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SCHUR-GEOMETRIC AND SCHUR-HARMONIC CONVEXITY OF WEIGHTED INTEGRAL MEAN

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Abstract. Recently, there have been many new results on Schur convexity of integral means. In this paper we investigate the necessary and sufficient conditions for the existence of Schur-geometric and Schur-harmonic properties in weighted integral means, weighted midpoint and weighted trapezoid quadrature formulas.

1. Introduction

Let us recall the definitions of convex, $n-$convex and Schur-convex functions.

Definition 1. A function f is *convex* on an interval I if for any two points $x, y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be *concave.*

Let $A \subset \mathbb{R}^n$. We introduce the following notion: for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in A$, we write $\mathbf{x} \prec \mathbf{y}$, if

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \quad \text{and} \quad \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for} \quad k = 1, \dots, n-1,$$

where $x_{[i]}$ denotes the $i-$th-largest component in $\mathbf{x}$.

Definition 2. Function $F : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be **Schur-convex** on A if for every $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in A$ such that $\mathbf{x} \prec \mathbf{y}$, we have

$$F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n).$$

Function F is said to be **Schur-concave** on A if $-F$ is Schur-convex.

Remark 1. Every convex and symmetric function is Schur-convex.

Numerous researchers have recently investigated Schur-geometric and Schur-harmonic convexities [2, 8, 9].

First, let us define for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\ln \mathbf{x} := (\ln x_1, \dots, \ln x_n)$ and $\frac{1}{\mathbf{x}} := \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right)$.

Let us give the following definitions:

Definition 3. Function $F : A \subset \mathbb{R}^n_+ \to \mathbb{R}$ is said to be **Schur-geometrically convex** on A if for every $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in A$ such that $\ln \mathbf{x} \prec \ln \mathbf{y}$, we have

$$F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n).$$

Function F is said to be **Schur-geometrically concave** on A if $-F$ is Schur-convex.

Definition 4. Function $F : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be **Schur-harmonically convex** on A if for every $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in A$ such that $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$, we have

$$F(x_1, \dots, x_n) \leq F(y_1, \dots, y_n).$$

Function F is said to be **Schur-harmonically concave** on A if $-F$ is Schur-convex.

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Schur-convexity has been investigated by numerous researchers. The following result was proved in [4] for the arithmetic integral mean.

Theorem 1. *Let f be a continuous function on an interval I with a non-empty interior. Then*

$$F(x,y)=\begin{cases}\frac{1}{y-x}\int_x^y f(t)dt & x,y\in I, x\neq y\\ f(x) & x=y\in I\end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

The next result for Schur-convexity of the weighted arithmetic integral mean was proved several years ago [7].

Theorem 2. *Let f be a continuous function on $I\subset\mathbb{R}$ and let w be a positive continuous weight on I. Then the function*

$$F_w(x,y)=\begin{cases}\frac{1}{\int_x^y w(t)dt}\int_x^y w(t)f(t)dt & x,y\in I, x\neq y\\ f(x) & x=y\in I\end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if the inequality

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{w(x)f(x)+w(y)f(y)}{w(x)+w(y)}$$

holds (reverses) for all $x,y\in I$.

The Schur-convex property of the functions

$$M(x,y)=\begin{cases}\frac{1}{y-x}\int_x^y f(t)dt-f\left(\frac{x+y}{2}\right) & x,y\in I, x\neq y\\ 0 & x=y\in I\end{cases}$$

$$T(x,y)=\begin{cases}\frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(t)dt & x,y\in I, x\neq y\\ 0 & x=y\in\end{cases}$$

has been recently investigated (see [1,3]).

