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ON AN ABSTRACT FORMULATION OF A THEOREM OF SIERPIŃSKI

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Abstract. In our earlier paper, we gave an abstract formulation of a theorem of Sierpiński in uncountable commutative groups. In this paper, we prove a result which generalizes the earlier formulation.

1. Introduction

Sierpiński [9] in one of his classical papers proved that there exist two Lebesgue measure zero sets in $\mathbb{R}$ whose algebraic sum is nonmeasurable. In establishing this result, he used the Hamel basis and Steinhaus famous theorem on a distance set. Several generalizations of Sierpiński's theorem are available in the literature. Kharazishvili [7] proved that for every σ-ideal $\mathcal{I}$ in $\mathbb{R}$ which is not closed with respect to the algebraic sum, and for every σ-algebra $\mathcal{S}(\supseteq \mathcal{I})$ for which the quotient algebra satisfies the countable chain condition, there exist X, $Y \in \mathcal{I}$ such that $X + Y \notin \mathcal{S}$. Now, instead of the real line $\mathbb{R}$, if we choose a commutative group G and any non-zero, σ-finite, complete, G-invariant (or, G-quasi-invariant) measure μ, then an analogue of Sierpiński's theorem can be established with respect to some extension of μ. In fact, it was shown by Kharazishvili [10] that for every uncountable commutative group G and for any σ-finite, left G-invariant (or, G-quasi-invariant) measure μ on G, there exists a left G-invariant (or, G-quasi-invariant) complete measure μ' extending μ and two sets $A, B \in \mathcal{I}(\mu')$ (the σ-ideal of μ'-measure zero sets) such that $A+B \notin \text{dom}(\mu')$. In [1], the authors gave an abstract and generalized formulation of Sierpiński's theorem in uncountable commutative groups which do not involve any use of measure.

Most of the notations, definitions and results of this paper are taken from [1] (see also [2, 3]). Throughout the paper, we identify every infinite cardinal with the least ordinal representing it as card(E) for the cardinality of any set E, and use the symbols such as ξ, ρ, α, k etc. for any arbitrary infinite cardinal k and k^+ for the successor of k. Further, given an infinite group G and a set $A \subseteq G$, we denote by gA $(g \in G)$ the set $\{gx : x \in A\}$ and call a class $\mathcal{C}$ of subsets of G as G-invariant if $gA \in \mathcal{C}$ for every $g \in G$ and $A \in \mathcal{C}$.

Definition 1.1. A pair $(\Sigma, \mathcal{I})$ consisting of two non-empty classes of subsets of G is called a G-invariant, k-additive measurable structure on G if:

(i) Σ is an algebra and $\mathcal{I}$ $(\subseteq \Sigma)$ is a proper ideal in G;

(ii) both Σ and $\mathcal{I}$ are k-additive. This means that the both classes Σ and $\mathcal{I}$ are closed with respect to the union of at most k number of sets;

(iii) Σ and $\mathcal{I}$ are G-invariant.

A k-additive algebra Σ is diffused if $\{x\} \in \Sigma$ for every $x \in G$ and a k-additive measurable structure $(\Sigma, \mathcal{I})$ is called k^+-saturated if the cardinality of any arbitrary collection of mutually disjoint sets from $\Sigma \setminus \mathcal{I}$ is atmost k.

In the sixtieth of the past century, Riecan and Neubrunn developed the notion of small systems and used the same to give abstract formulations of several well-known theorems in classical measure and integration (see [12–14], etc.). Small systems have been used by several other authors in the subsequent periods [5, 6, 11, 15]. The following Definition introduces a modified and generalized version.

