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RATIONAL PÁL TYPE $(0,1;0)$-INTERPOLATION AND QUADRATURE FORMULA WITH CHEBYSHEV–MARKOV FRACTIONS

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Abstract. We present a Pál-type $(0,1;0)$-interpolation on an inter-scaled set of nodes, when Hermite and Lagrange data are prescribed on the zeros of Chebyshev–Markov sine fraction $U_n(x)$ and its derivative $U_n'(x)$, respectively. A quadrature formula based on the obtained Pál-type interpolation has been constructed. Coefficients of this quadrature are obtained in the explicit form.

1. Introduction

The study of different type interpolation processes has been a subject of interest for several mathematicians. In almost all the cases the interpolatory polynomials are considered on the nodes which are the zeros of certain classical orthogonal polynomials. The main idea of the present paper is to construct a rational interpolation process and its corresponding quadrature formula.

Let $\mathcal{R}_{2n-1}(a_0,a_1,a_2,\dots,a_{2n-1})$ be a rational space defined as

$$\mathcal{R}_{2n-1}(a_0,a_1,\dots,a_{2n-1}) := \left\{\frac{p_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1+a_kx)}\right\},$$

where $p_{2n-1}(x)$ is a polynomial of degree $\le 2n-1$ and $\{a_k\}_{k=0}^{2n-1}$ are real and belong to $[-1,1]$, or are paired by a complex conjugation.

Chebyshev and Markov introduced rational cosine and sine fractions [9] which generalize Chebyshev polynomials, possess many similar properties [8, 16, 18] and are called Chebyshev–Markov rational fractions. More details on the rational generalization of Chebyshev polynomials can be found in [1–6, 19]. Let $U_n(x)$ be the rational Chebyshev–Markov sine fraction,

$$U_n(x) = \frac{\sin\mu_{2n}(x)}{\sqrt{1-x^2},} \tag{1.1}$$

where

$$\mu_{2n}(x) = \frac{1}{2}\sum_{k=0}^{2n-1}\arccos\frac{x+a_k}{1+a_kx}, \quad \mu_{2n}'(x) = -\frac{\lambda_{2n}(x)}{\sqrt{1-x^2}},$$
$$\lambda_{2n}(x) = \frac{1}{2}\sum_{k=0}^{2n-1}\frac{\sqrt{1-a_k^2}}{1+a_kx}, \quad n\in N. \tag{1.2}$$

The rational fraction $U_n(x)$ can be expressed as

$$U_n(x) = \frac{P_{n-1}(x)}{\sqrt{\Pi_{k=0}^{2n-1}(1+a_kx)}},$$

where $P_{n-1}(x)$ is an algebraic polynomial of degree $n-1$ with a real coefficient, and $\{a_k\}_{k=0}^{2n-1}$ are as defined above. The fraction $U_n(x)$ has $n-1$ zeros on the interval $(-1,1)$ given by

$$-1 < x_{n-1} < x_{n-2} < \dots < x_2 < x_1 < 1,$$

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with

$$\mu_{2n}(x_k) = k\pi,\ \ k = 1, 2, \dots, n-1.$$

Also, the rational function $\lambda_{2n}(x)$ can be expressed as

$$\lambda_{2n}(x) = \frac{p_{2n-1}(x)}{\prod_{k=0}^{2n-1}(1 + a_k x)},$$

where $p_{2n-1}(x)$ is a polynomial of degree atmost $2n-1$. It has no zeros in the interval $[-1, 1]$. On differentiating (1.1), we get

$$U_n'(x) = \frac{-\cos \mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2} + x \sin \mu_{2n}(x)}{(1-x^2)^{3/2}} \tag{1.3}$$

and

$$U_n'(x_k) = -\frac{\lambda_{2n}(x_k)}{(1-x_k^2)}. \tag{1.4}$$

In 1962, Rusak [15] initiated the study of interpolation processes by means of rational functions on the interval $[-1, 1]$. The nodes were taken to be the zeros of Chebyshev–Markov rational fractions. In [13], rational interpolation functions of Hermite–Fejér-type were constructed. Min [10] was the first, who considered the rational quasi-Hermite-type interpolation. He constructed the interpolated function and proved its uniform convergence for the continuous functions on the segment with the restriction that the poles of the approximating rational functions should not have limit points on the interval $[-1, 1]$. Based on the ideas of [13] and using the method, somewhat different from that of [10], Rouba et al. [12], [14] revisited the rational interpolation functions of Hermite–Fejér-type. They also proved the uniform convergence of the interpolation process for the function $f \in \mathcal{C}[-1, 1]$ and obtained explicitly its corresponding Lobatto type quadrature formula. Recently, Shrawan Kumar et al. [7] studied the Radau type quadrature for an almost quasi-Hermite–Fejér-type interpolation in rational spaces.

In this paper, we have considered the existence and explicit representation of a Pál type $(0, 1; 0)$-interpolation on the rational space $\mathcal{R}_{3n-3}(a_0, a_1, \dots, a_{2n-1})$, when the Hermite and Lagrange data are prescribed on the zeros of $U_n(x)$ $(\{x_k\}_{k=1}^{n-1})$ and its derivative $U_n'(x)$ $(\{t_k\}_{k=1}^{n-2})$, respectively. These zeros are inter-scaled such that

$$-1 = x_n < x_{n-1} < t_{n-2} < x_{n-2} < \cdots < x_2 < t_1 < x_1 < 1 = x_0.$$

A quadrature formula corresponding to the interpolation process has also been obtained.

