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INVERTIBILITY OF FOURIER CONVOLUTION OPERATORS WITH PIECEWISE CONTINUOUS SYMBOLS ON BANACH FUNCTION SPACES

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Dedicated to Professor Roland Duduchava on the occasion of his 75th birthday

Abstract. We extend results on the invertibility of Fourier convolution operators with piecewise continuous symbols on the Lebesgue space $L^p(\mathbb{R})$, $p \in (1,\infty)$, obtained by Roland Duduchava in the late 1970s, to the setting of a separable Banach function space $X(\mathbb{R})$ such that the Hardy–Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. We specify our results in the case of rearrangement-invariant spaces with suitable Muckenhoupt weights.

1. Introduction

Let PC be the C^*-algebra of all bounded piecewise continuous functions on the one-point compactification of the real line $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. By definition, $a \in PC$ if and only if $a \in L^\infty(\mathbb{R})$ and the finite one-sided limits

$$a(x_0 - 0) := \lim_{x \to x_0 - 0} a(x), \quad a(x_0 + 0) := \lim_{x \to x_0 + 0} a(x)$$

exist for each $x_0 \in \dot{\mathbb{R}}$. The set of all discontinuities (i.e., jumps) of a function $a \in PC$ is at most countable (see, e.g., [5, Chap. II. Section 3, Theorem 3]).

We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class of all infinitely differentiable and rapidly decaying functions (see, e.g., [16, Section 2.2.1]). Let $\mathcal{F}$ denote the Fourier transform defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}f)(x) := \int_{\mathbb{R}} f(t) e^{itx}\, dt, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1}$ be the inverse of $\mathcal{F}$ defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{-itx}\, dx, \quad t \in \mathbb{R}.$$

It is well known that these operators extend uniquely to the space $L^2(\mathbb{R})$. As usual, we will use the symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ for the direct and inverse Fourier transform on $L^2(\mathbb{R})$. It is well known (see, e.g., [16, Theorem 2.5.10]) that the Fourier convolution operator

$$W^0(a) := \mathcal{F}^{-1} a \mathcal{F} \tag{1.1}$$

is bounded on the space $L^2(\mathbb{R})$ for every $a \in L^\infty(\mathbb{R})$. The function a is called the symbol of the operator $W^0(a)$.

Let $X(\mathbb{R})$ be a Banach function space and $X'(\mathbb{R})$ be its associate space. Their technical definitions are postponed to Section 2.1. The class of Banach function spaces is very large. It includes Lebesgue, Orlicz, Lorentz spaces, variable Lebesgue spaces and their weighted analogues (see, e.g., [1, 7]). Let $\mathcal{B}(X(\mathbb{R}))$ denote the Banach algebra of all bounded linear operators acting on $X(\mathbb{R})$.

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Recall that the (non-centered) Hardy–Littlewood maximal function $\mathcal{M}f$ of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is defined by

$$(\mathcal{M}f)(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length containing x. The Hardy–Littlewood maximal operator $\mathcal{M}$ defined by the rule $f \mapsto \mathcal{M}f$ is a sublinear operator.

If $X(\mathbb{R})$ is separable, then $L^2(\mathbb{R}) \cap X(\mathbb{R})$ is dense in $X(\mathbb{R})$ (see, e.g., [12, Lemma 2.2]). A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $X(\mathbb{R})$ if the convolution operator $W^0(a)$ defined by (1.1) maps $L^2(\mathbb{R}) \cap X(\mathbb{R})$ into $X(\mathbb{R})$ and extends to a bounded linear operator on $X(\mathbb{R})$. The function a is called the symbol of the Fourier convolution operator $W^0(a)$. The set $\mathcal{M}_{X(\mathbb{R})}$ of all Fourier multipliers on $X(\mathbb{R})$ is a unital normed algebra under pointwise operations and the norm

$$\|a\|_{\mathcal{M}_{X(\mathbb{R})}} := \left\|W^0(a)\right\|_{\mathcal{B}(X(\mathbb{R}))}.$$

If, in addition, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$, or on its associate space $X'(\mathbb{R})$, then for all $a \in \mathcal{M}_{X(\mathbb{R})}$,

$$\|a\|_{L^\infty(\mathbb{R})} \le \|a\|_{\mathcal{M}_{X(\mathbb{R})}}. \tag{1.2}$$

The constant 1 on the right-hand side of (1.2) is best possible (see [19, Corollary 4.2] and [20, Theorem 2.3]). Once (1.2) is available, one can show that $\mathcal{M}_{X(\mathbb{R})}$ is a Banach algebra (see [20, Corollary 2.4]).

Suppose that $a : \mathbb{R} \to \mathbb{C}$ is a function of the finite total variation $V(a)$ given by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all partitions of $\mathbb{R}$ of the form

$$-\infty < x_0 < x_1 < \dots < x_n < +\infty$$

with $n \in \mathbb{N}$. The set $V(\mathbb{R})$ of all functions of finite total variation on $\mathbb{R}$ with the norm

$$\|a\|_{V(\mathbb{R})} := \|a\|_{L^\infty(\mathbb{R})} + V(a)$$

is a unital Banach algebra. By [13, Theorem 3.27], $V(\mathbb{R}) \subset PC$.

Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. It follows from [17, Theorem 4.3] that if $a \in V(\mathbb{R})$, then the convolution operator $W^0(a)$ is bounded on the space $X(\mathbb{R})$, and

$$\|W^0(a)\|_{\mathcal{B}(X(\mathbb{R}))} \le c_X \|a\|_{V(\mathbb{R})} \tag{1.3}$$

where c_X is a positive constant depending only on $X(\mathbb{R})$.

For the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, inequality (1.3) is usually called Stechkin's inequality. Its proofs can be found, e.g., in [3, Theorem 17.1], [9, Theorem 2.11], [10, Theorem 6.2.5].

For a subset S of a Banach space E, let $\mathrm{clos}_E(S)$ denote the closure of S with respect to the norm of E. Let $P\mathbb{C}^0$ denote the set of all piecewise constant functions with finitely many jumps. It is clear that $P\mathbb{C}^0 \subset V(\mathbb{R}) \subset PC$. It follows from [9, Lemma 2.10] that $PC = \mathrm{clos}_{L^\infty(\mathbb{R})}\left(P\mathbb{C}^0\right)$. Hence

$$PC = \mathrm{clos}_{L^\infty(\mathbb{R})}\left(P\mathbb{C}^0\right) = \mathrm{clos}_{L^\infty(\mathbb{R})}\left(V(\mathbb{R})\right). \tag{1.4}$$

For a separable Banach function space $X(\mathbb{R})$ such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$, consider the following Banach algebras of Fourier multipliers:

$$PC^0_{X(\mathbb{R})} := \mathrm{clos}_{\mathcal{M}_{X(\mathbb{R})}}\left(P\mathbb{C}^0\right), \quad PC_{X(\mathbb{R})} := \mathrm{clos}_{\mathcal{M}_{X(\mathbb{R})}}\left(V(\mathbb{R})\right).$$

It follows from (1.2) and (1.4) that

$$PC^0_{X(\mathbb{R})} \subset PC_{X(\mathbb{R})} \subset PC.$$

Therefore, it is natural to refer to $PC^0_{X(\mathbb{R})}$ and $PC_{X(\mathbb{R})}$ as algebras of piecewise continuous Fourier multipliers. For $1 < p < \infty$, the algebras $PC^0_{L^p(\mathbb{R})}$ and $PC_{L^p(\mathbb{R})}$ were introduced by Duduchava (see [9, Chap. 1, Section 2]).