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Definition 1.2. For any infinite cardinal k, a transfinite k-sequence $\{\mathcal{N}_\alpha\}_{\alpha<k}$ of nonempty classes of sets in G is called a G-invariant, k-small system on G if:

(i) $\emptyset \in \mathcal{N}_\alpha$ for all $\alpha < k$;

(ii) each $\mathcal{N}_\alpha$ is a G-invariant class;

(iii) $E \in \mathcal{N}_\alpha$ and $F \subseteq E$ implies $F \in \mathcal{N}_\alpha$;

(iv) $E \in \mathcal{N}_\alpha$ and $F \in \bigcap_{\alpha<k} \mathcal{N}_\alpha$ implies $E \cup F \in \mathcal{N}_\alpha$;

(v) for any $\alpha < k$, there exists $\alpha^* > \alpha$ such that for any one-to-one correspondence $\beta \to \mathcal{N}_\beta$ with $\beta > \alpha^*$, $\bigcup_\beta E_\beta \in \mathcal{N}_\alpha$ whenever $E_\beta \in \mathcal{N}_\beta$;

(vi) for any $\alpha, \beta < k$, there exists $\gamma > \alpha, \beta$ such that $\mathcal{N}_\gamma \subseteq \mathcal{N}_\alpha$ and $\mathcal{N}_\gamma \subseteq \mathcal{N}_\beta$.

We further define

Definition 1.3. A G-invariant k-additive algebra $\mathcal{S}$ on G is admissible with respect to the k-small system $\{\mathcal{N}_\alpha\}_{\alpha<k}$ if for every $\alpha < k$:

(i) $\mathcal{S} \setminus \mathcal{N}_\alpha \neq \emptyset \neq \mathcal{S} \cap \mathcal{N}_\alpha$;

(ii) $\mathcal{N}_\alpha$ has an S-base, i.e., $E \in \mathcal{N}_\alpha$ is contained in some $F \in \mathcal{N}_\alpha \cap \mathcal{S}$;

(iii) $\mathcal{S} \setminus \mathcal{N}_\alpha$ satisfies the k-chain condition, i.e., the cardinality of any arbitrary collection of mutually disjoint sets from $\mathcal{S} \setminus \mathcal{N}_\alpha$ is at most k.

The above two Definitions have been used by the present authors in some of their recently done works (see, for example, [1–3]). We set $\mathcal{N}_\infty = \bigcap_{\alpha<k} \mathcal{N}_\alpha$. From conditions (ii), (iii) and (v) of Definition 1.2, it follows that $\mathcal{N}_\infty$ is a G-invariant, k-additive ideal in G and denote by $\widetilde{\mathcal{S}}$ the G-invariant k-additive algebra generated by $\mathcal{S}$ and $\mathcal{N}_\infty$. Every element of $\widetilde{\mathcal{S}}$ is of the form $(X \setminus Y) \cup Z$, where $X \in \mathcal{S}$ and $Y, Z \in \mathcal{N}_\infty$, and $(\widetilde{\mathcal{S}}, \mathcal{N}_\infty)$ turns out to be a G-invariant, k-additive measurable structure on G. Moreover, we have the following

Theorem 1.4. *If $\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$, then the G-invariant, k-additive measurable structure $(\widetilde{\mathcal{S}}, \mathcal{N}_\infty)$ on G is k^+-saturated.*

A proof of the above theorem follows directly from condition (iv) of Definition 1.2 and from conditions (i), (ii) and (iii) of Definition 1.3 or, in short, from the admissibility of $\mathcal{S}$. Based on the above Definitions and Theorems, some combinatorial properties of sets [9, Ch. 7] and also on the important representation theorem for infinite commutative groups [9, Appendix 2], the present authors have proved in [1] the following

Theorem 1.5. *Let G be an uncountable commutative group with $\text{card}(G) = k^+$. Let $\{\mathcal{N}_\alpha\}_{\alpha<k}$ be a G-invariant, k-small system on G and let $\mathcal{S}$ be a diffused, k-additive algebra on G which is also admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$. Then there exists a subset A of G such that $A \in \mathcal{N}_\infty$, but $A + A \notin \widetilde{\mathcal{S}}$.*

2. Result

Theorem 1.5 is an abstract formulation of Sierpiński's theorem given in terms of any diffused, G-invariant, k-additive measurable structure on a commutative group G to which we have referred to in the Introduction. In this section we prove a result which extends our previous formulation to the groups that are not necessarily commutative.

Definition 2.1 ([4])**.** Let $\mathcal{R}$ be an equivalence relation on a set X and $E \subseteq X$. The saturation of E in X with respect to the equivalence relation is the union of all equivalence classes of $\mathcal{R}$ whose intersection with E is nonvoid.