2. Explicit Representation of Pál type $(0, 1; 0)$-interpolation

For any function $f \in C[-1, 1]$ the Pál type (0,1;0)-interpolation function $W_n(x, f)$ satisfying the conditions

$$\begin{cases} W_n(x_k, f) = f(x_k), & k = 0, 1, \dots, n, \\ W_n'(x_k, f) = \alpha_k, & k = 1, 2, \dots, n-1, \\ W_n(t_k, f) = f(t_k), & k = 1, 2, \dots, n-2, \end{cases} \tag{2.1}$$

can be explicitly represented as

$$W_n(x, f) = \sum_{k=0}^{n} f(x_k)E_k(x) + \sum_{k=1}^{n-1} \alpha_k D_k(x) + \sum_{k=1}^{n-2} f(t_k)C_k(x), \tag{2.2}$$

where α_k, $k = 1, 2, \dots, n-1$ are arbitrarily given real numbers, $\{E_k(x)\}_{k=0}^{n}$, $\{D_k(x)\}_{k=1}^{n-1}$ and $\{C_k(x)\}_{k=1}^{n-2}$ are fundamental functions of the Pál type (0,1;0) interpolation $W_n(x, f)$, satisfying the following conditions: for $k = 1, 2, \dots, n-2$,

$$\begin{cases} C_k(x_j) = 0, & j = 0, 1, \dots, n, \\ C_k'(x_j) = 0, & j = 1, 2, \dots, n-1, \\ C_k(t_j) = \delta_{jk}, & j = 1, 2, \dots, n-2, \end{cases} \tag{2.3}$$

for $k = 1, 2, \ldots, n-1$,

$$\begin{cases} D_k(x_j) = 0, & j = 0, 1, \ldots, n, \\ D'_k(x_j) = \delta_{jk}, & j = 1, 2, \ldots, n-1, \\ D_k(t_j) = 0, & j = 1, 2, \ldots, n-2 \end{cases} \tag{2.4}$$

and for $k = 0, 1, 2, \ldots, n$,

$$\begin{cases} E_k(x_j) = \delta_{jk}, & j = 0, 1, \ldots, n, \\ E'_k(x_j) = 0, & j = 1, 2, \ldots, n-1, \\ E_k(t_j) = 0, & j = 1, 2, \ldots, n-2. \end{cases} \tag{2.5}$$

In the following lemmas, we give the explicit representation of these fundamental functions of the Pál type $(0, 1; 0)$-interpolation $W_n(x, f)$.

Lemma 1. *The fundamental functions* $\{C_k(x)\}_{k=1}^{n-2}$ *satisfying conditions* (2.3) *can be explicitly represented for* $k = 1, 2, \ldots, n-2$, *as*

$$C_k(x) = \frac{(\lambda_{2n}(t_k))^{3/2}(1-x^2)U_n^2(x)L_k(x)}{(1-t_k^2)U_n^2(t_k)(\lambda_{2n}(x))^{3/2}}, \tag{2.6}$$

where $U_n(x)$ *are given by* (1.1), $\lambda_{2n}(x)$ *are given by* (1.2) *and* $\{L_k(x)\}_{k=1}^{n-2}$ *are given by*

$$L_k(x) = \frac{U'_n(x)}{(x-t_k)U''_n(t_k)}.$$

Proof. We will show that $\{C_k(x)\}_{k=1}^{n-2}$ given by (2.6) satisfies conditions (2.3). Obviously, for $k = 1, 2, \ldots, n-2$, $C_k(x_j) = 0$, $j = 0, 1, \ldots, n$ and $C'_k(x_j) = 0$, $j = 1, 2, \ldots, n-1$. Also, for $j \neq k$, $C_k(t_j) = 0$, $j = 1, \ldots, n-2$ and for $j = k$,

$$\lim_{x \to t_k} C_k(x) = \frac{(\lambda_{2n}(t_k))^{3/2}}{(1-t_k^2)U_n^2(t_k)} \frac{(1-t_k^2)U_n^2(t_k)}{(\lambda_{2n}(t_k))^{3/2}} \lim_{x \to t_k} L_k(x) = 1$$

which completes the proof of the Lemma. □

Lemma 2. *The fundamental functions* $\{D_k(x)\}_{k=1}^{n-1}$ *satisfying conditions* (2.4) *can be explicitly represented for* $k = 1, 2, \ldots, n-1$, *as*

$$D_k(x) = \frac{(\lambda_{2n}(x_k))^{3/2}}{(1-x_k^2)(U'_n(x_k))^2} \frac{(1-x^2)U_n(x)U'_n(x)\ell_k(x)}{(\lambda_{2n}(x))^{3/2}}, \tag{2.7}$$

where $U'_n(x)$ *are given by* (1.3), $\lambda_{2n}(x)$ *are given by* (1.2) *and* $\{\ell_k(x)\}_{k=1}^{n-1}$ *are given by*

$$\ell_k(x) = \frac{U_n(x)}{(x-x_k)U'_n(x_k)}. \tag{2.8}$$

Proof. Obviously, for $k = 1, 2, \ldots, n-1$, $D_k(x_j) = 0$, $j = 0, 1, \ldots, n$ and for $j \neq k$, $D'_k(x_j) = 0$, $j = 1, 2, \ldots, n-1$, for $j = k$,

$$\lim_{x \to y_k} D'_k(x) = \left(\frac{(\lambda_{2n}(x_k))^{3/2}}{(1-x_k^2)(U'_n(x_k))^2}\right)\left(\frac{(1-x_k^2)U'_n(x_k)}{(\lambda_{2n}(x_k))^{3/2}}\right) \lim_{x \to x_k}\left(\frac{U_n(x)}{x-x_k}\right) = 1.$$

Also, $D_k(t_j) = 0$, $j = 0, 1, \ldots, n$, which shows that $\{D_k(x)\}_{k=1}^{n-1}$, given by (2.7), satisfies all conditions (2.4) and hence completes the proof of the Lemma. □

Lemma 3. *The fundamental functions* $\{E_k(x)\}_{k=0}^{n}$ *satisfying conditions* (2.5) *can be explicitly represented as*

$$E_0(x) = \frac{(\lambda_{2n}(1))^{3/2}}{2U_n^2(1)U'_n(1)} \frac{(1+x)U_n^2(x)U'_n(x)}{(\lambda_{2n}(x))^{3/2}}, \tag{2.9}$$

for $k = 1, 2, \ldots, n-1$,

$$E_k(x) = \frac{(\lambda_{2n}(x_k))^{3/2}}{(1-x_k^2)U'_n(x_k)} \frac{(1-x^2)U'_n(x)}{(\lambda_{2n}(x))^{3/2}}\Big(1 + b_k(x-x_k)\Big)\ell_k^2(x), \tag{2.10}$$

where

$$b_k = -\frac{x_k}{1-x_k^2} - \frac{U_n''(x_k)}{U_n'(x_k)} + \frac{\lambda_{2n}'(x_k)}{2\lambda_{2n}(x_k)} \tag{2.11}$$

and

$$E_n(x) = \frac{(\lambda_{2n}(-1))^{3/2}}{2U_n^2(-1)U_n'(-1)} \frac{(1-x)U_n^2(x)U_n'(x)}{(\lambda_{2n}(x))^{3/2}}. \tag{2.12}$$

Proof. Obviously, for $j \neq k$, we have $E_k(x_j) = 0$, $j = 0, 1, \ldots, n$ and for $j = k$, using de L'Hospital's rule and (1.4), we have

$$\begin{aligned}\lim_{x\to x_k} E_k(x) =& \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)} \left(\lim_{x\to x_k} \frac{\sin\mu_{2n}(x)}{(x-x_k)} \right)^2 \\ =& \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)} \left(\lim_{x\to x_k} \frac{-\lambda_{2n}(x)\cos\mu_{2n}(x)}{\sqrt{1-x^2}} \right)^2 = 1.\end{aligned}$$

Also, for $k = 1, 2, \ldots, n-1$, we have $E_k(t_j) = 0$, $j = 1, 2, \ldots, n-2$.