The aim of this paper is to study the invertibility of convolution operators $W^0(a)$ with piecewise continuous symbols $a \in PC^0_{X(\mathbb{R})}$ on the Banach function spaces. Our main result is the following

Theorem 1.1. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Suppose that $a \in PC^0_{X(\mathbb{R})}$. For the operator $W^0(a)$ to be invertible on the space $X(\mathbb{R})$, it is necessary and sufficient that*

$$\operatorname{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0.$$

For the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, the above result was obtained by Roland Duduchava in [9, Theorem 2.18].

Question 1.2. Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. Is it true that $PC^0_{X(\mathbb{R})} = PC_{X(\mathbb{R})}$?

Note that for the Lebesgue spaces $L^p(\mathbb{R})$, the positive answer follows from [9, Remark 2.12]:

$$PC^0_{L^p(\mathbb{R})} = PC_{L^p(\mathbb{R})}, \quad 1 < p < \infty. \tag{1.5}$$

Let $\mathcal{P}(\mathbb{R})$ denote the set of all measurable a.e. finite functions $p(\cdot) : \mathbb{R} \to [1, \infty]$ such that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}} p(x) =: p_+ < \infty.$$

By $L^{p(\cdot)}(\mathbb{R})$ we denote the set of all complex-valued measurable functions f on $\mathbb{R}$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a separable and reflexive Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R})} := \inf\left\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1\right\},$$

and its associate space is isomorphic to the space $L^{p'(\cdot)}(\mathbb{R})$, where

$$1/p(x) + 1/p'(x) = 1 \quad \text{for a.e.} \quad x \in \mathbb{R}$$

(see, e.g., [7, Chap. 2] or [8, Chap. 3]). It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$. The space $L^{p(\cdot)}(\mathbb{R})$ is referred to as a variable Lebesgue space. By [8, Theorem 5.7.2], the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R})$ if and only if it is bounded on $L^{p'(\cdot)}(\mathbb{R})$. As it is shown in [18, Theorem 4.2], in this case

$$PC^0_{L^{p(\cdot)}(\mathbb{R})} = PC_{L^{p(\cdot)}(\mathbb{R})}.$$

The proof of this equality is based on an analogue of the Riesz-Thorin interpolation theorem for variable Lebesgue spaces. In Section 3, we show that the answer to Question 1.2 is positive also for rearrangement-invariant Banach function spaces with suitable Muckenhoupt weights. Our proof is based on the Boyd interpolation theorem [6].

For general Banach function spaces, interpolation tools are not available. Hence one cannot prevent that the answer to Question 1.2 might be negative. In this situation it would be interesting to answer the following.

Question 1.3. Does Theorem 1.1 remain true for the algebra $PC_{X(\mathbb{R})}$ in the place of $PC^0_{X(\mathbb{R})}$?

The paper is organized as follows. Section 2 contains definitions and properties of a Banach function space and its associate space (see, e.g., [23] and [1, Chap. 1]), of a rearrangement-invariant Banach function space (see, e.g., [1, Chap. 3]) and its Boyd indices [6], and of a weighted rearrangement-invariant Banach function space with a suitable Muckenhoupt weight (see, e.g., [2, Chap. 2]). In

Section 3, we first prove that the answer to Question 1.2 is positive for the Lebesgue spaces $L^p(\mathbb{R}, w)$, $1 < p < \infty$, with Muckenhoupt weights $w \in A_p(\mathbb{R})$ using the stability of Muckenhoupt weights and the Stein-Weiss interpolation theorem. Further, we extend this result to the case of a weighted Banach function space $X(\mathbb{R}, w)$ built upon a separable rearrangement-invariant space $X(\mathbb{R})$ with the Boyd indices $\alpha_X, \beta_X \in (0,1)$ and a suitable Muckenhoupt weight $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. In Section 4, we recall the definition of M-equivalence of elements of a Banach algebra and formulate the Gohberg-Krupnik local principle [14, 15]. We apply it two times. First, we show that the algebra $PC^0_{X(\mathbb{R})}$ is inverse closed in the algebra $L^\infty(\mathbb{R})$. Finally, we prove Theorem 1.1 employing the local principle.

We would like to dedicate this work to Roland Duduchava, whose ideas penetrate the entire paper. This work was started as the Undergraduate Research Opportunity Project of the third author at the NOVA University of Lisbon in January-February of 2020 under the supervision of the second author.

2. Preliminaries

2.1. **Banach function spaces.** Let $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{S} \in \{\mathbb{R}_+, \mathbb{R}\}$. The set of all Lebesgue measurable complex-valued functions on $\mathbb{S}$ is denoted by $\mathfrak{M}(\mathbb{S})$. Let $\mathfrak{M}^+(\mathbb{S})$ be the subset of functions in $\mathfrak{M}(\mathbb{S})$ whose values lie in $[0, \infty]$. The Lebesgue measure of a measurable set $E \subset \mathbb{S}$ is denoted by $|E|$ and its characteristic function is denoted by χ_E. Following [23, p. 3] and [1, Chap. 1, Definition 1.1], a mapping $\rho : \mathfrak{M}^+(\mathbb{S}) \to [0, \infty]$ is called a Banach function norm if, for all functions f, g, f_n $(n \in \mathbb{N})$ in $\mathfrak{M}^+(\mathbb{S})$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{S}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,
(A2) $0 \leq g \leq f$ a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$ (the lattice property),
(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),
(A4) E is bounded $\Rightarrow \rho(\chi_E) < \infty$,
(A5) E is bounded $\Rightarrow \int_E f(x)\,dx \leq C_E \rho(f)$

with $C_E \in (0, \infty)$ which may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\mathbb{S})$ of all functions $f \in \mathfrak{M}(\mathbb{S})$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathbb{S})$, the norm of f is defined by

$$\|f\|_{X(\mathbb{S})} := \rho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathbb{S})$ becomes a Banach space (see [23, Chap. 1, §1, Theorem 1] or [1, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathfrak{M}^+(\mathbb{S})$ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{S}} f(x)g(x)\,dx \ : \ f \in \mathfrak{M}^+(\mathbb{S}),\ \rho(f) \leq 1 \right\}, \quad g \in \mathfrak{M}^+(\mathbb{S}).$$

It is a Banach function norm itself [23, Chap. 1, §1] or [1, Chap. 1, Theorem 2.2]. The Banach function space $X'(\mathbb{S})$ determined by the Banach function norm ρ' is called the associate space (Köthe dual) of $X(\mathbb{S})$. The associate space $X'(\mathbb{S})$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathbb{S})]^*$.

