lambda_2)t} - 1\Big)\Big],$$
$$\Phi^-(x,0) = \frac{2\mu^* m[\alpha + i(1+2\alpha^2)]}{\varkappa^*(1+2\alpha^2)+2} f'(x),$$

$$\Omega^-(x) = \frac{e^{-2\pi i\delta}}{2}\Big[F(x) - \frac{1}{\pi i\chi(x)}\int\limits_{-1}^{1}\frac{\chi(\sigma)F(\sigma)}{\sigma - x}d\sigma\Big] - \frac{D_0 e^{-2\pi i\delta}}{\chi(x)},$$

$$F(x) = \frac{4i\mu^* mk(1+\alpha)}{(a-ib)(1+\alpha^2)}f'(x),$$

$$a = (\varkappa^* + 2)(1+\alpha^2) - 2\alpha^2, \quad b = -2\alpha,$$

$$\chi(x) = (1+x)^{\frac{1}{2}+\delta}(1-x)^{\frac{1}{2}-\delta}, \quad \delta = \frac{1}{\pi}\operatorname{arctg}\frac{\alpha\varkappa^*}{\varkappa^*+2}, \tag{6}$$

$$D_0 = \frac{(1+i\alpha)N_0}{2\pi}$$

$$\lambda_1 = \frac{[\varkappa^* m(1+\alpha^2) + 2k][\varkappa^*(1+\alpha^2)+2] + 4k\alpha^2}{[\varkappa^*(1+\alpha^2)+2]^2 + 4\alpha^2},$$

$$\lambda_2 = \frac{2\alpha\varkappa^*(m-k)(1+\alpha^2)}{[\varkappa^*(1+\alpha^2)+2]^2 + 4\alpha^2}.$$

Note that the second term of the formula M corresponding to the solution of the problem about the pressure of the rectangular stamp, plays a major "guiding" role, while the first term corresponds to the perturbation caused by the curvature of the punch profile, which we usually assume to be a slightly curved smooth curve. The integral that participates in this term is calculated based on the remainder theory.

On the basis of (5) and (6) we can conclude that in our case (i.e., in the case of pressure of a rigid punch with friction) the tangential and normal stresses have the character of damping oscillations with respect to time t. Also, taking into account (6), we can conclude that oscillations are absent in the following cases:

1) for $\alpha = 0$ (i.e., without friction);

2) for $m = k$ (i.e., the constants $\lambda, \dots, \mu^*$ are connected by the relation $\frac{\lambda}{\lambda^*} = \frac{\mu}{\mu^*}$).

As is mentioned, we calculate the following kind of integral

$$I(z) = \frac{1}{\pi}\int\limits_{-1}^{1}\frac{\chi(\sigma)f'(\sigma)}{\sigma - z}d\sigma$$

based of the remainder theory.

Let us examine some particular cases of the punch profile bearing in mind that the linear theory of viscoelasticity studies usually only small displacements, and in this connection we consider a slightly curved profile of the punch.

1. $f(x) = \frac{x^2-1}{R}$, i.e., $f'(x) = \frac{2x}{R}$, where R is a large enough number because we will have small deformations.

Given large $|z|$ (i.e., $z \to \infty$), we have

$$\begin{aligned} z\chi(z) &= z(1+z)^{\frac{1}{2}+\delta}(1-z)^{\frac{1}{2}-\delta} = -ie^{\pi i\delta}z(z+1)^{\frac{1}{2}+\delta}(z-1)^{\frac{1}{2}-\delta} \\ &= -ie^{\pi i\delta}\Big[z^2 + 2\delta z + \frac{1}{2}(4\delta^2 - 1) + O\Big(\frac{1}{z}\Big)\Big], \end{aligned}$$

so,

$$I(z) = \frac{2}{R\cos\pi\delta}\Big[(z+1)^{\frac{1}{2}+\delta}(z-1)^{\frac{1}{2}-\delta} - z^2 - 2\delta z - \frac{1}{2}(4\delta^2-1)\Big].$$

2. In the case $f(x) = -\frac{1}{R}\sqrt{1-x^2}$, $f'(x) = \frac{x}{R\sqrt{1-x^2}}$, we obtain

$$I(z) = \frac{1}{R}\frac{e^{\pi i\delta}}{\sin\pi\delta}\Big[z\Big(\frac{z+1}{z-1}\Big)^{\delta} - z - 2\delta\Big].$$

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