The objective of this paper is to give the necessary and sufficient condition for the function $F_w(x,y)$, function $M_w: I^2\to\mathbb{R}$ defined by

$$M_w(x,y)=\begin{cases}\frac{1}{\int_x^y w(t)dt}\int_x^y w(t)f(t)dt-f\left(\frac{x+y}{2}\right) & x,y\in I, x\neq y\\ 0 & x=y\in I\end{cases}$$

and function $T_w: I^2\to\mathbb{R}$ defined by

$$T_w(x,y)=\begin{cases}\frac{f(x)+f(y)}{2}-\frac{1}{\int_x^y w(t)dt}\int_x^y w(t)f(t)dt & x,y\in I, x\neq y\\ 0 & x=y\in I\end{cases}$$

to be Schur-geometrically convex (Schur-geometrically concave) and Schur-harmonically convex (Schur-harmonically concave) on I^2. The necessary and sufficient condition for the functions $M_w(x,y)$ and $T_w(x,y)$ to be Schur-convex on I^2 is given in [5].

Let us recall the weighted one-point quadrature formula [6]. If $f:[x,y]\to\mathbb{R}$ is such that $f^{(n)}$ is a piecewiese continuous function, then we have

$$\int_x^y w(t)f(t)dt=\sum_{j=1}^n A_j(z)f^{(j-1)}(z)+(-1)^n\int_x^y W_{n,w}(t,z)f^{(n)}(t)dt, \tag{1.2}$$

where for $j=1,\ldots,n$

$$A_j(z)=\frac{(-1)^{j-1}}{(j-1)!}\int_x^y (z-s)^{j-1}w(s)ds$$

and

$$W_{n,w}(t,z)=\begin{cases} w_{1n}(t)=\frac{1}{(n-1)!}\int_x^t (t-s)^{n-1}w(s)ds & t\in[x,z],\\ w_{2n}(t)=\frac{1}{(n-1)!}\int_y^t (t-s)^{n-1}w(s)ds & t\in(z,y].\end{cases}$$

In order to prove our results, we shall use the following characterization of the Schur-geometric convexity and Schur-harmonic convexity [9]:

Lemma 1. *Let $f : I^2 \subset \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on I^2 and differentiable in the interior of I^2. Then f is Schur-geometrically convex (Schur-geometrically concave) on I^2 if and only if it is symmetric and*

$$(\log b-\log a)\left(b\frac{\partial f}{\partial b}-a\frac{\partial f}{\partial a}\right)\geq 0(\leq 0) \tag{1.3}$$

for all $a, b \in I$.

Lemma 2. *Let $f : I^2 \subset \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on I^2 and differentiable in the interior of I^2. Then f is Schur-harmonically convex (Schur-harmonically concave) on I^2 if and only if it is symmetric and*

$$(b-a)\left(b^2\frac{\partial f}{\partial b}-a^2\frac{\partial f}{\partial a}\right)\geq 0(\leq 0) \tag{1.4}$$

for all $a, b \in I$.

2. Main result

Theorem 3. *The function $F_w(x,y)$ is Schur-geometrically convex (concave) on $I^2 \subset \mathbb{R}^2_+$ if and only if the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)} \tag{2.1}$$

holds (reverses) for every $x, y \in I$.

Proof. Obviously, $F_w(x,y)$ is continuous on I^2, differentiable in the interior of I^2 and symmetric. Let $x, y \in I$, and without loss of generality, we can assume that $x \leq y$. After direct computation we get

$$\begin{aligned}
&(\log y-\log x)\left(y\frac{\partial f}{\partial y}-x\frac{\partial f}{\partial x}\right)\\
&=(\log y-\log x)\cdot\left(y\cdot\frac{w(y)f(y)\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot w(y)}{\left(\int_x^y w(t)dt\right)^2}\right.\\
&\left.-\ x\cdot\frac{-w(x)f(x)\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot(-w(x))}{\left(\int_x^y w(t)dt\right)^2}\right)\\
&=\frac{\log y-\log x}{\int_x^y w(t)dt}\cdot\left(yw(y)f(y)+xw(x)f(x)-\frac{(xw(x)+yw(y))\cdot\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right)\\
&=\frac{(\log y-\log x)\,(xw(x)+yw(y))}{\int_x^y w(t)dt}\left(\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right),
\end{aligned} \tag{2.2}$$

so, the sign of the expression (2.2) depends on the sign of the term in the brackets. According to Lemma 1, the function F_w is Schur-geometrically convex (concave) if and only if (2.1) holds (reverses), so we have proved the theorem. ∎

Theorem 4. *The function $F_w(x,y)$ is Schur-harmonically convex (concave) on $I^2 \subset \mathbb{R}^2_+$ if and only if the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\leq\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)} \tag{2.3}$$

holds (reverses) for every $x, y \in I$.