Remark 2.1. We note that our definition of a Banach function space is slightly different from that found in [1, Chap. 1, Definition 1.1]. In particular, in Axioms (A4) and (A5) we assume that the set E is a bounded set, whereas it is sometimes assumed that E merely satisfies $|E| < \infty$. We do this so that the weighted Lebesgue spaces with Muckenhoupt weights satisfy Axioms (A4) and (A5). Moreover, it is well known that all main elements of the general theory of Banach function spaces work with (A4) and (A5) stated for bounded sets [23] (see also the discussion at the beginning of Chapter 1 on page 2 of [1]). Unfortunately, we overlooked that the definition of a Banach function space in our previous works [11, 12, 17, 19, 21] had to be changed by replacing in Axioms (A4) and (A5) the requirement of $|E| < \infty$ by the requirement that E is a bounded set to include weighted Lebesgue spaces in our considerations. However, the results proved in the above papers remain true.

2.2. **Rearrangement-invariant Banach function spaces.** Suppose that $\mathbb{S} \in \{\mathbb{R}, \mathbb{R}_+\}$. Let $\mathfrak{M}_0(\mathbb{S})$ and $\mathfrak{M}_0^+(\mathbb{S})$ be the classes of a.e. finite functions in $\mathfrak{M}(\mathbb{S})$ and $\mathfrak{M}^+(\mathbb{S})$, respectively. The distribution function μ_f of $f \in \mathfrak{M}_0(\mathbb{S})$ is given by

$$\mu_f(\lambda) := \big|\{x \in \mathbb{S} : |f(x)| > \lambda\}\big|, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathfrak{M}_0(\mathbb{S})$ are said to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$. The non-increasing rearrangement of $f \in \mathfrak{M}_0(\mathbb{S})$ is the function defined by

$$f^*(t) := \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

We here use the standard convention that $\inf \emptyset = +\infty$.

A Banach function norm $\rho : \mathfrak{M}^+(\mathbb{S}) \to [0, \infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(\mathbb{S})$ the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space $X(\mathbb{S})$ generated by ρ is said to be a rearrangement-invariant Banach function space (or simply, rearrangement-invariant space). The Lebesgue, Orlicz, and Lorentz spaces are classical examples of rearrangement-invariant Banach function spaces (see, e.g., [1] and references therein). By [1, Chap. 2, Proposition 4.2], if a Banach function space $X(\mathbb{S})$ is rearrangement-invariant, then its associate space $X'(\mathbb{S})$ is rearrangement-invariant, too.

2.3. **Boyd indices.** Suppose $X(\mathbb{R})$ is a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm ρ. In this case, the Luxemburg representation theorem [1, Chap. 2, Theorem 4.10] provides a unique rearrangement-invariant Banach function norm $\overline{\rho}$ over the half-line $\mathbb{R}_+$ equipped with the Lebesgue measure, defined by

$$\overline{\rho}(h) := \sup\left\{ \int_{\mathbb{R}_+} g^*(t) h^*(t)\, dt : \ \rho'(g) \leq 1 \right\},$$

and such that $\rho(f) = \overline{\rho}(f^*)$ for all $f \in \mathfrak{M}_0^+(\mathbb{R})$. The rearrangement-invariant Banach function space generated by $\overline{\rho}$ is denoted by $\overline{X}(\mathbb{R}_+)$.

For each $t > 0$, let E_t denote the dilation operator defined on $\mathfrak{M}(\mathbb{R}_+)$ by

$$(E_t f)(s) = f(st), \quad 0 < s < \infty.$$

With $X(\mathbb{R})$ and $\overline{X}(\mathbb{R}_+)$ as above, let $h_X(t)$ denote the operator norm of $E_{1/t}$ as an operator on $\overline{X}(\mathbb{R}_+)$. By [1, Chap. 3, Proposition 5.11], for each $t > 0$, the operator E_t is bounded on $\overline{X}(\mathbb{R}_+)$ and the function h_X is increasing and submultiplicative on $(0, \infty)$. The Boyd indices of $X(\mathbb{R})$ are the numbers α_X and β_X defined by

$$\alpha_X := \sup_{t \in (0,1)} \frac{\log h_X(t)}{\log t}, \quad \beta_X := \inf_{t \in (1,\infty)} \frac{\log h_X(t)}{\log t}.$$

By [1, Chap. 3, Proposition 5.13], $0 \leq \alpha_X \leq \beta_X \leq 1$. The Boyd indices are said to be nontrivial if $\alpha_X, \beta_X \in (0, 1)$. The Boyd indices of the Lebesgue space $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, are both equal to $1/p$. Note that the Boyd indices of a rearrangement-invariant space may be different [1, Chap. 3, Exercises 6, 13].

The next theorem follows from the Boyd interpolation theorem [6, Theorem 1] for quasi-linear operators of weak types (p, p) and (q, q). Its proof can also be found in [1, Chap. 3, Theorem 5.16] and [22, Theorem 2.b.11].

Theorem 2.2. *Let $1 \leq q < p \leq \infty$ and $X(\mathbb{R})$ be a rearrangement-invariant Banach function space with the Boyd indices α_X, β_X satisfying*

$$1/p < \alpha_X \leq \beta_X < 1/q.$$

Then there exists a constant $C_{p,q} \in (0, \infty)$ with the following property. If a linear operator $T : \mathfrak{M}(\mathbb{R}) \to \mathfrak{M}(\mathbb{R})$ is bounded on the Lebesgue spaces $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, then it is also bounded on the rearrangement-invariant Banach function space $X(\mathbb{R})$ and

$$\|T\|_{\mathcal{B}(X(\mathbb{R}))} \leq C_{p,q} \max\left\{ \|T\|_{\mathcal{B}(L^p(\mathbb{R}))}, \|T\|_{\mathcal{B}(L^q(\mathbb{R}))} \right\}. \tag{2.1}$$

Notice that estimate (2.1) is not stated explicitly in [1, 6, 22]. However, it can be extracted from the proof of the Boyd interpolation theorem.

