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BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF G-FRACTIONAL INTEGRAL AND G-FRACTIONAL MAXIMAL OPERATORS WITH $G-BMO$ FUNCTIONS

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Abstract. In this paper we introduce the Gegenbauer BMO ($G-BMO$) space and study its basic properties, analogous to the classical case. The John-Nirenberg type theorem is proved for $f \in BMO_G(\mathbb{R}_+)$. Moreover, the notions of a higher order commutator of Gegenbauer fractional (G-fractional) integral $J_G^{b,\alpha,k}$ and Gegenbauer fractional (G-fractional) maximal operator $M_G^{b,\alpha,k}$ with $G-BMO$ function are studied. When commutator b is a ($G-BMO$) function, the necessary and sufficient conditions for $(L_p; L_q)$ boundedness of commutators $J_G^{b,\alpha,k}$ and $M_G^{b,\alpha,k}$ are obtained.

Introduction

The boundedness of the fractional maximal operator, fractional integral and its commutators plays an important role in harmonic analysis and their applications. In recent decades, many authors have proved the boundedness of the commutators with BMO functions of fractional maximal operator and fractional integral operator on some function spaces (see, e.g., [1–4, 6–8, 13, 19]).

The fractional integral operator I_α and fractional maximal operator M_α are defined as follows:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}, \quad n \geq 1, \quad 0 < \alpha < n,$$

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|\leq r} |f(y)|dy.$$

Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, then the commutator is generated by the function $b(x)$ and I_α is defined as the form

$$[b, I_\alpha]f(x) = b(x)I_\alpha(x) - I_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{[b(x)-b(y)]}{|x-y|^{n-\alpha}} f(y)dy.$$

In [2] and [19], the following theorem is proved by a somewhat different method.

Theorem A. *Let $0 < \alpha < n$, $1 < p < \dfrac{n}{\alpha}$ and $\dfrac{1}{p} - \dfrac{1}{q} = \dfrac{\alpha}{n}$. Then $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

Define the commutator $[b, M_\alpha]$ of the fractional maximal operator M_α as

$$[b, M_\alpha](f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|\leq r} |b(x)-b(y)||f(y)|dy.$$

In [19], it is proved that under the conditions of Theorem A $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.

In the present paper, we prove theorems on the boundedness of commutators both of the G-fractional integral and of the G-fractional maximal operator on $G-BMO$ space. The results obtained here are analogous to the corresponding theorem obtained for the $[b, I_\alpha]$ and $[b, M_\alpha]$ in [2] and [19].

The paper is organized as follows.

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In Section 1, we present some definitions, notations and auxiliary results. In Section 2, the $G-BMO$ space is introduced and its properties are proved. In Sections 3 and 4 we prove the $(L_{p,\lambda};\, L_{q,\lambda})$ boundedness of the commutator of G-fractional integrals and the $(L_{p,\lambda};\, L_{q,\lambda})$ boundedness of the commutator of G-fractional maximal operator on $G-BMO$ space, respectively.

1. Definitions, Notations and Auxiliary Results

Our investigation is based on the Gegenbauer differential operator G_λ (see [5])

$$G_\lambda \equiv G = (x^2-1)^{\frac{1}{2}-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}, x\in(1,\infty), \lambda\in(0,\frac{1}{2}).$$

The shift operator $A^\lambda_{\mathrm{ch}y}$ generated by G_λ is given in the form (see [10, 11])

$$A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)=\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\int\limits_0 f(\mathrm{ch}x\mathrm{ch}y-\mathrm{sh}x\mathrm{sh}y\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi$$

and it possesses all properties of the generalized shift operator given in the monograph due to B.M.Levitan [16, 17].