On differentiating (2.10) with respect to x and using (1.4), we get

$$\begin{aligned}E_k'(x) =& \frac{(1-x_k^2)}{U_n'(x_k)(\lambda_{2n}(x_k))^{1/2}} \left[\frac{2U_n'(x)\{1+b_k(x-x_k)\}}{(\lambda_{2n}(x))^{3/2}} \left(\frac{\sin\mu_{2n}(x)}{x-x_k} \right)' \right. \\ &+ \left(\frac{b_kU_n'(x) + \{1+b_k(x-x_k)\}U_n''(x)}{(\lambda_{2n}(x))^{3/2}} \right. \\ &\left.\left. - \frac{3\lambda_{2n}'(x)U_n'(x)\{1+b_k(x-x_k)\}}{2(\lambda_{2n}(x))^{5/2}} \right) \left(\frac{\sin\mu_{2n}(x)}{x-x_k} \right) \right] \left(\frac{\sin\mu_{2n}(x)}{x-x_k} \right),\end{aligned}$$

then for $j \neq k$, we have $E_k'(x_j) = 0$, $j = 1, 2, \ldots, n-1$ and for $j = k$,

$$\begin{aligned}\lim_{x\to x_k} E_k'(x) =& \frac{(1-x_k^2)}{U_n'(x_k)(\lambda_{2n}(x_k))^{1/2}} \left[\frac{2U_n'(x_k)}{(\lambda_{2n}(x_k))^{3/2}} \left(\lim_{x\to x_k} \left(\frac{\sin\mu_{2n}(x)}{x-x_k} \right) \left(\frac{\sin\mu_{2n}(x)}{x-x_k} \right)' \right) \right. \\ &\left. + \left(\frac{b_kU_n'(x_k) + U_n''(x_k)}{(\lambda_{2n}(x_k))^{3/2}} - \frac{3\lambda_{2n}'(x_k)U_n'(x_k)}{2(\lambda_{2n}^2(x_k))^{5/2}} \right) \left(\lim_{x\to x_k} \frac{\sin\mu_{2n}(x)}{x-x_k} \right)^2 \right].\end{aligned}$$

We know that

$$\lim_{x\to x_k} \frac{\sin\mu_{2n}(x)}{(x-x_k)} = \mu_{2n}'(x_k)\cos\mu_{2n}(x_k) = -\frac{\lambda_{2n}(x_k)}{\sqrt{1-x_k^2}}$$

and

$$\lim_{x\to x_k} \left(\frac{\sin\mu_{2n}(x)}{x-x_k} \right)' = \frac{1}{2}\cos\mu_{2n}(x_k)\mu_{2n}''(x_k),$$

where

$$\mu_{2n}''(x) = -\frac{x\lambda_{2n}(x) + (1-x^2)\lambda_{2n}'(x)}{(1-x^2)^{3/2}},$$

therefore

$$\lim_{x\to x_k} E_k'(x) = \left[\frac{x_k}{1-x_k^2} + \frac{U_n''(x_k)}{U_n'(x_k)} - \frac{\lambda_{2n}'(x_k)}{2\lambda_{2n}(x_k)} + b_k \right] = 0,$$

due to (2.11) which shows that $\{E_k(x)\}_{k=1}^{n-1}$ given by (2.10) satisfy all the conditions given by (2.5) for $k = 1, 2, \ldots, n-1$.

Similarly, we can show that E_0 and $E_n(x)$ given by (2.9) and (2.12), respectively, satisfy conditions (2.5) for $k = 0$ and (2.5), for $k = n$, respectively, which completes the proof of the Lemma. □

Remark 4. The Pál type (0, 1; 0)-interpolation $W_n(f, x)$, satisfying conditions (2.1) can be explicitly represented as (2.2) with the help of Lemmas 1–3. Taking all a_i's as zero, $W_n(f, x)$ reduces to the interpolated polynomials of degree $\leq 3n-3$.

Theorem 5. *The function $W_n(f,x)$ is a rational function, that is,*

$$W_n(f,x) \in \mathcal{R}_{3n-3}(a_0,a_1,a_2,\dots,a_{2n-1}).$$

Proof. Since $U_n \in \mathcal{R}_{n-1}(a_0,a_1,\dots,a_{2n-1})$, we can express it as

$$U_n(x) := \frac{S_{n-1}(x)}{S_n^*(x)},$$

where $S_n^*(x) := \sqrt{\prod_{k=0}^{2n-1}(1+xa_k)}$, $S_{n-1}(x) := c_{n-1}(x-x_1)(x-x_2)\dots(x-x_{n-1})$ and c_{n-1} depends on n and $\{a_k\}_{k=0}^{2n-1}$. So, we have

$$\ell_k(x) = \frac{S_n^*(x_k)}{S_n^*(x)} q_k(x), \quad k=1,2,\dots,n-1,$$

where

$$q_k(x) := \frac{S_{n-1}(x)}{S'_{n-1}(x_k)(x-x_k)}, \quad k=1,2,\dots,n-1.$$

Thus $\ell_k(x) \in \mathcal{R}_{n-2}(a_0,a_1,\dots,a_{2n-1})$. Similarly, we can express

$$U'_n(x) := \frac{Q_{n-2}(x)}{S_n^*(x)},$$

where $Q_{n-1}(x) := d_{n-1}(x-t_1)(x-t_2)\dots(x-t_{n-2})$ and d_{n-1} depends on n and $\{a_k\}_{k=0}^{2n-1}$. Then

$$L_k(x) = \frac{S_n^*(t_k)}{S_n^*(x)} q_k^*(x), \quad k=1,2,\dots,n-1,$$

where

$$q_k^*(x) := \frac{Q_{n-2}(x)}{Q'_{n-2}(t_k)(x-t_k)}, \quad k=1,2,\dots,n-2.$$

Thus $L_k(x) \in \mathcal{R}_{n-3}(a_0,a_1,\dots,a_{2n-1})$. Hence, by (2.6), (2.7) and (2.10) the lemma follows. □

Remark 6. Notice that the poles of the rational function $W_n(f,x)$ can be found from the equality $\lambda_{2n}(x)=0$. They depend on the parameters $a_k, k=0,1,\dots,2n-1$. The relationship between the zeros of the function $\lambda_{2n}(x)$ and the parameters a_k is described in [17].