2.4. **Lebesgue spaces with Muckenhoupt weights.** A measurable function $w : \mathbb{R} \to [0, \infty]$ is called a weight if the set $w^{-1}(\{0, \infty\})$ has measure zero. For $1 < p < \infty$, the Muckenhoupt class $A_p(\mathbb{R})$ is defined as the class of all weights $w : \mathbb{R} \to [0, \infty]$ such that $w \in L^p_{\rm loc}(\mathbb{R})$, $w^{-1} \in L^{p'}_{\rm loc}(\mathbb{R})$ and

$$\sup_I \left(\frac{1}{|I|} \int_I w^p(x)\, dx \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{-p'}(x)\, dx \right)^{1/p'} < \infty,$$

where $1/p + 1/p' = 1$ and the supremum is taken over all intervals $I \subset \mathbb{R}$ of finite length $|I|$. Since $w \in L^p_{\rm loc}(\mathbb{R})$ and $w^{-1} \in L^{p'}_{\rm loc}(\mathbb{R})$, the weighted Lebesgue space

$$L^p(\mathbb{R}, w) := \{ f \in \mathfrak{M}(\mathbb{R}) : fw \in L^p(\mathbb{R}) \}$$

is a separable Banach function space (see, e.g., [21, Lemma 2.4]) with the norm

$$\|f\|_{L^p(\mathbb{R},w)} := \left(\int_{\mathbb{R}} |f(x)|^p w^p(x)\, dx \right)^{1/p}.$$

2.5. **Rearrangement-invariant Banach function spaces with suitable Muckenhupt weights.** Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm ρ. We say that $f \in X_{\rm loc}(\mathbb{R})$ if $f\chi_E \in X(\mathbb{R})$ for every bounded measurable set $E \subset \mathbb{R}$.

Lemma 2.3 ([21, Lemma 2.4])**.** *Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm ρ, let $X'(\mathbb{R})$ be its associate space, and let $w : \mathbb{R} \to [0, \infty]$ be a weight. Suppose that $w \in X_{\rm loc}(\mathbb{R})$ and $1/w \in X'_{\rm loc}(\mathbb{R})$. Then*

$$\rho_w(f) := \rho(fw), \quad f \in \mathfrak{M}^+(\mathbb{R}),$$

is a Banach function norm and

$$X(\mathbb{R}, w) := \{ f \in \mathfrak{M}(\mathbb{R}) : fw \in X(\mathbb{R}) \}$$

is a Banach function space generated by the Banach function norm ρ_w. The space $X'(\mathbb{R}, w^{-1})$ is the associate space of $X(\mathbb{R}, w)$.

Lemma 2.4 ([11, Lemma 2.3])**.** *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space and $X'(\mathbb{R})$ be its associate space. Suppose that the Boyd indices of $X(\mathbb{R})$ satisfy $0 < \alpha_X, \beta_X < 1$ and let*

$$w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R}). \tag{2.2}$$

Then

(a) *$w \in X_{\rm loc}(\mathbb{R})$ and $1/w \in X'_{\rm loc}(\mathbb{R})$;*

(b) *the Banach function space $X(\mathbb{R}, w)$ is separable;*

(c) *the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the Banach function space $X(\mathbb{R}, w)$ and on its associate space $X'(\mathbb{R}, w^{-1})$.*

We say that a weight w is suitable for a rearrangement-invariant Banach function space $X(\mathbb{R})$ with the Boyd indices α_X, β_X satisfying $\alpha_X, \beta_X \in (0, 1)$ if (2.2) is fulfilled.

3. $PC_{X(\mathbb{R},w)}$ as the Closure of the Set of Piecewise Constant Functions

3.1. **The case of Lebesgue spaces with Muckenhoupt weights.** Let us start with an important lemma due to Duduchava.

Lemma 3.1 ([9, Lemma 2.10])**.** *For every function $a \in V(\mathbb{R})$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $P\mathbb{C}^0$ such that*

$$\lim_{n\to\infty} \|a_n - a\|_{L^\infty(\mathbb{R})} = 0, \quad \sup_{n\in\mathbb{N}} V(a_n) \le V(a).$$

We now extend equality (1.5) to the case of Lebesgue spaces with Muckenhoupt weights.

Theorem 3.2. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{R})$. Then*

$$PC^0_{L^p(\mathbb{R},w)} = PC_{L^p(\mathbb{R},w)}. \tag{3.1}$$

Proof. The proof is analogous to that of [18, Theorem 4.2] (see also [11, Lemma 3.1]). First of all, we observe that if $w \in A_p(\mathbb{R})$, then the Stechkin-type inequality (1.3) is fulfilled in $L^p(\mathbb{R}, w)$ (see [3, Theorem 17.1] and also Lemma 2.4).

Since $P\mathbb{C}^0 \subset V(\mathbb{R})$, we, obviously, have

$$PC^0_{L^p(\mathbb{R},w)} \subset PC_{L^p(\mathbb{R},w)}. \tag{3.2}$$

If $w \in A_p(\mathbb{R})$, then $w^{1+\delta_2} \in A_{p(1+\delta_1)}(\mathbb{R})$ whenever $|\delta_1|$ and $|\delta_2|$ are sufficiently small (see, e.g., [2, Theorem 2.31]). If $p \geq 2$, then one can find sufficiently small $\delta_1, \delta_2 > 0$ and a small number $\theta \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p(1+\delta_1)}, \quad w = 1^{1-\theta} w^{(1+\delta_2)\theta}, \quad w^{1+\delta_2} \in A_{p(1+\delta_1)}(\mathbb{R}). \tag{3.3}$$

If $1 < p < 2$, then one can find a sufficiently small number $\delta_2 > 0$, a number $\delta_1 < 0$ with sufficiently small $|\delta_1|$, and a number $\theta \in (0, 1)$ such that all conditions in (3.3) are fulfilled. Let us use the following abbreviations:

$$\mathcal{M}_p := \mathcal{M}_{L^p(\mathbb{R},w)}, \quad \mathcal{M}_{p_\theta} := \mathcal{M}_{L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})},$$

$$\mathcal{B}_p := \mathcal{B}(L^p(\mathbb{R}, w)), \quad \mathcal{B}_{p_\theta} := \mathcal{B}(L^{p(1+\delta_1)}(\mathbb{R}, w^{1+\delta_2})).$$

Let $a \in PC_{L^p(\mathbb{R},w)}$ and $\varepsilon > 0$. Then there exists $b \in V(\mathbb{R})$ such that

$$\|a - b\|_{\mathcal{M}_p} < \varepsilon/2. \tag{3.4}$$

By Lemma 3.1, there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $P\mathbb{C}^0$ such that

$$\lim_{n\to\infty} \|b_n - b\|_{L^\infty(\mathbb{R})} = 0, \quad \sup_{n\in\mathbb{N}} V(b_n) \leq V(b). \tag{3.5}$$

Then there exists $N \in \mathbb{N}$ such that

$$\sup_{n\geq N} \|b_n\|_{V(\mathbb{R})} \leq 2\|b\|_{V(\mathbb{R})}. \tag{3.6}$$

It follows from the Stechkin-type inequality (1.3) and inequality (3.6) that for all $n \geq N$,

$$\|b_n - b\|_{\mathcal{M}_{p_\theta}} \leq \|b_n\|_{\mathcal{M}_{p_\theta}} + \|b\|_{\mathcal{M}_{p_\theta}} \leq 3c_\theta \|b\|_{V(\mathbb{R})}, \tag{3.7}$$

where $c_\theta := c_{L^{p(1+\delta_1)}(\mathbb{R},w^{1+\delta_2})}$.