Proof. The function $F_w(x,y)$ is continuous on I^2, differentiable in the interior of I^2 and symmetric. Let $x,y\in I$, and without loss of generality, we can assume that $x\le y$. We compute

$$\begin{aligned}
&(y-x)\left(y^2\frac{\partial F_w}{\partial y}-x^2\frac{\partial F_w}{\partial x}\right)\\
&=(y-x)\cdot\left(y^2\cdot\frac{w(y)f(y)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot w(y)}{\left(\int_x^y w(t)dt\right)^2}\right.\\
&\left.-x^2\cdot\frac{-w(x)f(x)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot(-w(x))}{\left(\int_x^y w(t)dt\right)^2}\right)\\
&=\frac{y-x}{\int_x^y w(t)dt}\cdot\left(x^2w(x)f(x)+y^2w(y)f(y)-\frac{\left(x^2w(x)+y^2w(y)\right)\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right)\\
&=\frac{(y-x)\left(x^2w(x)+y^2w(y)\right)}{\int_x^y w(t)dt}\left(\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)}-\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right). \qquad (2.4)
\end{aligned}$$

The term $\frac{(y-x)\left(x^2w(x)+y^2w(y)\right)}{\int_x^y w(t)dt}$ is always positive, so the sign of the expression (2.4) depends only on the sign of the term in brackets. According to Lemma 2, function F_w is Schur-harmonically convex (concave) if and only if (2.3) holds (reverses), so we have proved the theorem. □

Remark 2. If $w(t)=\frac{1}{y-x}$ (the case of a uniform weight function), we get the following classification of Schur-geometrically and Schur-harmonically convexity (concavity):

$F(x,y)$ is Schur-geometrically convex (concave) $\Leftrightarrow \frac{\int_x^y f(t)dt}{y-x}\le\frac{xf(x)+yf(y)}{x+y}$, holds (reverses) for every $x,y\in I$.

$F(x,y)$ is Schur-harmonically convex (concave) $\Leftrightarrow \frac{\int_x^y f(t)dt}{y-x}\le\frac{x^2f(x)+y^2f(y)}{x^2+y^2}$, holds (reverses) for every $x,y\in I$.

Theorem 5. *The function $M_w(x,y)$ is Schur-geometrically convex (concave) if $f:I\to\mathbb{R}$ is decreasing (increasing) and the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\le\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)} \qquad (2.5)$$

holds (reverses) for all $x,y\in I$.

Proof. It is easy to check that $M_w(x,y)$ is symmetric, contionuous on I^2 and differentiable on the interior of I^2. According to Lemma 1, we have to check that $M_w(x,y)$ satisfies condition (1.3). Let $x,y\in I$, and without loss of generallity we can assume that $x\le y$. Then we have

$$\begin{aligned}
&(\log y-\log x)\left(y\frac{\partial M_w}{\partial y}-x\frac{\partial M_w}{\partial x}\right)\\
&=(\log y-\log x)\left(y\cdot\frac{w(y)f(y)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot w(y)}{\left(\int_x^y w(t)dt\right)^2}-\frac{y}{2}f'\left(\frac{x+y}{2}\right)\right.\\
&\left.-x\cdot\frac{-w(x)f(x)\cdot\int_x^y w(t)dt-\int_x^y w(t)f(t)dt\cdot(-w(x))}{\left(\int_x^y w(t)dt\right)^2}+\frac{x}{2}f'\left(\frac{x+y}{2}\right)\right)\\
&=\frac{\log y-\log x}{\int_x^y w(t)dt}\left(xw(x)f(x)+yw(y)f(y)-(xw(x)+yw(y))\cdot\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt}\right.\\
&\left.-\frac{y-x}{2}\int\limits_x^y w(t)dt\cdot f'\left(\frac{x+y}{2}\right)\right)=\frac{(\log y-\log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}
\end{aligned}$$