Let $H=H(0,r)=(0,r)$. For any measurable set E, $\mu E=|E|_\lambda=\int\limits_E \mathrm{sh}^{2\lambda}ydy$. For $1\le p<\infty$, let $L_p(\mathbb{R}_+,G)=L_{p,\lambda}(\mathbb{R}_+)$, $\mathbb{R}_+=(0,\infty)$ be the space of measurable functions on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}}=\Bigg(\int\limits_{\mathbb{R}_+}|f(\mathrm{ch}y)|^p\mathrm{sh}^{2\lambda}ydy\Bigg)^{\frac{1}{p}},\quad 1\le p<\infty,$$

$$\|f\|_{\infty,\lambda}\equiv\|f\|_\infty=ess\sup_{x\in\mathbb{R}_+}|f(\mathrm{ch}x)|,\quad p=\infty.$$

For $f\in L^{\mathrm{loc}}_{1,\lambda}(\mathbb{R}_+)$, the G-fractional maximal operator M^α_G and the G-fractional integral J^α_G are defined in [14] as follows:

$$M^\alpha_G f(\mathrm{ch}x)=\sup_{r>0}\frac{1}{|H|_\lambda^{1-\frac{\alpha}{2\lambda+1}}}\int\limits_H A^\lambda_{\mathrm{ch}y}|f(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}ydy.$$

Here $|H(0,r)|_\lambda=\int\limits_0^r \mathrm{sh}^{2\lambda}ydy$ is the measure that is absolutely continuous with respect to the Lebesgue measure of the interval H

$$J^\alpha_G f(\mathrm{ch}x)=\int\limits_0^\infty\frac{A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)}{(\mathrm{sh}y)^{2\lambda+1-\alpha}}\mathrm{sh}^{2\lambda}ydy.$$

The next result has been obtained in [14] and gives us the $(L_{p,\lambda},L_{q,\lambda})$ boundedness of M^α_G and J^α_G (see also [13, 15]).

Theorem B. *Suppose that* $0<\lambda<\dfrac{1}{2}, 0<\alpha<2\lambda+1,$ *and* $1\le p<\dfrac{2\lambda+1}{\alpha}$.

(a) *If* $1<p<\frac{2\lambda+1}{\alpha}$, *then the condition* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$ *is necessary and sufficient for the boundedness of* M^α_G *and* J^α_G *from* $L_{p,\lambda}(\mathbb{R}_+)$ *to* $L_{q,\lambda}(\mathbb{R}_+)$.

(b) *If* $p=1$, *then the condition* $1-\frac{1}{q}=\frac{\alpha}{2\lambda+1}$ *is necessary and sufficient for the boundedness of* M^α_G *and* J^α_G *from* $L_{1,\lambda}(\mathbb{R}_+)$ *to* $WL_{q,\lambda}(\mathbb{R}_+)$.

We denote by $WL_{q,\lambda}(\mathbb{R}_+)$ the spaces of all locally integrable functions $f(\mathrm{ch}x)$, $x\in\mathbb{R}_+$, with the finite norm

$$\|f\|_{WL_{q,\lambda}(\mathbb{R}_+)}=\sup_{r>0}r|\{x\in\mathbb{R}_+:|f(\mathrm{ch}x)|>r\}|_\lambda^{\frac{1}{p}},\ 1\le p<q.$$

Throughout the paper $A\lesssim B$ mean that $A\le CB$ with some positive constant C, which may depend on some parameters. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

Let $H(x,r)=(x-r,x+r)\cap[0,\infty)$, $r\in(0,\infty)$, $x\in[0,\infty)$. Thus,

$$H(x,r)=\begin{cases}(0,x+r), & 0\le x<r,\\ (x-r,x+r), & x\ge r.\end{cases}$$

We will need the following lemmas.

Lemma 1.1 ([14]). *For any $\mu>0$, the following relation is true:*

$$|H(x,r)|_{\frac{\mu}{2}}\approx\begin{cases}\left(\mathrm{sh}\dfrac{x+r}{2}\right)^{\mu+1}, & 0<x+r<2,\\ \left(\mathrm{sh}\dfrac{x+r}{2}\right)^{2\mu}, & 2\le x+r<\infty.\end{cases}$$

For $x=0$ and $\mu=2\lambda$, we have

$$|H(0,r)|_{\lambda}\approx\left(\mathrm{sh}\frac{r}{2}\right)^{\gamma},$$

where $\gamma=\gamma_\lambda(r)=\begin{cases}2\lambda+1, & \text{if} \quad 0<r<2,\\ 4\lambda, & \text{if} \quad 2\le r<\infty.\end{cases}$

Lemma 1.2 ([11]). *If $f\in L_{p,\lambda}(\mathbb{R}_+)$, then for any $y\in[0,\infty)$, the inequality*

$$\|A_{\mathrm{ch}y}f\|_{L_{p,\lambda}}\le\|f\|_{L_{p,\lambda}},1\le p\le\infty \tag{1.1}$$

holds.