Taking into account (3.3), we obtain from the Stein-Weiss interpolation theorem (see, e.g., [1, Chap. 3, Theorem 3.6]) that for all $n \in \mathbb{N}$,

$$\begin{aligned}\|b_n - b\|_{\mathcal{M}_p} &= \|W^0(b_n - b)\|_{\mathcal{B}_p} \\ &\leq \|W^0(b_n - b)\|^{1-\theta}_{\mathcal{B}(L^2(\mathbb{R}))} \|W^0(b_n - b)\|^{\theta}_{\mathcal{B}_{p_\theta}} \\ &= \|b_n - b\|^{1-\theta}_{L^\infty(\mathbb{R})} \|b_n - b\|^{\theta}_{\mathcal{M}_{p_\theta}}.\end{aligned} \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we see that there exists $n_0 \in \mathbb{N}$ such that

$$\|b_{n_0} - b\|_{\mathcal{M}_p} < \varepsilon/2. \tag{3.9}$$

Inequalities (3.4) and (3.9) imply that for every $\varepsilon > 0$, there exists $c = b_{n_0} \in P\mathbb{C}^0$ such that $\|a - c\|_{\mathcal{M}_p} < \varepsilon$, whence $a \in \mathrm{clos}_{\mathcal{M}_p}\left(P\mathbb{C}^0\right) = PC^0_{L^p(\mathbb{R},w)}$. Then

$$PC_{L^p(\mathbb{R},w)} \subset PC^0_{L^p(\mathbb{R},w)}. \tag{3.10}$$

Gathering embeddings (3.2) and (3.10), we arrive at equality (3.1). ☐

3.2. **The case of separable rearrangement-invariant spaces with suitable Muckenhoupt weights.** We are now in a position to prove the main result of this section and to answer Question 1.2 for separable rearrangement-invariant spaces Banach function spaces with suitable Muckenhoupt weights.

Theorem 3.3. *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X,\ \beta_X < 1$. Suppose that a weight w belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Then*

$$PC^0_{X(\mathbb{R},w)} = PC_{X(\mathbb{R},w)}. \tag{3.11}$$

Proof. Since $P\mathbb{C}^0 \subset V(\mathbb{R})$, we, obviously, have

$$PC^0_{X(\mathbb{R},w)} \subset PC_{X(\mathbb{R},w)}. \tag{3.12}$$

To prove the reverse inclusion, let $a \in PC_{X(\mathbb{R},w)}$ and $\varepsilon > 0$. Then there exists $b \in V(\mathbb{R})$ such that

$$\|a - b\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon/2. \tag{3.13}$$

Since $\alpha_X, \beta_X \in (0,1)$ and $w \in A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$, it follows from [2, Theorem 2.31] that there exist p and q such that

$$1 < q < 1/\beta_X \le 1/\alpha_X < p < \infty, \quad w \in A_p(\mathbb{R}) \cap A_q(\mathbb{R}). \tag{3.14}$$

Let $C_{p,q}$ be the constant from estimate (2.1). As in the proof of inequality (3.9) (see the proof of the previous theorem), it can be shown that there exists $b_{n_0} \in P\mathbb{C}^0$ for some $n_0 \in \mathbb{N}$ such that

$$\|b - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}, \quad \|b - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}} < \frac{\varepsilon}{2C_{p,q}}. \tag{3.15}$$

It follows from (3.14), (3.15) and Theorem 2.2 that

$$\begin{aligned}
\|b - b_{n_0}\|_{\mathcal{M}_{X(\mathbb{R},w)}} &= \|W^0(b - b_{n_0})\|_{\mathcal{B}(X(\mathbb{R},w))} \\
&= \|wW^0(b - b_{n_0})w^{-1}I\|_{\mathcal{B}(X(\mathbb{R}))} \\
&\le C_{p,q} \max\left\{\|wW^0(b - b_{n_0})w^{-1}I\|_{\mathcal{B}(L^p(\mathbb{R}))}, \|wW^0(b - b_{n_0})w^{-1}I\|_{\mathcal{B}(L^q(\mathbb{R}))}\right\} \\
&= C_{p,q} \max\left\{\|W^0(b - b_{n_0})\|_{\mathcal{B}(L^p(\mathbb{R},w))}, \|W^0(b - b_{n_0})\|_{\mathcal{B}(L^q(\mathbb{R},w))}\right\} \\
&= C_{p,q} \max\left\{\|b - b_{n_0}\|_{\mathcal{M}_{L^p(\mathbb{R},w)}}, \|b - b_{n_0}\|_{\mathcal{M}_{L^q(\mathbb{R},w)}}\right\} < \varepsilon/2.
\end{aligned} \tag{3.16}$$

Inequalities (3.13) and (3.16) imply that for every $\varepsilon > 0$, there exists $c = b_{n_0} \in P\mathbb{C}^0$ such that $\|a - c\|_{\mathcal{M}_{X(\mathbb{R},w)}} < \varepsilon$, whence $a \in \operatorname{clos}_{\mathcal{M}_{X(\mathbb{R},w)}}\left(P\mathbb{C}^0\right) = PC^0_{X(\mathbb{R},w)}$. Then

$$PC_{X(\mathbb{R},w)} \subset PC^0_{X(\mathbb{R},w)}. \tag{3.17}$$

The desired equality (3.11) follows now from the embeddings (3.12) and (3.17). □

Combining Theorem 3.3 with Theorem 1.1, we arrive at the following

Corollary 3.4. *Let $X(\mathbb{R})$ be a separable rearrangement-invariant Banach function space with the Boyd indices satisfying $0 < \alpha_X,\ \beta_X < 1$. Suppose that a weight w belongs to $A_{1/\alpha_X}(\mathbb{R}) \cap A_{1/\beta_X}(\mathbb{R})$. Suppose that $a \in PC_{X(\mathbb{R},w)}$. For the operator $W^0(a)$ to be invertible on the Banach function space $X(\mathbb{R}, w)$, it is necessary and sufficient that*

$$\operatorname{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0.$$

4. The Gohberg-Krupnik Local Principle in Action

4.1. **M-equivalence.** Let $\mathcal{A}$ be a unital Banach algebra. A subset $M \subset \mathcal{A}$ is called a localizing class if $0 \notin M$ and for any $f_1, f_2 \in M$ there exists a third element $f \in M$ such that $f_j f = f f_j = f$ for $j = 1, 2$.

Two elements $a, b \in \mathcal{A}$ are said to be M-equivalent from the left (resp., from the right) if

$$\inf_{f\in M} \|(a - b)f\| = 0 \quad \Big(\text{resp.} \quad \inf_{f\in M} \|f(a - b)\| = 0\Big).$$

If a and b are both M-equivalent from the left and from the right, then they are said to be M-equivalent. In this case we write $a \overset{M}{\sim} b$.

Because of the completeness, let us give a simple proof of the continuity of M-equivalence, which was mentioned implicitly in [9, p. 21].