3. Quadrature Formula

Under the same assumption on the parameters $a_1,a_2,\dots,a_{2n1}$, we consider the following Pál type $(0,1;0)$-interpolation.

For the given function f defined on $[-1,1]$, we define the function

$$V_n(f,x) = \sum_{k=0}^{n} f(x_k)\Omega_k(x) + \sum_{k=1}^{n-1} \alpha_k\sigma_k(x) + \sum_{k=1}^{n-2} f(t_k)\gamma_k(x), \tag{3.1}$$

where, for $k=1,2,\dots,n-1$,

$$\begin{aligned}\Omega_k(x) =& \frac{(1-x^2)U'_n(x)}{(1-x_k^2)U'_n(x_k)}\left[1-2\left(\frac{U''_n(x_k)}{U'_n(x_k)}+\frac{x_k}{(1-x_k^2)}\right)(x-x_k)\right]\ell_k^2(x),\\ \Omega_0(x) =& \frac{(1+x)U_n^2(x)U'_n(x)}{2U_n^2(1)U'_n(1)}, \quad \Omega_n(x) = \frac{(1-x)U_n^2(x)U'_n(x)}{2U_n^2(-1)U'_n(-1)},\end{aligned}$$

for $k=1,2,\dots,n-1$,

$$\sigma_k(x) = \frac{(1-x^2)U_n(x)U'_n(x)\ell_k(x)}{(1-x_k^2)(U'_n(x_k))^2}$$

and for $k=1,2,\dots,n-2$,

$$\gamma_k(x) = \frac{(1-x^2)U_n^2(x)L_k(x)}{(1-t_k^2)U_n^2(t_k)}.$$

The function $V_n(x)$ given by (3.1), satisfies conditions (2.1) and hence is the Pál type $(0,1;0)$-interpolation, and

$$V_n(f,x) \in \mathcal{R}_{3n-3}(a_1, a_2, \dots, a_{2n-1}).$$

The quadrature formula corresponding to the interpolatory function (3.1) is given by

$$\begin{aligned}\int_{-1}^{1}(1-x^2)f(x)dx &\approx \sum_{k=0}^{n} f(x_k)\int_{-1}^{1}(1-x^2)\Omega_k(x)dx \\ &+ \sum_{k=1}^{n-1} f'(x_k)\int_{-1}^{1}(1-x^2)\sigma_k(x)dx + \sum_{k=1}^{n-2} f(y_k)\int_{-1}^{1}(1-x^2)\gamma_k(x)dx \\ &\approx \sum_{k=0}^{n} E_k f(x_k) + \sum_{k=1}^{n-1} D_k f'(x_k) + \sum_{k=1}^{n-2} C_k f(y_k), \end{aligned} \tag{3.2}$$

where

$$E_k = \int_{-1}^{1}(1-x^2)\Omega_k(x)dx, \quad k=0,1,\dots,n, \tag{3.3}$$

$$D_k = \int_{-1}^{1}(1-x^2)\sigma_k(x)dx, \quad k=1,2,\dots,n-1, \tag{3.4}$$

$$C_k = \int_{-1}^{1}(1-x^2)\gamma_k(x)dx, \quad k=1,2,\dots,n-2. \tag{3.5}$$

Theorem 7. *The quadrature formula* (3.2) *can be expressed as*

$$\int_{-1}^{1}(1-x^2)f(x)dx = \sum_{k=1}^{n-1}\left(\frac{2\pi(1-x_k^2)^{3/2}}{\lambda_{2n}(x_k)}\right)f(x_k). \tag{3.6}$$

Remark 8. The quadrature formula (3.6) can be evaluated by finding the value of the integrals (3.3), (3.4) and (3.5). These integrals have singularities lying in the interval $[-1,1]$. The integrals are evaluated by performing suitable transformations and using the Cauchy residue theorem at the poles which lie in the interval.

To prove Theorem 7, we shall need the following lemmas below.

Lemma 9. *For D_k, $k=1,2,\dots,n-1$, given by* (3.4), *we have*

$$D_k = \frac{1}{(1-x_k^2)(U_n'(x_k))^3}\int_{-1}^{1}\frac{(1-x^2)^2U_n^2(x)U_n'(x)}{(x-x_k)}dx = 0.$$

Proof. D_k for $k=1,2,\dots,n-1$, given by (3.4), can be represented as

$$D_k = \frac{1}{(1-x_k^2)(U_n'(x_k))^3}I_k, \tag{3.7}$$

where

$$I_k = \int_{-1}^{1}\frac{(1-x^2)^2U_n^2(x)U_n'(x)}{(x-x_k)}dx$$

$$= \int_{-1}^{1} \frac{\sin^2 \mu_{2n}(x)}{(x-x_k)} \left(\frac{x \sin \mu_{2n}(x) - \cos \mu_{2n}(x) \lambda_{2n}(x) \sqrt{1-x^2}}{\sqrt{1-x^2}} \right) dx = I_{k1} - I_{k2},$$

where

$$I_{k1} = \int_{-1}^{1} \frac{x \sin^3 \mu_{2n}(x)}{(x-x_k)\sqrt{1-x^2}} dx \tag{3.8}$$

and

$$I_{k2} = \int_{-1}^{1} \frac{\sin^2 \mu_{2n}(x) \cos \mu_{2n}(x) \lambda_{2n}(x)}{(x-x_k)} dx. \tag{3.9}$$