2. The Gegenbauer BMO-Space

The space of functions of bounded mean oscillation, or BMO_G, naturally arises as the class of functions whose deviation from their means over intervals is bounded. The L_∞ functions have this property, but there exist unbounded functions with a bounded mean oscillation. Such functions are slowly growing, and they typically have at most logarithmic blow up. The space BMO_G shares similar properties with the space L_∞ and often serves as its substitute. What exactly is a bounded mean oscillation and what kind of functions have this property?

The mean of a locally integrable function over a set is another word for its average over that set. The oscillation of a function over a set is the absolute value of the difference of the function from its mean over this set. The mean oscillation is therefore the average of this oscillation over a set. A function is said to be of bounded mean oscillation if its mean oscillation over all intervals is bounded. Precisely, given a locally integrable function f on $\mathbb{R}_+=(0,\infty)$, denote by

$$f_H(\mathrm{ch}x)=\frac{1}{|H|_\lambda}\int\limits_H A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y dy,$$

where $H=H(0,r)$, the mean (or average) of f over H. Then the oscillation of f over H are the functions $|A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)-f_H(\mathrm{ch}x)|$, and the mean oscillation of f over H is

$$\frac{1}{|H|_\lambda}\int\limits_H |A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)-f_H(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}y dy.$$

2.1. Definition and some properties of the $G-BMO$ space.

Definition 2.1. We denote by $BMO_G(\mathbb{R}_+)$ the Gegenbauer-BMO space ($G-BMO$ space) as the set of locally integrable functions on $\mathbb{R}_+=(0,\infty)$ such that

$$\|f\|_{BMO_G(\mathbb{R}_+)}=\sup_{x,r\in\mathbb{R}_+}\frac{1}{|H|_\lambda}\int\limits_H |A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)-f_H(\mathrm{ch}x)|\mathrm{sh}^{2\lambda}y dy<\infty.$$

We set

$$BMO_G(\mathbb{R}_+)=\left\{f\in L^{\mathrm{loc}}_{1,\lambda}(\mathbb{R}_+):\|f\|_{BMO_G(\mathbb{R}_+)}<\infty\right\}.$$

Several remarks are in order. First, it is a simple fact that $BMO_G(\mathbb{R}_+)$ is a linear space, that is, if $f, g \in BMO_G(\mathbb{R}_+)$ and $\mu \in R$, then $f+g$ and μf in $BMO_G(\mathbb{R}_+)$, and

$$\|f+g\|_{BMO_G} \le \|f\|_{BMO_G} + \|g\|_{BMO_G}, \ \|\mu f\|_{BMO_G} = |\mu| \|f\|_{BMO_G}.$$

But $\|\cdot\|_{BMO_G}$ is not a norm. The problem is that if $\|\cdot\|_{BMO_G} = 0$, this does not imply that $f = 0$, but that f is a constant. From Proposition 2.2, every constant function C satisfies $\|C\|_{BMO_G} = 0$, then the functions f and $f + c$ have the same BMO_G norms. In the sequel, we keep in mind that elements of BMO_G whose difference is a constant are identified. Although $\|\cdot\|_{BMO_G}$ is only a seminorm, we occasionally refer to it as a norm when there is no possibility of confusion.

We begin with the basic properties of BMO_G.

Proposition 2.2. *The following properties of the $BMO_G(\mathbb{R}_+)$ space are valid:*

1) *If $\|f\|_{BMO_G} = 0$, then f is a.e. equal to a constant.*

2) *$L_\infty(\mathbb{R}_+)$ is contained in $BMO_G(\mathbb{R}_+)$ and $\|f\|_{BMO_G} \le 2\|f\|_{L_\infty}$.*

3) *Suppose that there exist a constant $A > 0$ and for all intervals H in $\mathbb{R}_+$ a constant C_H such that*

$$\sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| \mathrm{sh}^{2\lambda} y dy \le A, \tag{2.1}$$

then $f \in BMO_G(\mathbb{R}_+)$ and $\|f\|_{BMO_G} \le 2A$.