Proposition 4.1. *Let $\mathcal{A}$ be a unital Banach algebra, M a localizing class of $\mathcal{A}$ and $\{x_n\}_{n\in\mathbb{N}}$, and $\{y_n\}_{n\in\mathbb{N}}$ be the sequences of elements of $\mathcal{A}$, convergent to x and y, respectively. Suppose that*

$$\sup_{a\in M}\|a\| < \infty.$$

If $x_n \overset{M}{\sim} y_n$ for all $n \in \mathbb{N}$, then $x \overset{M}{\sim} y$.

Proof. Fix $\varepsilon > 0$. Let $a \in M$ and $L := \sup_{a\in M}\|a\|$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\|(x-y)a\| &= \|(x-x_n)a + (y_n - y)a + (x_n - y_n)a\| \\ &\le \|(x-x_n)a\| + \|(y_n-y)a\| + \|(x_n-y_n)a\| \\ &\le \|x-x_n\|\,\|a\| + \|y_n-y\|\,\|a\| + \|(x_n-y_n)a\| \\ &\le L(\|x-x_n\| + \|y-y_n\|) + \|(x_n-y_n)a\|. \end{aligned} \tag{4.1}$$

Since $x_n \to x$ and $y_n \to y$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\|x - x_n\| < \frac{\varepsilon}{4L} \quad \text{and} \quad \|y - y_n\| < \frac{\varepsilon}{4L}. \tag{4.2}$$

On the other hand, $x_n \overset{M}{\sim} y_n$ for every $n \in \mathbb{N}$. In particular, $x_{N+1} \overset{M}{\sim} y_{N+1}$. Therefore, by the definition of M-equivalence from the left, there is an $a' \in M$ such that

$$\|(x_{N+1} - y_{N+1})a'\| < \frac{\varepsilon}{2}. \tag{4.3}$$

Combining inequalities (4.1)–(4.3), we get

$$\forall \varepsilon > 0 \quad \exists a' \in M \,:\, \|(x-y)a'\| < \varepsilon,$$

i.e., x and y are M-equivalent from the left. Similarly, we prove that x and y are M-equivalent from the right. Thus x and y are M-equivalent. □

4.2. **The local principle.** Let $\mathcal{A}$ be a unital Banach algebra and M be a localizing class in $\mathcal{A}$. An element $a \in \mathcal{A}$ is called M-invertible from the left (resp., from the right) if there are the elements $b \in \mathcal{A}$ and $f \in M$ such that $baf = f$ (resp., $fab = f$). Finally, $a \in \mathcal{A}$ is said to be M-invertible if it is M-invertible from the left and from the right.

Let T be an index set. A system $\{M_\tau\}_{\tau\in T}$ of localizing classes is said to be covering if from each choice $\{f_\tau\}_{\tau\in T}$ with $f_\tau \in M_\tau$ there can be selected a finite number of elements $f_{\tau_1}, \dots, f_{\tau_m}$ whose sum is invertible in $\mathcal{A}$.

Let $M := \cup_{\tau\in T} M_\tau$ and let $\operatorname{Com} M$ stand for the commutant of M, that is, the set of all $a \in \mathcal{A}$ which commute with every element in M.

The following theorem was obtained by Gohberg and Krupnik [15]. Its proof can be found in several books (see, e.g., [4, Theorem 1.32], [14, Section 5.1], [24, Theorem 2.4.5]).

Theorem 4.2 (Gohberg, Krupnik)**.** *Let $\mathcal{A}$ be a unital Banach algebra, let T be an index set, let $\{M_\tau\}_{\tau\in T}$ be a covering system of localizing classes, and let $a \in \operatorname{Com} M$. Suppose that, for each $\tau \in T$, the element a is M_τ-equivalent from the left (resp., from the right) to $a_\tau \in \mathcal{A}$. Then the element a is left-invertible (resp., right-invertible) in $\mathcal{A}$ if and only if a_τ is M_τ-invertible from the left (resp., from the right) for all $\tau \in T$.*

4.3. **The algebra $PC^0_{X(\mathbb{R})}$ is inverse closed in the algebra $L^\infty(\mathbb{R})$.** The first step in the proof of Theorem 1.1 consists in establishing the inverse closedness of the Banach algebra of piecewise continuous Fourier multipliers $PC^0_{X(\mathbb{R})}$ in the C^*-algebra $L^\infty(\mathbb{R})$. Although the proof of the following lemma is similar to that of [9, Lemma 2.17], because of the completeness of presentation, we give it here.

Lemma 4.3. *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $X(\mathbb{R})$ and on its associate space $X'(\mathbb{R})$. If $a \in PC^0_{X(\mathbb{R})}$ and*

$$\operatorname{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0, \tag{4.4}$$

then $a^{-1} \in PC^0_{X(\mathbb{R})}$.

Proof. For each $x \in \dot{\mathbb{R}}$, consider the sets of characteristic functions of intervals of $\dot{\mathbb{R}}$ given by

$$\mathrm{M}_x^- := \left\{\chi_{[c,x]} : c \in \mathbb{R}, c < x\right\}, \quad \mathrm{M}_x^+ := \left\{\chi_{[x,d]} : d \in \mathbb{R}, x < d\right\}, \quad x \in \mathbb{R}, \tag{4.5}$$

and

$$\mathrm{M}_\infty^+ := \left\{\chi_{\{\infty\}\cup(-\infty,d]} : d \in \mathbb{R}\right\}, \quad \mathrm{M}_\infty^- := \left\{\chi_{[c,+\infty)\cup\{\infty\}} : c \in \mathbb{R}\right\}, \quad x = \infty. \tag{4.6}$$

We claim that $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}$ constitutes a covering system of localizing classes of $PC^0_{X(\mathbb{R})}$ (here the index set T coincides with the union of two copies of $\dot{\mathbb{R}}$, one of which corresponds to the left neighborhoods and the other corresponds to the right neighborhoods of $x \in \dot{\mathbb{R}}$). First note that every element of $\mathrm{M}_x^\pm$ is a characteristic function that, obviously, belongs to $P\mathbb{C}^0$. Therefore $\mathrm{M}_x^\pm \subset PC^0_{X(\mathbb{R})}$ for all $x \in \dot{\mathbb{R}}$. Moreover, by the definition of $\mathrm{M}_x^\pm$, we have $0 \notin \mathrm{M}_x^\pm$ for all $x \in \dot{\mathbb{R}}$.

Now fix $x \in \dot{\mathbb{R}}$ and $\chi_1, \chi_2 \in \mathrm{M}_x^\pm$. By the definition, χ_1 and χ_2 are characteristic functions of intervals I_1 and I_2, respectively. Let χ_3 be the characteristic function of $I_3 := I_1 \cap I_2$. We find that $\chi_3 \in \mathrm{M}_x^\pm$ and

$$\chi_1\chi_3 = \chi_2\chi_3 = \chi_3 = \chi_3\chi_2 = \chi_3\chi_1.$$

Therefore, $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}$ is a family of localizing classes of $PC^0_{X(\mathbb{R})}$.