$$\times \left(\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)} - \frac{y-x}{2} \cdot \frac{\int_x^y w(t)dt}{xw(x)+yw(y)} \cdot f'(\frac{x+y}{2}) - \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \right)$$

(If the function f is decreasing (increasing),

then the middle term in the upper identity is ≥ 0 (≤ 0) so)

$$\geq (\leq) \frac{(\log y - \log x)(xw(x)+yw(y))}{\int_x^y w(t)dt} \left(\frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)} - \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \right).$$

Since (2.5) holds (reverses), the condition in Lemma 1 is satisfied and the proof is completed. □

Theorem 6. *The function $M_w(x,y)$ is Schur-harmonically convex (concave) if f is decreasing (increasing) and the inequality*

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \leq \frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)} \tag{2.6}$$

holds (reverses) for all $x, y \in I$.

Proof. Since $M_w(x,y)$ is symmetric, continuous on I^2 and differentiable on the interior of I^2, according to Lemma 2 we have to check that $M_w(x,y)$ satisfies condition (1.4). Let $x, y \in I$, and without loss of generality, we can assume that $x \leq y$. Then we have

$$(y-x)\left(y^2\frac{\partial M_w}{\partial y} - x^2\frac{\partial M_w}{\partial x}\right)$$

$$= \frac{y-x}{\int_x^y w(t)dt} \cdot \left(x^2w(x)f(x)+y^2w(y)f(y) - \left(x^2w(x)+y^2w(y)\right) \cdot \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \right.$$

$$\left. - \frac{y^2-x^2}{2}\int_x^y w(t)dt \cdot f'\left(\frac{x+y}{2}\right)\right)$$

$$= \frac{(y-x)(x^2w(x)+y^2w(y))}{\int_x^y w(t)dt} \cdot \left(\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)} - \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \right.$$

$$\left. - \frac{y^2-x^2}{2} f'(\frac{x+y}{2}) \frac{\int_x^y w(t)dt}{x^2w(x)+y^2w(y)} \right)$$

(If the function f is decreasing (increasing),

then the last term in the upper identity is ≥ 0 (≤ 0) so)

$$\geq (\leq) \frac{(y-x)(x^2w(x)+y^2w(y))}{\int_x^y w(t)dt} \left(\frac{x^2w(x)f(x)+y^2w(y)f(y)}{x^2w(x)+y^2w(y)} - \frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} \right).$$

Since (2.6) holds (reverses), the condition in Lemma 2 is satisfied and the proof is completed. □

Remark 3. For the case of the uniform weight function we have:

$M(x,y)$ is Schur-geometrically convex (concave) if f is decreasing (increasing) and $\frac{\int_x^y f(t)dt}{y-x} \leq \frac{xf(x)+yf(y)}{x+y}$, holds (reverses) for every $x, y \in I$.

$M(x,y)$ is Schur-harmonically convex (concave) if f is decreasing (increasing) and $\frac{\int_x^y f(t)dt}{y-x} \leq \frac{x^2f(x)+y^2f(y)}{x^2+y^2}$, holds (reverses) for every $x, y \in I$.

Theorem 7. *The function $T_w(x,y)$ is Schur-geometrically convex (concave) if $f: I \to \mathbb{R}$ is convex (concave), twice differentiable and*

$$\frac{\int_x^y tw(t)dt}{\int_x^y w(t)dt} = \frac{xw(x)+yw(y)}{w(x)+w(y)} \tag{2.7}$$

and

$$2\frac{w(x)w(y)(y-x)}{w(x)+w(y)} \leq \int_x^y w(t)dt \tag{2.8}$$

holds (reverses) for all $x, y \in I$.