4) *If $f \in BMO_G(\mathbb{R}_+)$, $y \in \mathbb{R}_+$, then $A^\lambda_{\mathrm{ch}y} f$ is also in $BMO_G(\mathbb{R}_+)$ and*

$$\|A^\lambda_{\mathrm{ch}y} f\|_{BMO_G} \le \|f\|_{BMO_G}.$$

5) *Let f be in $BMO_G(\mathbb{R}_+)$. Given an interval H and a positive integer m, we have*

$$|b_H(\mathrm{ch}x) - b_{2^m H}(\mathrm{ch}x)| \le 2m\|b\|_{BMO_G}.$$

Proof. To prove 1), we note that f is a.e. equal to its average C_N over every segment $[0, N]$. Since $[0, N] \subset [0, N + 1]$, it follows that $C_N = C_{N+1}$ for all N. This implies the required conclusion.

To prove 2), we using (1.1). Then

$$A^\lambda_{\mathrm{ch}y} |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \le A^\lambda_{\mathrm{ch}y} \left(|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x)| + |f_H(\mathrm{ch}x)|\right)$$
$$\le 2A^\lambda_{\mathrm{ch}y} |f(\mathrm{ch}x)| \le 2\|f\|_{L_\infty}.$$

For item 3), we get

$$|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \le |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| + |f_H(\mathrm{ch}x) - C_H|$$
$$\le |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| + \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - C_H| \mathrm{sh}^{2\lambda} y dy.$$

Averaging over H and using (2.1), one has

$$\|f\|_{BMO_G} \le 2A.$$

Let us prove property 4). Applying Lemma 1.2, we have

$$\|A^\lambda_{\mathrm{ch}y} f\|_{BMO_G} \le \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - A^\lambda_{\mathrm{ch}y} f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$
$$\le \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H A^\lambda_{\mathrm{ch}y} |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$
$$\le \sup_{x,r\in\mathbb{R}_+} \frac{1}{|H|_\lambda} \int\limits_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy = \|f\|_{BMO_G}.$$

Finally, we prove 5). In fact,

$$|b_H(\mathrm{ch}x) - b_{2H}(\mathrm{ch}x)| \le \frac{1}{|H|_\lambda}\left|\int\limits_H \left(A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_{2H}(\mathrm{ch}x)\right)\mathrm{sh}^{2\lambda} y dy\right|$$

$$\le \frac{2}{|2H|_\lambda}\int\limits_H \left|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - b_{2H}(\mathrm{ch}x)\right|\mathrm{sh}^{2\lambda} y dy \le 2\|f\|_{BMO_G}.$$

Then A_n iteration yields

$$|b_H - b_{2H} + b_{2H} - b_{2^2H} + \cdots + b_{2^{m-1}H} - b_{2^m H}| \le 2m\|f\|_{BMO_G}. \qquad \square$$

Example. We show that $L_\infty(\mathbb{R}_+)$ is a proper subspace of $BMO_G(\mathbb{R}_+)$. We claim that the function $\log(\mathrm{sh}x)$ is in $BMO_G(\mathbb{R}_+)$, but not in $L_\infty(\mathbb{R}_+)$. To prove that it is in $BMO_G(\mathbb{R}_+)$, for every $x_0 \in \mathbb{R}_+$ and $r > 0$, we choose a constant $C_{x_0,r}$ such that the average $|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x) - C_{x_0,r}|$ for all $y \in [0, x_0 + r]$ is uniformly bounded.

Consider the integral

$$\frac{1}{|H(0, x_0 + r)|_\lambda}\int\limits_0^{x_0+r} \left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x) - C_{x_0,r}\right|\mathrm{sh}^{2\lambda} y dy,$$

where $C_{x_0,r} = (\log r)(\log x_0)$, $0 \le x_0 \le 2$ and $0 \le x_0 \le \mathrm{arcsh}1$. We may take $r = 1$, then

$$\frac{1}{|H(0, x_0 + 1)|_\lambda}\int\limits_0^{x_0+1} \left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x)\right|\mathrm{sh}^{2\lambda} y dy$$

$$= \frac{1}{|H(0, x_0 + 1)|_\lambda}\int\limits_0^{x_0+1} \left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{ch}^2 x - 1)^{\frac{1}{2}}\right|\mathrm{sh}^{2\lambda} y dy$$