Consider now an arbitrary choice of elements

$$\left\{\chi_x^-, \chi_x^+\right\}_{x\in\dot{\mathbb{R}}} \subseteq \left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}.$$

In view of the compactness of $\dot{\mathbb{R}}$, there exist a finite number of points $x_1, x_2, \dots, x_n$ in $\dot{\mathbb{R}}$ such that the functions $\chi_{x_1}^-, \chi_{x_2}^-, \dots, \chi_{x_n}^-, \chi_{x_1}^+, \chi_{x_2}^+, \dots, \chi_{x_n}^+$ satisfy the following property:

$$g(t) := \sum_{j=1}^n \left(\chi_{x_j}^-(t) + \chi_{x_j}^+(t)\right) \geq 1 \quad \text{for all} \quad t \in \dot{\mathbb{R}}. \tag{4.7}$$

Since g is a linear combination of characteristic functions of intervals of $\dot{\mathbb{R}}$, we see that $g : \dot{\mathbb{R}} \to \mathbb{N}$ and $g \in P\mathbb{C}^0$. Moreover, since $g \geq 1$, we have $g^{-1} = 1/g \in P\mathbb{C}^0$. Hence, by the definition of $PC^0_{X(\mathbb{R})}$, we conclude that $g, g^{-1} \in PC^0_{X(\mathbb{R})}$. Therefore, $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x\in\dot{\mathbb{R}}}$ is a covering system of localizing classes of $PC^0_{X(\mathbb{R})}$.

We have

$$a \overset{\mathrm{M}_x^\pm}{\sim} a(x \pm 0) \quad \text{for} \quad x \in \dot{\mathbb{R}}, \quad a \in P\mathbb{C}^0, \tag{4.8}$$

since for each $x \in \dot{\mathbb{R}}$, there exist the functions $\chi_x^\pm \in \mathrm{M}_x^\pm$ such that

$$[a(t) - a(x \pm 0)]\, \chi_x^\pm(t) = 0 \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

It is clear that if $x \in \dot{\mathbb{R}}$, then for each $\chi_x^\pm \in \mathrm{M}_x^\pm$, we have $\|\chi_x^\pm\|_{V(\mathbb{R})} = 3$. Therefore, by the Stechkin-type inequality (1.3),

$$\sup\left\{\|\chi_x^\pm\|_{\mathcal{M}_{X(\mathbb{R})}} : \chi_x^\pm \in \mathrm{M}_x^\pm\right\} \leq 3c_X < \infty. \tag{4.9}$$

If $a \in PC^0_{X(\mathbb{R})}$, then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $P\mathbb{C}^0$ such that

$$\|a - a_n\|_{\mathcal{M}_{X(\mathbb{R})}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.10}$$

Then, in view of inequality (1.2), we conclude that

$$\|a - a_n\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.11}$$

Since $a, a_n \in PC$, for each $x \in \dot{\mathbb{R}}$ there exist finite one-sided limits $a(x \pm 0)$ and $a_n(x \pm 0)$ and the sets of discontinuities of a and a_n are at most countable. Hence (4.11) implies that for $x \in \dot{\mathbb{R}}$ one has $a_n(x \pm 0) \to a(x \pm 0)$, as $n \to \infty$. Thus

$$\|a_n(x \pm 0) - a(x \pm 0)\|_{\mathcal{M}_{X(\mathbb{R})}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.12}$$

Combining (4.8)–(4.10) and (4.12) with Proposition 4.1, we conclude that

$$a \overset{\mathrm{M}_x^{\pm}}{\sim} a(x \pm 0) \quad \text{for} \quad x \in \dot{\mathbb{R}}, \quad a \in PC^0_{X(\mathbb{R})}. \tag{4.13}$$

On the other hand, by the hypothesis (4.4), we get $a(x \pm 0) \neq 0$ for $x \in \dot{\mathbb{R}}$. Therefore the constant functions $a(x \pm 0)$ are invertible in the Banach algebra $PC^0_{X(\mathbb{R})}$ and $a(x \pm 0)^{-1} \in PC^0_{X(\mathbb{R})}$ for $x \in \dot{\mathbb{R}}$. Hence $a(x \pm 0)$ is $\mathrm{M}_x^{\pm}$-invertible for every $x \in \dot{\mathbb{R}}$. Finally, taking into account (4.13) and applying the Gohberg-Krupnik Local Principle (Theorem 4.2), we get that a is invertible in the algebra $PC^0_{X(\mathbb{R})}$, i.e., $a^{-1} \in PC^0_{X(\mathbb{R})}$. □

4.4. **Proof of Theorem 1.1.** The proof presented below follows that of [9, Theorem 2.18].

If (4.4) is fulfilled, then by Lemma 4.3, we have $a^{-1} \in PC^0_{X(\mathbb{R})}$. From the general properties of the Fourier convolution operators on $X(\mathbb{R})$, we get

$$W^0(a)W^0(a^{-1}) = W^0(a^{-1})W^0(a) = I.$$

Therefore, the operator $W^0(a)$ is invertible on $X(\mathbb{R})$ and $(W^0(a))^{-1} = W^0(a^{-1})$.

Suppose now that the operator $W^0(a)$ is invertible on the space $X(\mathbb{R})$. For each $x \in \dot{\mathbb{R}}$, let $\mathrm{M}_x^{\pm}$ be defined by (4.5)–(4.6) and

$$\mathrm{M}_x^{0,\pm} := \left\{ W^0(g) \in \mathcal{B}(X(\mathbb{R})) \ : \ g \in \mathrm{M}_x^{\pm} \right\}.$$

We claim that $\left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}$ constitutes a covering system of localizing classes in the Banach algebra of bounded linear operators $\mathcal{B}(X(\mathbb{R}))$. Knowing that $\mathrm{M}_x^{\pm}$ is a localizing class in $PC^0_{X(\mathbb{R})}$, we have $0 \notin \mathrm{M}_x^{\pm}$. Therefore, $0 \notin \mathrm{M}_x^{0,\pm}$ for all $x \in \dot{\mathbb{R}}$.

Consider now $W^0(g_1), W^0(g_2) \in \mathrm{M}_x^{0,\pm}$. Then $g_1, g_2 \in \mathrm{M}_x^{\pm}$. Since $\mathrm{M}_x^{\pm}$ is a localizing class of $PC^0_{X(\mathbb{R})}$, there exists $g_3 \in \mathrm{M}_x^{\pm}$ such that

$$g_1 g_3 = g_2 g_3 = g_3 = g_3 g_2 = g_3 g_1.$$

Therefore, $W^0(g_3) \in \mathrm{M}_x^{0,\pm}$ and

$$W^0(g_1)W^0(g_3) = W^0(g_2)W^0(g_3) = W^0(g_3) = W^0(g_3)W^0(g_2) = W^0(g_3)W^0(g_1).$$

Hence $\left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}$ is a family of localizing classes in the Banach algebra of bounded linear operators $\mathcal{B}(X(\mathbb{R}))$.