Proof. The function $T_w(x,y)$ is symmetric, continuous on I^2 and differentiable on the interior of I^2, so according to Lemma 1, we have to check if the condition (1.3) holds. Let us assume $x, y \in I$, $x < y$. We have

$$\begin{aligned}
(\log y - \log x)\left(y\frac{\partial T_w}{\partial y} - x\frac{\partial T_w}{\partial x}\right) &= (\log y - \log x)\cdot\left(\frac{yf'(y)}{2} - \frac{yw(y)f(y)}{\int_x^y w(t)dt}\right. \\
&+ \frac{yw(y)\int_x^y w(t)f(t)dt}{\left(\int_x^y w(t)dt\right)^2} - \frac{xf'(x)}{2} - \frac{xw(x)f(x)}{\int_x^y w(t)dt} + \left.\frac{xw(x)\int_x^y w(t)f(t)dt}{\left(\int_x^y w(t)dt\right)^2}\right) \\
&= \frac{(\log y - \log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}\cdot\left(\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}\right. \\
&+ \left.\frac{\int_x^y w(t)dt}{xw(x)+yw(y)}\cdot\frac{yf'(y)-xf'(x)}{2}\right).
\end{aligned} \tag{2.9}$$

From (2.7), we have

$$\begin{aligned}
&(w(x)+w(y))\int_x^y tw(t)dt = (xw(x)+yw(y))\int_x^y w(t)dt \\
&\Rightarrow w(y)\int_x^y (y-t)w(t)dt = w(x)\int_x^y (t-x)w(t)dt.
\end{aligned} \tag{2.10}$$

Further, from (2.10), we have

$$\begin{aligned}
&w(y)\int_x^y (y-t)w(t)dt = w(x)\int_x^y (y-x-y+t)w(t)dt \\
&\Rightarrow w(y)\int_x^y (y-t)w(t)dt = (y-x)w(x)\int_x^y w(t)dt - w(x)\int_x^y (y-t)w(t)dt \\
&\Rightarrow (w(x)+w(y))\cdot\int_x^y (y-t)w(t)dt = (y-x)w(x)\int_x^y w(t)dt \\
&\Rightarrow \frac{w(y)\int_x^y (y-t)w(t)dt}{\int_x^y w(t)dt} = \frac{w(x)w(y)(y-x)}{w(x)+w(y)}.
\end{aligned} \tag{2.11}$$

Applying (2.11) and according to the inequality (2.7), we have

$$\frac{\int_x^y w(t)dt}{2} - \frac{w(y)\int_x^y (y-t)w(t)dt}{\int_x^y w(t)dt} \geq 0.$$

If f is convex, we have $f''(t) \geq 0$, so function f' is increasing, and we have

$$0 < x < y \Rightarrow f'(x) \leq f'(y) \Rightarrow xf'(x) \leq xf'(y) \leq yf'(y). \tag{2.12}$$

Applying (2.10), (2.11) and (2.12), we have

$$\begin{aligned}
&\frac{yw(y)f'(y)\int_x^y (y-t)w(t)dt - xw(x)f'(x)\int_x^y (t-x)w(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}\\
&= \frac{w(y)\int\limits_x^y (y-t)w(t)dt}{(xw(x)+yw(y))\int\limits_x^y w(t)dt}\cdot(yf'(y)-xf'(x))\\
&\le \frac{\int_x^y w(t)dt}{xw(x)+yw(y)}\cdot\frac{yf'(y)-xf'(x)}{2}. \qquad (2.13)
\end{aligned}$$