In other words, it is $\bigcup\{C : C \cap E \neq \varnothing \text{ and } C \in X/\mathcal{R}\}$.

It is easy to check that if H is a normal subgroup of any group G, then the saturation of any set E in G with respect to the equivalence relation generated by the quotient group G/H is the set HE. If E coincides with its saturation, then it is called saturated. Thus E is saturated if $HE = E$. A saturated set is also called H-invariant [8].

Theorem 2.2. *Let G be any uncountable group with* $\mathrm{card}(G) = k^+$. *Let* $\{\mathcal{N}_\alpha\}_{\alpha<k}$ *be a G-invariant, k-small system on G and $\mathcal{S}$ be a G-invariant, k-additive algebra on G which is admissible with respect to* $\{\mathcal{N}_\alpha\}_{\alpha<k}$. *We further assume that G has a normal subgroup $H \in \mathcal{S}$ such that G/H is commutative with* $\mathrm{card}(G/H) = k^+$ *and the saturation of any set E in G with respect to G/H also belongs to $\mathcal{S}$.*

Then there exists a subset A of G such that $A \in \mathcal{N}_\infty$ and $AA \notin \widetilde{\mathcal{S}}$.

Proof. We write $\Gamma = G/H$. By the hypothesis, Γ is commutative. Let $f : G \to \Gamma$ be the canonical homomorphism. We set $\mathcal{S}' = \{Y \subseteq \Gamma : f^{-1}(Y) \in \mathcal{S}\}$ and $\mathcal{N}'_\alpha = \{Y \subseteq \Gamma : f^{-1}(Y) \in \mathcal{N}_\alpha\}$ for any $\alpha < k$.

Since $\mathcal{S}$ is a G-invariant, k-additive algebra on G and f is a canonical homomorphism, so $\mathcal{S}'$ is a Γ-invariant, k-additive algebra on Γ. Also, since $H \in \mathcal{S}$, therefore $\mathcal{S}'$ is diffused.

Condition (i) of Definition 1.2 for $\{\mathcal{N}'_\alpha\}_{\alpha<k}$ is obvious. Let $h \in \Gamma$ and $F \in \mathcal{N}'_\alpha$. Then $h = f(x)$ for every $x \in gH$, where $g \in G$ and $f^{-1}(F) \in \mathcal{N}_\alpha$. Since $\mathcal{N}_\alpha$ is G-invariant, therefore $f^{-1}(hF) = xf^{-1}(F) \in \mathcal{N}_\alpha$. Hence $hF \in \mathcal{N}'_\alpha$ which proves condition (ii) of Definition 1.2 for $\{\mathcal{N}'_\alpha\}_{\alpha<k}$. Finally, from the Definition of $\mathcal{N}'_\alpha$ and some simple properties of inverse images of any function, it follows that conditions (iii)-(vi) of Definition 1.2 also hold for $\{\mathcal{N}'_\alpha\}_{\alpha<k}$. Thus $\{\mathcal{N}'_\alpha\}_{\alpha<k}$ is a Γ-invariant, k-small system on Γ.

We shall now show that $\mathcal{S}'$ is admissible with respect to $\{\mathcal{N}'_\alpha\}_{\alpha<k}$. Clearly, $\emptyset \in \mathcal{S}' \cap \mathcal{N}'_\alpha$ for $\alpha < k$. Since $\mathcal{S}$ is admissible with respect to $\{\mathcal{N}_\alpha\}_{\alpha<k}$, so by (i) of Definition 1.3, there exists for every $\alpha < k$, a set $A_\alpha \in \mathcal{S}\setminus\mathcal{N}_\alpha$. If A_α is saturated with respect to the equivalence relation generated by the quotient group G/H, then $A_\alpha = f^{-1}(B_\alpha)$ for some $B_\alpha \in \mathcal{S}' \setminus \mathcal{N}'_\alpha$. If A_α is not saturated, we replace it by HA_α which is saturated, and choose B_α such that $HA_\alpha = f^{-1}(B_\alpha)$. Consequently, $B_\alpha \in \mathcal{S}' \setminus \mathcal{N}'_\alpha$ and condition (i) of Definition 1.3 is satisfied.