Consider the transformation

$$x = \frac{1-y^2}{1+y^2} \tag{3.10}$$

which gives

$$dx = -\frac{4y}{(1+y^2)^2} dy, \tag{3.11}$$

$$\sqrt{1-x^2} = \frac{2y}{(1+y^2)}, \tag{3.12}$$

$$(x-x_k) = \frac{-2(y^2-y_k^2)}{(1+y^2)(1+y_k^2)}. \tag{3.13}$$

We know that

$$\sin \mu_{2n} \left(\frac{1-y^2}{1+y^2} \right) = \sin \phi_{2n}(y), \tag{3.14}$$

where $\sin \phi_{2n}(y)$ is Bernstein's sine fraction

$$\sin \phi_{2n}(y) = \frac{1}{2i} \Big(\chi_n(y) - \chi_n^{-1}(y) \Big), \tag{3.15}$$

where

$$\chi_n(y) = \prod_{j=0}^{2n-1} \frac{y-z_j}{y-\bar{z_j}}$$

and z_k are the roots of the equations $y^2 + (1+a_k)(1-a_k)^{-1} = 0$, $\mathcal{I} z_k > 0$, $k = 0, 1, \ldots, 2n-1$. Taking into account the assumptions on the parameters a_k, $k = 0, 1, \ldots, 2n-1$, we have the following: 1) $z_0 = i$, 2) if a_k and a_l are paired by a complex conjugation, then the corresponding numbers z_k and z_l are symmetric with respect to the imaginary axis. Besides, the function $\sin \phi_{2n}(y)$ has zeros at $\pm y_k,\ y_k = \sqrt{(1-x_k)/(1+x_k)}$, $k = 1, 2, \ldots, n-1$. Thus, by using transformation (3.10)–(3.13) and (3.14) in (3.8), we get

$$\begin{aligned} I_{k1} &= -\frac{1+y_k^2}{2} \int_{-\infty}^{\infty} \left(\frac{1-y^2}{1+y^2} \right) \frac{\sin^3 \phi_{2n}(y)}{(y^2-y_k^2)} dy \\ &= -\frac{1+y_k^2}{2} \lim_{z \to y_k, \mathcal{I} z_k > 0} J_{k1}(z), \end{aligned}$$

where

$$J_{k1}(z) = \int_{-\infty}^{\infty} \left(\frac{1-y^2}{1+y^2} \right) \frac{\sin^3 \phi_{2n}(y)}{(y^2-z^2)} dy.$$

From (3.14), we have

$$\sin^3 \phi_{2n}(y) = -\frac{1}{8i} \left(\chi_n^3(y) - 3\chi_n(y) + 3\chi_n^{-1}(y) - \chi_n^{-3}(y) \right). \tag{3.16}$$

Thus

$$J_{k1}(z) = \frac{1}{8i}\Big(J_{k11}(z) - 3J_{k12}(z) + 3J_{k13}(z) - J_{k14}(z)\Big), \tag{3.17}$$

where

$$J_{k11}(z) = \int_{-\infty}^{\infty} \left(\frac{1-y^2}{1+y^2}\right) \frac{\chi_n^3(y)}{(y^2-z^2)} dy,$$

$$J_{k12}(z) = \int_{-\infty}^{\infty} \left(\frac{1-y^2}{1+y^2}\right) \frac{\chi_n^{-3}(y)}{(y^2-z^2)} dy,$$

$$J_{k13}(z) = \int_{-\infty}^{\infty} \left(\frac{1-y^2}{1+y^2}\right) \frac{\chi_n(y)}{(y^2-z^2)} dy$$

and

$$J_{k14}(z) = \int_{-\infty}^{\infty} \left(\frac{1-y^2}{1+y^2}\right) \frac{\chi_n^{-1}(y)}{(y^2-z^2)} dy.$$

Since $z_0 = i$, thus the integrand of $J_{k11}(z)$ has only a singular point $y = z$ in the upper half plane. Thus by the residue theorem, we have

$$\begin{aligned} J_{k11}(z) =& 2\pi i \lim_{y\to z} \left(\frac{1-y^2}{1+y^2}\right) \frac{\chi_n^3(y)}{(y+z)} \\ =& \left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n^3(z)}{z} \pi i. \end{aligned} \tag{3.18}$$

Similarly,

$$J_{k12}(z) = \left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n^{-3}(z)}{z} \pi i, \tag{3.19}$$

$$J_{k13}(z) = \left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n(z)}{z} \pi i \tag{3.20}$$

and

$$J_{k14}(z) = \left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n^{-1}(z)}{z} \pi i \tag{3.21}$$

Using (3.18), (3.19), (3.20) and (3.21) in (3.17), we get

$$\begin{aligned} J_{k1}(z) =& \frac{1}{8i}\Bigg(\left(\left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n^3(z)}{z} \pi i - \left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n^{-3}(z)}{z} \pi i \right. \\ & - 3\left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n(z)}{z} \pi i + 3\left(\frac{1-z^2}{1+z^2}\right) \frac{\chi_n^{-1}(z)}{z} \pi i \Bigg). \end{aligned}$$

Taking the limit as $\lim\limits_{z\to y_k, \mathcal{I}z_k>0}$ and using $\chi_n(y_k) = 1$, it follows that

$$I_{k1} = 0. \tag{3.22}$$

Now we evaluate I_{k2}, given by (3.9). Using (3.11) and (3.13) in (3.9), we get

$$I_{k2} \quad = \quad -(1+y_k^2) \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \sin^2 \mu_{2n}\left(\frac{1-y^2}{1+y^2}\right) \cos \mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)}{(1+y^2)(y^2-y_k^2)} \; dy.$$

We know that

$$\cos \mu_{2n}\left(\frac{1-y^2}{1+y^2}\right) = \cos \phi_{2n}(y),$$

where $\cos\phi_{2n}(y)$ is Bernstein's cosine fraction

$$\cos\phi_{2n}(y) = \frac{1}{2}\left(\chi_n(y) + \chi_n^{-1}(y)\right). \tag{3.23}$$

Thus by (3.14) and (3.23), we have

$$I_{k2} = -(1+y_k^2) \lim_{z\to y_k, \Im z_k>0} \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\phi_{2n}(y)\cos\phi_{2n}(y)}{(1+y^2)(y^2-z^2)}\,dy.$$

By virtue of (3.14) and (3.23), we have

$$\sin^2\phi_{2n}(y)\cos\phi_{2n}(y) = -\frac{1}{8}\left(\chi_n^3(y) + \chi_n^{-3}(y) - \chi_n(y) - \chi_n^{-1}(y)\right). \tag{3.24}$$