On the other hand, if we apply (1.2) for $n=2$ and $z=x$ and multiply by $\frac{xw(x)}{xw(x)+yw(y)}$, and also for $z=y$, multiply by $\frac{yw(y)}{xw(x)+yw(y)}$, and then add those two identities, we obtain

$$\begin{aligned}
&\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{xw(x)f(x)+yw(y)f(y)}{xw(x)+yw(y)}\\
&+\frac{yw(y)f'(y)\int_x^y (y-t)w(t)dt - xw(x)f'(x)\int_x^y (t-x)w(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}\\
&= \frac{\int_x^y \left[xw(x)\cdot\int_t^y (s-t)w(s)ds + yw(y)\cdot\int_x^t (t-s)w(s)ds\right] f''(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}. \qquad (2.14)
\end{aligned}$$

Now, we apply (2.13) in (2.9) and use (2.14) to get

$$\begin{aligned}
&(\log y - \log x)\left(y\frac{\partial T_w}{\partial y} - x\frac{\partial T_w}{\partial x}\right) \ge \frac{(\log y-\log x)(xw(x)+yw(y))}{\int_x^y w(t)dt}\\
&\times\frac{\int_x^y \left[xw(x)\cdot\int_t^y (s-t)w(s)ds + yw(y)\cdot\int_x^t (t-s)w(s)ds\right] f''(t)dt}{(xw(x)+yw(y))\cdot\int_x^y w(t)dt}\\
&= \frac{(\log y-\log x)\cdot\int_x^y \left[xw(x)\cdot\int_t^y (s-t)w(s)ds + yw(y)\cdot\int_x^t (t-s)w(s)ds\right] f''(t)dt}{\left(\int_x^y w(t)dt\right)^2}.
\end{aligned}$$

Since f is convex and the integrals in the brackets are non negative, we have proved that $(\log y - \log x)\left(y\frac{\partial T_w}{\partial y} - x\frac{\partial T_w}{\partial x}\right) \ge 0$, for all $x, y \in I$, $x<y$, so, the function T_w is Schur-geometrically convex.

The proof for the Schur-geometrically concave case is similar. □

Theorem 8. *The function $T_w(x,y)$ is Schur-harmonically convex (concave) if $f: I \to \mathbb{R}$ is convex (concave), twice differentiable and*

$$\frac{\int_x^y tw(t)dt}{\int_x^y w(t)dt} = \frac{xw(x)+yw(y)}{w(x)+w(y)} \qquad (2.15)$$

and

$$2\frac{w(x)w(y)(y-x)}{w(x)+w(y)} \le \int\limits_x^y w(t)dt \qquad (2.16)$$

holds (reverses) for all $x, y \in I$.

Proof. Since the function $T_w(x,y)$ is symmetric, continuous on I^2 and differentiable on the interior of I^2, according to Lemma 2, we have to check if the condition (1.4) holds. Let us assume $x, y \in I$,

$x < y$. We have

$$(y-x)\left(y^2\frac{\partial T_w}{\partial y} - x^2\frac{\partial T_w}{\partial x}\right) = (y-x)\cdot\left(\frac{y^2 f'(y)}{2} - \frac{y^2 w(y) f(y)}{\int_x^y w(t)dt}\right.$$
$$+ \frac{y^2 w(y)\int_x^y w(t)f(t)dt}{\left(\int\limits_x^y w(t)dt\right)^2} - \frac{x^2 f'(x)}{2} - \frac{x^2 w(x) f(x)}{\int\limits_x^y w(t)dt} + \left.\frac{x^2 w(x)\int\limits_x^y w(t)f(t)dt}{\left(\int\limits_x^y w(t)dt\right)^2}\right)$$
$$= \frac{(y-x)(x^2 w(x) + y^2 w(y))}{\int_x^y w(t)dt}\cdot\left(\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{x^2 w(x) f(x) + y^2 w(y) f(y)}{x^2 w(x) + y^2 w(y)}\right.$$
$$+ \left.\frac{\int_x^y w(t)dt}{x^2 w(x) + y^2 w(y)}\cdot\frac{y^2 f'(y) - x^2 f'(x)}{2}\right). \tag{2.17}$$

Again, as in the proof of Theorem 7, we conclude that (2.10), (2.11) and

$$\frac{\int_x^y w(t)dt}{2} - \frac{w(y)\int_x^y (y-t)w(t)dt}{\int_x^y w(t)dt} \geq 0$$

hold.