$$= \frac{1}{|H(0, x_0 + 1)|_\lambda}\int\limits_0^{x_0+1} \left|\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\int\limits_0^\pi \log[(\mathrm{ch}x\mathrm{ch}y - \mathrm{sh}x\mathrm{sh}y\cos\varphi)^2 - 1]^{\frac{1}{2}}\right|\mathrm{sh}^{2\lambda} y dy$$

$$\le \frac{1}{|H(0, x_0 + 1)|_\lambda}\int\limits_0^{x_0+1} |\log\mathrm{sh}(x + y)|\,\mathrm{sh}^{2\lambda} y dy \le \log\mathrm{sh}(x + x_0 + 1)$$

$$\le \log\mathrm{sh}(x_0 + 1 + \mathrm{arcsh}1) \le \log\mathrm{sh}(x_0 + 2) \le \log\mathrm{sh}4.$$

Now, let $C_{x_0,1} = \log(2x_0)$, $\mathrm{arcsh}1 \le x \le x_0$, $x_0 > 2$. In this case, we have

$$\frac{1}{|H(0, x_0 + 1)|_\lambda}\int\limits_0^{x_0+1} \left|A^\lambda_{\mathrm{ch}y}\log(\mathrm{sh}x) - \log(2x_0)\right|\mathrm{sh}^{2\lambda} y dy$$

$$\le \log\frac{\mathrm{sh}(x + x_0 + 1)}{\mathrm{sh}(2x_0)} < \log\frac{\mathrm{sh}(2x_0 + 2)}{\mathrm{sh}2x_0} = \log\frac{(\mathrm{sh}2x_0)\mathrm{ch}2 + (\mathrm{ch}2x_0)\mathrm{sh}2}{\mathrm{sh}2x_0}$$

$$= \log\left(\mathrm{ch}2 + \frac{\mathrm{ch}2x_0}{\mathrm{sh}(2x_0)}\mathrm{sh}2\right) \le \log(\mathrm{ch}2 + 2\mathrm{sh}2) \le \log(3\mathrm{ch}2),$$

since $\mathrm{ch}x \le 2\mathrm{sh}x$ if $x \le 1$.

Thus, according to property 3), $\log(\mathrm{sh}x)$ is in $BMO_G(\mathbb{R}_+)$. It is obvious that $\log(\mathrm{sh}x)$ is not in $L_\infty(\mathbb{R}_+)$.

Below, we will need some property of $BMO_G(\mathbb{R}_+)$ functions. Observe that if an interval H_1 is contained in the interval H_2, then

$$|f_{H_1} - f_{H_2}| \le \frac{1}{|H_1|_\lambda} \int\limits_{H_1} |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_{H_2}(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$
$$\le \frac{1}{|H_1|_\lambda} \int\limits_{H_2} |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_{H_2}(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$
$$\le \frac{|H_2|_\lambda}{|H_1|_\lambda} \|f\|_{BMO_G}.$$

Theorem 2.3. *$BMO_G(\mathbb{R}_+)$ is a complete space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $BMO_G(\mathbb{R}_+)$. Thus $\|f_n - f_m\|_{BMO_G} \to 0$, for $n, m \to \infty$. We choose a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_{BMO_G} < \frac{1}{2^k}$ for all $k \ge 1$. From this it follows that

$$\sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_{BMO_G} < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Then for a.e. $x \in \mathbb{R}_+$,

$$\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| < \infty,$$

and, consequently, the series

$$f_{n_1}(\mathrm{ch}x) + \sum_{k=1}^{\infty} \{f_{n_{k+1}}(\mathrm{ch}x) - f_{n_k}(\mathrm{ch}x)\}$$

converges, this is equivalent to the existence of

$$\lim_{k\to\infty} f_{n_k}(\mathrm{ch}x), \text{for a.e. } x \in \mathbb{R}_+.$$

We define the function f as follows:

$$f(\mathrm{ch}x) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(\mathrm{ch}x), & \text{for a.e.} \quad x \in \mathbb{R}_+, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we prove that

$$\lim_{k\to\infty} f_{n_k}(\mathrm{ch}x) = f(\mathrm{ch}x), \text{a.e. } x \in \mathbb{R}_+.$$