Let $F \in \mathcal{N}'_\alpha$ and $E = f^{-1}(F)$. Then $E \in \mathcal{N}_\alpha$ by (ii) of Definition 1.3 there exists $A \in \mathcal{S} \cap \mathcal{N}_\alpha$ such that $E \subseteq A$. If A is saturated, then $A = f^{-1}(B)$ for some $B \in \mathcal{S}' \cap \mathcal{N}'_\alpha$ and $F \subseteq B$. If A is not saturated, we choose the saturation of $G \setminus A$, i.e., $H(G \setminus A)$ with respect to the equivalence relation generated by the quotient group G/H. But $H(G \setminus A) \in \mathcal{S}$ and so, $G \setminus H(G \setminus A) \in \mathcal{S}$. Moreover, $G \setminus H(G \setminus A)$ is a subset of A. Therefore $G \setminus H(G \setminus A) \in \mathcal{N}_\alpha \cap \mathcal{S}$. We choose $B (\subseteq \Gamma)$ such that $G \setminus H(G \setminus A) = f^{-1}(B)$. Then $F \subseteq B$ and $B \in \mathcal{S}' \cap \mathcal{N}'_\alpha$. This shows that $\mathcal{N}'_\alpha$ has an $\mathcal{S}'$-base for every $\alpha < k$ and condition (ii) of Definition 1.3 is proved. Lastly, any arbitrary collection of mutually disjoint sets from $\mathcal{S}' \setminus \mathcal{N}'_\alpha$ is at most k which follows directly from the fact that a similar result is true for the sets from $\mathcal{S}\setminus\mathcal{N}_\alpha$. This shows that $\mathcal{S}' \setminus \mathcal{N}'_\alpha$ satisfies the k-chain condition for every $\alpha < k$ which proves (iii) of Definition 1.3.

Thus we find that $\mathcal{S}'$ is a Γ-invariant, k-additive algebra on Γ which is diffused and admissible with respect to the Γ-invariant, k-small system $\{\mathcal{N}'_\alpha\}_{\alpha<k}$ on Γ.

Let $\mathcal{N}'_\infty = \bigcap_{\alpha<k} \mathcal{N}'_\alpha$ and $\widetilde{\mathcal{S}'}$ be the Γ-invariant, k-additive algebra generated by $\mathcal{S}'$ and $\mathcal{N}'_\infty$. Thus $(\widetilde{\mathcal{S}'}, \mathcal{N}'_\infty)$ is a Γ-invariant, k-additive, measurable structure on the quotient group Γ which is k^+-saturated. Hence by Theorem 1.5, there exists $B \in \mathcal{N}'_\infty$ such that $BB \notin \widetilde{\mathcal{S}'}$. Let $A = f^{-1}(B)$. Then $AA = f^{-1}(B)f^{-1}(B) = f^{-1}(BB)$. So, AA is saturated. If possible, let $AA \in \widetilde{\mathcal{S}}$. Then $AA = E\Delta P$, where $E \in \mathcal{S}$, $P \in \mathcal{N}_\infty$ and E, P are both saturated. Hence $E = f^{-1}(F)$, $P = f^{-1}(Q)$, where $F \in \mathcal{S}'$, $Q \in \mathcal{N}'_\infty$ and therefore $AA = E\Delta P = f^{-1}(F)\Delta f^{-1}(Q) = f^{-1}(F\Delta Q) = f^{-1}(BB)$. But this implies that $BB \in \widetilde{\mathcal{S}'}$ – a contradiction. □

Remark. In general for Theorem 2.2, G need not be commutative. Let H' be a noncommutative group with $\mathrm{card}(H') = \omega$ (the first infinite cardinal) and A' be a commutative group with $\mathrm{card}(A') = \omega_1$ (the first uncountable cardinal). We set $G = H' \times A'$ as the external direct product of H' and A'. Then G is isomorphic with the internal direct product HA, where $H = \{(h, e_{A'}) : h \in H'\}$ and $A = \{(e_{H'}, a) : a \in A'\}$. Moreover, G is noncommutative having H as a normal subgroup and $G/H = A$ is commutative with $\mathrm{card}(G/H) = \omega_1$.

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