If f is convex, we have $f''(t) \geq 0$, so, the function f' is increasing, and we have

$$0 < x < y \Rightarrow f'(x) \leq f'(y) \Rightarrow x^2 f'(x) \leq x^2 f'(y) \leq y^2 f'(y). \tag{2.18}$$

Applying (2.10), (2.11) and (2.18), we have

$$\frac{y^2 w(y) f'(y)\int_x^y (y-t)w(t)dt - x^2 w(x) f'(x)\int_x^y (t-x)w(t)dt}{(x^2 w(x) + y^2 w(y))\cdot\int_x^y w(t)dt}$$
$$= \frac{w(y)\int\limits_x^y (y-t)w(t)dt}{(x^2 w(x) + y^2 w(y))\int\limits_x^y w(t)dt}\cdot\left(y^2 f'(y) - x^2 f'(x)\right)$$
$$\leq \frac{\int_x^y w(t)dt}{x^2 w(x) + y^2 w(y)}\cdot\frac{y^2 f'(y) - x^2 f'(x)}{2}. \tag{2.19}$$

On the other hand, if we apply (1.2) for $n = 2$ and $z = x$ and multiply by $\frac{x^2 w(x)}{x^2 w(x) + y^2 w(y)}$, and also for $z = y$, multiply by $\frac{y^2 w(y)}{x^2 w(x) + y^2 w(y)}$, and then add those two identities, we obtain

$$\frac{\int_x^y w(t)f(t)dt}{\int_x^y w(t)dt} - \frac{x^2 w(x) f(x) + y^2 w(y) f(y)}{x^2 w(x) + y^2 w(y)}$$
$$+ \frac{y^2 w(y) f'(y)\int_x^y (y-t)w(t)dt - x^2 w(x) f'(x)\int_x^y (t-x)w(t)dt}{(x^2 w(x) + y^2 w(y))\cdot\int_x^y w(t)dt}$$
$$= \frac{\int_x^y \left[x^2 w(x)\cdot\int_t^y (s-t)w(s)ds + y^2 w(y)\cdot\int_x^t (t-s)w(s)ds\right] f''(t)dt}{(x^2 w(x) + y^2 w(y))\cdot\int_x^y w(t)dt}. \tag{2.20}$$

Now, we apply (2.19) in (2.17) and use (2.20) to get

$$(y-x)\left(y^2\frac{\partial T_w}{\partial y} - x^2\frac{\partial T_w}{\partial x}\right) \geq \frac{(y-x)(x^2 w(x) + y^2 w(y))}{\int_x^y w(t)dt}$$
$$\times \frac{\int_x^y \left[x^2 w(x)\cdot\int_t^y (s-t)w(s)ds + y^2 w(y)\cdot\int_x^t (t-s)w(s)ds\right] f''(t)dt}{(x^2 w(x) + y^2 w(y))\cdot\int_x^y w(t)dt}$$

$$= \frac{(y-x)\cdot\int_x^y \left[x^2w(x)\cdot\int_t^y(s-t)w(s)ds + y^2w(y)\cdot\int_x^t(t-s)w(s)ds\right]f''(t)dt}{\left(\int_x^y w(t)dt\right)^2}.$$

Since f is convex and the integrals in the brackets are non negative, we have proved that $(y-x)\left(y^2\frac{\partial T_w}{\partial y} - x^2\frac{\partial T_w}{\partial x}\right) \geq 0$, for all $x, y \in I, x < y$, so, the function T_w is Schur-harmonically convex.

The proof for the Schur-harmonically concave case is similar. □

Remark 4. For $w(t) = \frac{1}{y-x}$ it is easy to check that conditions (2.7), (2.8), (2.15) and (2.16) are valid, so, if f is convex, then T is Schur-geometrically and Schur-harmonically convex.

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