Thus

$$I_{k2} = \frac{(1+y_k^2)}{8} \lim_{z\to y_k, \Im z_k>0} \Big(J_{k21}(z) + J_{k22}(z) - J_{k23}(z) - J_{k24}(z)\Big), \tag{3.25}$$

where

$$J_{k21}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)(y^2-z^2)}dy,$$

$$J_{k22}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-3}(y)}{(1+y^2)(y^2-z^2)}dy,$$

$$J_{k23}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(1+y^2)(y^2-z^2)}dy$$

and

$$J_{k24}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)(y^2-z^2)}dy.$$

Since $z_0 = i$, thus the integrand of $J_{k21}(z)$ has only a singular point $y = z$ in the upper half-plane. Hence by the residue theorem, we have

$$\begin{aligned} J_{k21}(z) =& 2\pi i \lim_{y\to z} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(1+y^2)(y+z)} \\ =& \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^3(z)}{(1+z^2)}\pi i. \end{aligned} \tag{3.26}$$

Similarly,

$$J_{k22}(z) = \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^{-3}(z)}{(1+z^2)}\pi i, \tag{3.27}$$

$$J_{k23}(z) = \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n(z)}{(1+z^2)}\pi i \tag{3.28}$$

and

$$J_{k24}(z) = \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^{-1}(z)}{(1+z^2)}\pi i. \tag{3.29}$$

Putting the values of $J_{k21}(z), J_{k22}(z), J_{k23}(z)$ and $J_{k24}(z)$ from (3.26), (3.27), (3.28) and (3.29), respectively, in (3.25), we get

$$I_{k2} = \frac{(1+y_k^2)}{8} \lim_{z\to y_k, \Im z_k>0} \frac{\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)}{(1+z^2)} \left(\chi_n^3(z) + \chi_n^{-3}(z) - \chi_n(z) - \chi_n^{-1}(z)\right)\pi i.$$

Since $\chi_n(y_k) = 1$, thus

$$I_{k2} = 0. \tag{3.30}$$

Using (3.22) and (3.30) in (3.7), the Lemma follows. □

Lemma 10. *For $E_k, k = 1, 2, \ldots, n-1$ given by (3.3), we have*

$$E_k = \frac{2\pi(1-x_k^2)^{3/2}}{\lambda_{2n}(x_k)}.$$

Proof. E_k for $k = 1, 2, \ldots, n-1$ given by (3.3), due to (2.8) and Lemma 2 can be represented as

$$\begin{aligned} E_k =& \frac{1}{(1-x_k^2)U_n'(x_k)} \int_{-1}^{1} (1-x^2)^2 U_n'(x)\ell_k^2(x)dx \\ =& \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)U_n'(x_k)} \int_{-1}^{1} \frac{(1-x^2)U_n'(x)\sin^2\mu_{2n}(x)}{(x-x_k)^2}dx. \end{aligned}$$

Since

$$U_n'(x) = \frac{-\cos\mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2} + x\sin\mu_{2n}(x)}{(1-x^2)^{3/2}},$$

we have

$$E_k(x) = \frac{(1-x_k^2)}{\lambda_{2n}^2(x_k)U_n'(x_k)} I_k, \tag{3.31}$$

where

$$I_k = I_{k1} - I_{k2} \tag{3.32}$$

with

$$I_{k1} = \int_{-1}^{1} \frac{x\sin^3\mu_{2n}(x)}{\sqrt{1-x^2}(x-x_k)^2}dx \tag{3.33}$$

and

$$I_{k2} = \int_{-1}^{1} \frac{\lambda_{2n}(x)\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)}{(x-x_k)^2}dx. \tag{3.34}$$

Using transformation (3.10) and due to (3.11), (3.13) and (3.14), (3.33) can be transformed to

$$\begin{aligned} I_{k1} =& \frac{(1+y_k)^2}{2} \int_{-\infty}^{\infty} \frac{(1-y^2)\sin^3\phi_{2n}(y)}{(y^2-y_k^2)^2}\ dy \\ =& \frac{(1+y_k^2)^2}{2} \lim_{z\to y_k, \Im z_k>0} \int_{-\infty}^{\infty} \frac{(1-y^2)\sin^3\phi_{2n}(y)}{(y^2-z^2)^2}\ dy. \end{aligned} \tag{3.35}$$

Due to (3.16), (3.35) can be represented as

$$I_{k1} = -\frac{(1+y_k)^2}{16i} \lim_{z\to y_k, \Im z_k>0} \left(I_{k11}(z) - 3I_{k12}(z) + 3I_{k13}(z) - I_{k14}(z)\right), \tag{3.36}$$

where

$$I_{k11}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^3(y)}{(y^2-z^2)^2}dy,$$

$$I_{k12}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n(y)}{(y^2-z^2)^2}dy,$$

$$I_{k13}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-1}(y)}{(y^2-z^2)^2}dy$$

and

$$I_{k14}(z) = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-3}(y)}{(y^2-z^2)^2}dy.$$

Since $z_0 = i$, the integrand of $I_{k11}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus by the residue theorem, we have

$$I_{k11}(z) = 2\pi i \lim_{y\to z} \frac{d}{dy} \frac{(1-y^2)\chi_n^3(y)}{(y+z)^2}$$

which implies

$$I_{k11}(z) = 2\pi i\left(\frac{z\{3(1-z^2)\chi_n^2(z)\chi_n'(z) - 2z\chi_n^3(z)\} - (1-z^2)\chi_n^3(z)}{4z^3} \right).$$

On simple calculations and using $\chi_n(y_k) = 1$, we get

$$\lim_{z\to y_k, \Im z_k>0} I_{k11}(z) = \frac{\pi i}{2y_k^3}\left(3y_k(1-y_k^2)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right). \tag{3.37}$$

Similarly,

$$\lim_{z\to y_k, \Im z_k>0} I_{k14}(z) = \frac{\pi i}{2y_k^3}\left(3y_k(1-y_k^2)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right). \tag{3.38}$$

Since the integrand of $I_{k12}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus again, using the residue theorem, we have

$$I_{k12}(z) = 2\pi i \lim_{y\to z} \frac{d}{dy} \frac{(1-y^2)\chi_n(y)}{(y+z)^2}$$

which gives

$$I_{k12}(z) = 2\pi i\left(\frac{z\{(1-z^2)\chi_n'(z) - 2z\chi_n(z)\} - (1-z^2)\chi_n(z)}{4z^3} \right).$$

On simple calculations and using $\chi_n(y_k) = 1$, we get

$$\lim_{z\to y_k, \Im z_k>0} I_{k12}(z) = \frac{\pi i}{2y_k^3}\left(y_k(1-y_k^2)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right). \tag{3.39}$$