By the triangle inequality,

$$\|f_{n_k}\|_{BMO_G} = \left\| f_{n_1} + \sum_{\nu=1}^{k-1} \left(f_{n_{k+1}} - f_{n_k}\right) \right\|_{BMO_G}$$
$$\le \left\| |f_{n_1}| + \sum_{\nu=1}^{k-1} \left|f_{n_{k+1}} - f_{n_k}\right| \right\|_{BMO_G}$$
$$\le \|f_{n_1}\|_{BMO_G} + \left\| \sum_{\nu=1}^{k-1} \left|f_{n_{k+1}} - f_{n_k}\right| \right\|_{BMO_G} \le \|f_{n_1}\|_{BMO_G} + 1.$$

From this it follows that

$$\|f_{n_k}\|_{BMO_G} \le \text{const}, \quad \text{at} \quad k \to \infty,$$

i.e., $f \in BMO_G(\mathbb{R}_+)$.

Now, we show that

$$\|f - f_{n_k}\|_{BMO_G} \to 0, \quad \text{at} \quad k \to \infty.$$

In fact,

$$\|f - f_{n_k}\|_{BMO_G} = \left\| \sum_{\nu=k}^{\infty} (f_{n_{\nu+1}} - f_{n_\nu}) \right\|_{BMO_G}$$
$$\leq \left\| \sum_{\nu=k}^{\infty} |f_{n_{\nu+1}} - f_{n_\nu}| \right\|_{BMO_G} \leq \sum_{\nu=1}^{\infty} \|f_{n_{\nu+1}} - f_{n_\nu}\|_{BMO_G} < 1.$$

By the dominated convergence theorem,

$$\|f - f_{n_k}\|_{BMO_G} \to 0, \quad \text{at} \quad k \to \infty.$$

Finally, we have to show that $\{f_n\}$ is the Cauchy. Given $\varepsilon > 0$, there exists N_ε so, for all $n, m > N_\varepsilon$, we have

$$\|f_n - f_m\|_{BMO_G} < \frac{\varepsilon}{2}.$$

We choose a number $n_k > N_\varepsilon$ such that

$$\|f - f_{n_k}\|_{BMO_G} < \frac{\varepsilon}{2}.$$

Then we have

$$\|f - f_n\|_{BMO_G} \leq \|f - f_{n_k}\|_{BMO_G} + \|f_n - f_{n_k}\|_{BMO_G} < \varepsilon.$$

This completes the proof. □

The next section needs the following statement.

Theorem 2.4 (Calderon–Zygmund decomposition of $\mathbb{R}_+$). *Suppose that f is a non-negative integrable function on $\mathbb{R}_+$. Then for any fixed number $\beta > 0$, there exists a sequence $\{(j-1)r, jr\} = \{H_j\}$ of disjoint intervals such that*

(1) $f(\text{ch}x) \leq \beta$, $x \notin \bigcup_j H_j$;

(2) $\left|\bigcup_j H_j\right|_\lambda \leq \frac{1}{\beta} \|f\|_{L_{1,\lambda}}$;

(3) $\beta < \frac{1}{|H_j|_\lambda} \int_{H_j} A^\lambda_{\text{ch}y} f(\text{ch}x) \text{sh}^{2\lambda} y dy \leq 2^{(2\lambda+1)n}\beta$, $n = 1, 2, \ldots$.

Proof. Since $f \in L_{1,\lambda}(\mathbb{R}_+)$, by Lemma 1.2, $A^\lambda_{\text{ch}y} f \in L_{1,\lambda}(\mathbb{R}_+)$ and by the integral continuity, we can decompose $\mathbb{R}_+$ into a net of equal intervals (by the Lindelöf covering theorem (see [18]), this is possible)) such that for every H from the net

$$\frac{1}{|H|_\lambda} \int_H A^\lambda_{\text{ch}y} f(\text{ch}x) \text{sh}^{2\lambda} y dy \leq \beta. \tag{2.2}$$