Consider an arbitrary choice of elements

$$\left\{W^0(g_x^-), W^0(g_x^+)\right\}_{x \in \dot{\mathbb{R}}} \subseteq \left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}.$$

Bearing in mind that $\left\{\mathrm{M}_x^-, \mathrm{M}_x^+\right\}_{x \in \dot{\mathbb{R}}}$ is a covering system of localizing classes of the Banach algebra $PC^0_{X(\mathbb{R})}$ (see the proof of Lemma 4.3), there exist the points $x_1, x_2, \dots, x_n \in \dot{\mathbb{R}}$ such that $g_{x_i}^- \in \mathrm{M}_{x_i}^-$ and $g_{x_i}^+ \in \mathrm{M}_{x_i}^+$ for $i \in \{1, 2, \dots, n\}$, and the function

$$g := \sum_{i=1}^{n} \left(g_{x_i}^- + g_{x_i}^+\right)$$

is invertible in the algebra $PC^0_{X(\mathbb{R})}$. It follows that the operator $W^0(g)$ is invertible in the algebra $\mathcal{B}(X(\mathbb{R}))$ and its inverse is equal to $W^0(g^{-1}) \in \mathcal{B}(X(\mathbb{R}))$. Thus, $\left\{\mathrm{M}_x^{0,-}, \mathrm{M}_x^{0,+}\right\}_{x \in \dot{\mathbb{R}}}$ forms a covering system of localizing classes in the Banach algebra $\mathcal{B}(X(\mathbb{R}))$.

It follows from (4.13) that for all $x \in \dot{\mathbb{R}}$,

$$W^0(a) \overset{\mathrm{M}_x^{0,\pm}}{\sim} W^0(a(x \pm 0)) = a(x \pm 0)I.$$

If there exists some $x^* \in \dot{\mathbb{R}}$ such that $a(x^*-0)=0$ or $a(x^*+0)=0$, then $W^0(a) \overset{\mathrm{M}^{0,-}_{x^*}}{\sim} 0$ or $W^0(a) \overset{\mathrm{M}^{0,+}_{x^*}}{\sim} 0$. Since $W^0(a)$ is invertible, applying Gohberg-Krupnik's local principle (Theorem 4.2), we conclude that 0 is $\mathrm{M}^{0,-}_{x^*}$-invertible or $\mathrm{M}^{0,+}_{x^*}$-invertible. Therefore, $0 \in \mathrm{M}^{0,-}_{x^*} \cup \mathrm{M}^{0,+}_{x^*}$ which is a contradiction, since $\mathrm{M}^{0,-}_{x^*}$ and $\mathrm{M}^{0,+}_{x^*}$ are localizing classes of $\mathcal{B}(X(\mathbb{R}))$. Thus, $a(x \pm 0) \neq 0$ for all $x \in \dot{\mathbb{R}}$. Consequently, (4.4) is fulfilled. □

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References

1. C. Bennett, R. Sharpley, *Interpolation of Operators.* Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.
2. A. Böttcher, Yu. I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators.* Progress in Mathematics, 154. Birkhäuser Verlag, Basel, 1997.
3. A. Böttcher, Yu. I. Karlovich, I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions.* Operator Theory: Advances and Applications, 131. Birkhäuser Verlag, Basel, 2002.
4. A. Böttcher, B. Silbermann, *Analysis of Toeplitz Operators.* Second edition. Prepared jointly with Alexei Karlovich. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.
5. N. Bourbaki, *Functions of a Real Variable. Elementary Theory.* Springer-Verlag, Berlin, 2004.
6. D. W. Boyd, Indices of function spaces and their relationship to interpolation. *Canadian J. Math.* **21** (1969), 1245–1254.
7. D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces.* Foundations and harmonic analysis. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013.
8. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents.* Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
9. R. V. Duduchava, *Integral Equations with Fixed Singularities.* Teubner, Leipzig, 1979.
10. R. E. Edwards, G. I. Gaudry, *Littlewood-Paley and Multiplier Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 90. Springer-Verlag, Berlin-New York, 1977.
11. C. A. Fernandes, A. Yu. Karlovich, Semi-almost periodic Fourier multipliers on rearrangement-invariant spaces with suitable Muckenhoupt weights. *Bol. Soc. Mat. Mex. (3)* **26** (2020), no. 3, 1135–1162.
12. C. A. Fernandes, A. Yu. Karlovich, Yu. I. Karlovich, Noncompactness of Fourier convolution operators on Banach function spaces. *Ann. Funct. Anal.* **10** (2019), no. 4, 553–561.
13. G. Folland, *Real Analysis. Modern Techniques and Applications.* Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.
14. I. Gohberg, N. Krupnik, *One-Dimensional Linear Singular Integral Equations. I. Introduction.* Translated from the 1979 German translation by Bernd Luderer and Steffen Roch and revised by the authors. Operator Theory: Advances and Applications, 53. Birkhäuser Verlag, Basel, 1992.
15. I. Gohberg, N. Krupnik, On a local principle and algebras generated by Toeplitz matrices. In: *Convolution equations and singular integral operators*, pp. 157–184, Oper. Theory Adv. Appl., 206, Birkhäuser Verlag, Basel, 2010.
16. L. Grafakos, *Classical Fourier Analysis.* Third edition. Graduate Texts in Mathematics, 249. Springer, New York, 2014.
17. A. Yu. Karlovich, Maximally modulated singular integral operators and their applications to pseudodifferential operators on Banach function spaces. In: *Function spaces in analysis*, pp. 165–178, Contemp. Math., 645, Amer. Math. Soc., Providence, RI, 2015.
18. A. Yu. Karlovich, Algebras of continuous Fourier multipliers on variable Lebesgue spaces. *Mediterr. J. Math.* **17** (2020), no. 4, Paper No. 102, 19 pp.
19. A. Karlovich, E. Shargorodsky, When does the norm of a Fourier multiplier dominate its L^∞ norm? *Proc. Lond. Math. Soc. (3)* **118** (2019), no. 4, 901–941.
20. A. Karlovich, E. Shargorodsky, Algebras of convolution type operators with continuous data do not always contain all rank one operators. *Integral Equ. Oper. Theory*, to appear. Preprint is available at arXiv: 2007.10266.
21. A. Yu. Karlovich, I. M. Spitkovsky, The Cauchy singular integral operator on weighted variable Lebesgue spaces. In: *Concrete operators, spectral theory, operators in harmonic analysis and approximation*, pp. 275–291, Oper. Theory Adv. Appl., 236, Birkhäuser/Springer, Basel, 2014.
22. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces. II: Function Spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 97. Springer-Verlag, Berlin-New York, 1979.

23. W. Luxemburg, *Banach Function Spaces*. Thesis, Technische Hogeschool te Delft, 1955.
24. S. Roch, P. A. Santos, B. Silbermann, *Non-Commutative Gelfand Theories. A Tool-kit for Operator Theorists and Numerical Analysts*. Universitext. Springer-Verlag London, Ltd., London, 2011.

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