Similarly,

$$\lim_{z\to y_k, \Im z_k>0} I_{k13}(z) = \frac{\pi i}{2y_k^3}\left(y_k(1-y_k^2)\sum_{j=0}^{2n-1} \frac{z_j - \bar{z}_j}{(y_k - z_j)(y_k - \bar{z}_j)} - (1+y_k^2) \right) \tag{3.40}$$

Using (3.37), (3.38), (3.39), (3.40) in (3.36), we get

$$I_{k1} = 0. \tag{3.41}$$

Now we evaluate I_{k2} given by (3.34). Using the transformation (3.10) and due to (3.11), (3.13) and (3.14), (3.34) can be written as

$$I_{k2} \quad = \quad (1+y_k^2)^2 \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)\cos\mu_{2n}\left(\frac{1-y^2}{1+y^2}\right)}{(y^2-y_k^2)^2}\,dy.$$

Thus by (3.15) and (3.23), we have

$$I_{k2} = (1+y_k^2)^2 \lim_{z\to y_k,\Im z_k>0} \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\sin^2\phi_{2n}(y)\cos\phi_{2n}(y)}{(y^2-z^2)^2}\,dy.$$

Using (3.24) in the above equation, we get

$$I_{k2} = -\frac{(1+y_k^2)^2}{8} \lim_{z\to y_k,\Im z_k>0} \left(I_{k21}(z)+I_{k22}(z)-I_{k23}(z)-I_{k24}(z)\right), \tag{3.42}$$

where

$$I_{k21}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(y^2-z^2)^2}dy,$$

$$I_{k22}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-3}(y)}{(y^2-z^2)^2}dy,$$

$$I_{k23}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(y^2-z^2)^2}dy$$

and

$$I_{k24}(z) = \int_{-\infty}^{\infty} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^{-1}(y)}{(y^2-z^2)^2}dy.$$

Since $z_0 = i$, the integrand of $I_{k21}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus by the residue theorem, we have

$$\begin{aligned} I_{k21}(z) =& 2\pi i \lim_{y\to z} \frac{d}{dy} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n^3(y)}{(y+z)^2} \\ =& \frac{\pi i}{2z}\left(3\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n^2(z)\chi_n'(z) + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_z \chi_n^3(z)\right). \end{aligned}$$

On simple calculations and using $\chi_n(y_k) = 1$, we get

$$\begin{aligned} \lim_{z\to y_k,\Im z_k>0} I_{k21}(z) =& \frac{\pi i}{2y_k}\left(3\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right. \\ &\left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right). \end{aligned} \tag{3.43}$$

Similarly,

$$\begin{aligned} \lim_{z\to y_k,\Im z_k>0} I_{k22}(z) =& \frac{\pi i}{2y_k}\left(3\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right. \\ &\left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right). \end{aligned} \tag{3.44}$$

The integrand of $I_{k23}(z)$ has only a singular point $y = z$ in the upper half-plane. Thus by the residue theorem, we have

$$\begin{aligned} I_{k23}(z) =& 2\pi i \lim_{y\to z} \frac{d}{dy} \frac{y\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\chi_n(y)}{(y+z)^2} \\ =& \frac{\pi i}{2z}\left(\lambda_{2n}\left(\frac{1-z^2}{1+z^2}\right)\chi_n'(z) + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_z \chi_n(z)\right) \end{aligned}$$

which on simple calculations and using $\chi_n(y_k) = 1$, gives

$$\begin{aligned} \lim_{z\to y_k, \Im z_k>0} I_{k23}(z) =& \frac{\pi i}{2y_k}\left(\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right. \\ &\left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right) \end{aligned} \tag{3.45}$$

$$\begin{aligned} \lim_{z\to y_k, \Im z_k>0} I_{k24}(z) =& \frac{\pi i}{2y_k}\left(\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right. \\ &\left. + \left(\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)\right)'_{y_k}\right). \end{aligned} \tag{3.46}$$

Using (3.43), (3.44), (3.45) and (3.46) in (3.42), we get

$$I_{k2} = -\frac{(1+y_k^2)^2}{4y_k}\pi i\left(\lambda_{2n}\left(\frac{1-y_k^2}{1+y_k^2}\right)\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)}\right).$$

We know that

$$\sum_{j=0}^{2n-1}\frac{z_j-\bar{z}_j}{(y_k-z_j)(y_k-\bar{z}_j)} = -\frac{4\lambda_{2n}(x_k)}{i(1+y_k^2)},$$

hence

$$I_{k2} = \frac{2\pi\lambda_{2n}^2(x_k)}{\sqrt{1-x_k^2}}. \tag{3.47}$$

Putting the values of I_{k1} and I_{k2} from (3.41) and (3.47), respectively, in (3.32), we get

$$I_k = -\frac{2\pi\lambda_{2n}^2(x_k)}{\sqrt{1-x_k^2}}.$$

Substituting this value of I_k in (3.31), the Lemma follows. □

Lemma 11. *For E_0 defined by (3.3) for $k = 0$, we have*

$$E_0 = 0.$$

Proof. For $k = 0$, (3.3) can be represented as

$$E_0 = \frac{1}{2U_n^2(1)U_n'(1)} I_0, \tag{3.48}$$

where

$$\begin{aligned} I_0 =& \int_{-1}^{1}(1+x)(1-x^2)U_n^2(x)U_n'(x)dx \\ =& \int_{-1}^{1}(1+x)\sin^2\mu_{2n}(x)\left(\frac{x\sin\mu_{2n}(x)-\cos\mu_{2n}(x)\lambda_{2n}(x)\sqrt{1-x^2}}{(1-x^2)^{3/2}}\right)dx, \end{aligned}$$

$$I_0 = I_{01} - I_{02}, \tag{3.49}$$

where

$$I_{01} = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \left(\frac{x \sin^3 \mu_{2n}(x)}{(1-x^2)} \right) dx \tag{3.50}$$

and

$$I_{02} = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \left(\frac{\sin^2 \mu_{2n}(x) \cos \mu_{2n}(x) \lambda_{2n}(x)}{\sqrt{1-x^2}} \right) dx. \tag{3.51}$$

First, we evaluate I_{01}. Using transformations (3.10) and (3.11), (3.12) and (3.14) in (3.50), we have

$$I_{01} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{y^2} \left(\frac{1-y^2}{1+y^2} \right) \sin^3 \phi_{2n}(y) dy,$$

and due to (3.15), I_1 can be represented as

$$I_{01} = -\frac{1}{16i} \int_{-\infty}^{\infty} \Big(I_{011} - I_{012} - 3I_{013} + 3I_{014} \Big), \tag{3.52}$$

where

$$I_{011} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^3(y)}{y^2(1+y^2)} dy, \tag{3.53}$$

$$I_{012} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-3}(y)}{y^2(1+y^2)} dy,$$

$$I_{013} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n(y)}{y^2(1+y^2)} dy \tag{3.54}$$

and

$$I_{014} = \int_{-\infty}^{\infty} \frac{(1-y^2)\chi_n^{-1}(y)}{y^2(1+y^2)} dy.$$

Since $z_0 = i$, the integrand of I_{011}, given by (3.53), has only a singular point $y = 0$ in the upper half-plane. Thus by the residue theorem, we have

$$I_{011} = 2\pi i \lim_{y \to 0} \frac{d}{dy} \frac{1-y^2}{1+y^2} \chi_n^3(y) = 6\pi i \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z}_j} - \frac{1}{z_j} \right) = -24\pi \lambda_{2n}(1). \tag{3.55}$$

Similarly,

$$I_{012} = -24\pi \lambda_{2n}(1). \tag{3.56}$$

Again, using the residue theorem for I_{013}, given by (3.54), we get

$$I_{013} = 2\pi i \lim_{y \to 0} \frac{d}{dy} \frac{1-y^2}{1+y^2} \chi_n(y) = 2\pi i \sum_{j=0}^{2n-1} \left(\frac{1}{\bar{z}_j} - \frac{1}{z_j} \right) = -8\pi \lambda_{2n}(1). \tag{3.57}$$

and, similarly,

$$I_{014} = -8\pi \lambda_{2n}(1). \tag{3.58}$$

Thus using (3.55), (3.56), (3.57) and (3.58) in (3.52), we get

$$I_{01} = 0. \tag{3.59}$$

Now, for I_{02}, given by (3.51), due to (3.10)–(3.12) and (3.14), on a simple calculation, we have

$$I_{02} = \int_{-\infty}^{\infty} \frac{\sin^2 \phi_{2n}(y) \cos \phi_{2n}(y) \lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right)}{y(1+y^2)} dy$$

which, due to (3.24), can be represented as

$$I_{02} = -\frac{1}{8}\bigg(I_{021} + I_{022} - I_{023} - I_{024}\bigg), \tag{3.60}$$

where

$$I_{021} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \chi_n^3(y)}{y(1+y^2)} dy,$$

$$I_{022} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \chi_n^{-3}(y)}{y(1+y^2)} dy,$$

$$I_{023} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \chi_n(y)}{y(1+y^2)} dy$$

and

$$I_{024} = \int_{-\infty}^{\infty} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \chi_n^{-1}(y)}{y(1+y^2)} dy.$$

Now, since for I_{021}, the only singularity on the upper half plane is $y = 0$, hence by residue theorem, we have

$$I_{021} = 2\pi i \lim_{y\to 0} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \chi_n^3(y)}{(1+y^2)} = 2\pi i \lambda_{2n}(1). \tag{3.61}$$

Similarly,

$$I_{022} = 2\pi i \lambda_{2n}(1). \tag{3.62}$$

Again, for I_{023}, by the residue theorem, we have

$$I_{023} = 2\pi i \lim_{y\to 0} \frac{\lambda_{2n}\left(\frac{1-y^2}{1+y^2}\right) \chi_n(y)}{(1+y^2)} = 2\pi i \lambda_{2n}(1) \tag{3.63}$$

and

$$I_{024} = 2\pi i \lambda_{2n}(1). \tag{3.64}$$

Substituting the values of (3.61), (3.62), (3.63) and (3.64) in (3.60), we get

$$I_{02} = 0. \tag{3.65}$$

Putting the value of I_{01} and I_{02} from (3.59) and (3.65), respectively, in (3.49), we get $I_0 = 0$ which, due to (3.48), implies

$$E_0 = 0.$$

□

Lemma 12. *For E_n defined by* (3.3) *for $k = n$, we have*

$$E_n = 0.$$

Proof. For $k = n$, (3.3) can be represented as

$$E_n(x) = \frac{1}{2U_n^2(-1)U_n'(-1)} I_n,$$

where

$$\begin{aligned} I_n &= \int_{-1}^{1} (1-x)(1-x^2)U_n^2(x)U_n'(x)dx \\ &= \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \left(\frac{x\sin^3\mu_{2n}(x)}{(1-x^2)} - \frac{\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x)}{\sqrt{1-x^2}} \right) dx = I_{n1} - I_{n2}. \end{aligned}$$

Following the same steps as in Lemma 11, we get $I_{n1} = I_{n2} = 0$ which implies that $I_n = 0$ and hence the Lemma follows. □

Lemma 13. *For C_k given by* (3.5)*, we have $C_k = 0$, $k = 1, 2, \dots, n-2$.*

Proof. C_k given by (3.5), can be represented as

$$C_k = \frac{1}{(1-t_k^2)U_n^2(t_k)U_n''(t_k)} I_k,$$

where

$$I_k = \int_{-1}^{1} \frac{(1-x^2)^2 U_n^2(x)U_n'(x)}{(x-t_k)} dx,$$

which reduces to

$$I_k = \int_{-1}^{1} \frac{1}{(x-t_k)} \left(\frac{x\sin^3\mu_{2n}(x)}{\sqrt{1-x^2}} - \sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x) \right) dx = I_{k1} - I_{k2},$$

where

$$I_{k1} = \int_{-1}^{1} \frac{x\sin^3\mu_{2n}(x)}{(x-t_k)\sqrt{1-x^2}} dx$$

and

$$I_{k2} = \int_{-1}^{1} \frac{\sin^2\mu_{2n}(x)\cos\mu_{2n}(x)\lambda_{2n}(x)}{(x-t_k)} dx.$$

Proceeding as in the above lemmas, it follows that $I_{k1} = I_{k2} = 0$ which implies that $I_k = 0$, from which the Lemma follows. □

From Lemma 9–13 and (3.2), Theorem 7 follows.

4. Conclusion

Here, a quadrature formula corresponding to the Pál type $(0, 1; 0)$-interpolation in rational spaces has been obtained. This study may further be extended to the case of lacunary interpolation in rational spaces.

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