In fact, for any $\beta > 0$, there exists $\delta = \delta(\beta) > 0$ such that for every H_j with measure $|H_j|_\lambda = |H|_\lambda < \delta$,

$$\int_{H_j} A^\lambda_{\text{ch}y} f(\text{ch}x) \text{sh}^{2\lambda} y dy < \beta, \quad (j = 1, 2, \ldots),$$

where

$$|H_j|_\lambda = \int_{H_j} \text{sh}^{2\lambda} y dy, \quad (j = 1, 2, \ldots).$$

First, we prove (3). Let $H_1 = (0, r)$ be a fixed interval in the net. Then by (2.2), we can write

$$\frac{1}{H_1} \int_{H_1} A^\lambda_{\text{ch}y} f(\text{ch}x) \text{sh}^{2\lambda} y dy \leq \beta. \tag{2.3}$$

We divide the interval H_1 into 2^n equal intervals and let $H_1'=\left(0,\frac{r}{2^n}\right)$ be one from this intervals. By Lemma 1.1 (then $\mu=2\lambda$), one has

$$|H_1'|_\lambda=\int\limits_0^{\frac{r}{2^n}}\mathrm{sh}^{2\lambda}y\,dy\approx\left(\mathrm{sh}\frac{r}{2^{n+1}}\right)^{2\lambda+1},\quad 0<\frac{r}{2^n}<2.$$

Since for $0<t<1$, $\mathrm{sh}t\approx t$, we have

$$|H_1'|_\lambda\approx\left(\mathrm{sh}\frac{r}{2^{n+1}}\right)^{2\lambda+1}\approx\left(\frac{r}{2^{n+1}}\right)^{2\lambda+1}\approx\left(\frac{1}{2^n}\mathrm{sh}\frac{r}{2}\right)^{2\lambda+1}\approx 2^{-(2\lambda+1)n}|H_1'|_\lambda. \tag{2.4}$$

Concerning H_1', there may possibly be two cases:

(A) $\frac{1}{|H_1'|_\lambda}\int_{H_1'}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy>\beta.$

(B) $\frac{1}{|H_1'|_\lambda}\int_{H_1'}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\le\beta.$

For case (A), from (2.4) and (2.3), we have

$$\begin{aligned}\beta&<\frac{1}{|H_1'|_\lambda}\int\limits_{H_1'}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\\ &\approx\frac{2^{(2\lambda+1)n}}{|H_1'|_\lambda}\int\limits_{H_1'}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\\ &\lesssim\frac{2^{(2\lambda+1)n}}{|H_1|_\lambda}\int\limits_{H_1}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\lesssim 2^{(2\lambda+1)n}\beta.\end{aligned}$$

Here H_1' we choose as one of the sequences $\{H_j\}$.

We consider case (B). Suppose that $H_1'=H_2(r,2r)$. Dividing the interval into 2^n equal partials and reasoning however, we obtain

$$\begin{aligned}\beta&<\frac{1}{|H_2'|_\lambda}\int\limits_{H_2'}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\\ &\lesssim\frac{2^{(2\lambda+1)n}}{|H_1|_\lambda}\int\limits_{H_1}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\lesssim 2^{(2\lambda+1)n}\beta,\end{aligned}$$

where H_2' we choose as one of the sequences $\{H_j\}$. Further reasoning of the process, we obtain a sequence of disjoint $\{H_j\}$ such that

$$\beta<\frac{1}{|H_j|_\lambda}\int\limits_{H_j}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\lesssim 2^{(2\lambda+1)n}\beta,\quad(n=1,2,\ldots).$$

Proof of (1). Taking into account (2.4), from the Lebesgue differentiation theorem (see [12, Corollary 2.1]), we have

$$f(\mathrm{ch}x)=\lim_{r\to 0}\frac{1}{|H(0,r)|_\lambda}\int\limits_{H(0,r)}A^\lambda_{\mathrm{ch}y}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}y\,dy\le\beta$$

for a.e. $x\notin\bigcup_j H_j$. It remains to prove (2). Passing to the limit by $n\to\infty$ in the inequality

$$\Bigg|\bigcup_{j=1,2,\ldots,n}H_j\Bigg|_\lambda\le\sum_{j=1}^n|H_j|_\lambda\le\frac{1}{\beta}\sum_{j=1}^n\int\limits_{H_j}f(\mathrm{ch}x)\mathrm{sh}^{2\lambda}x\,dx,$$

which is contained in the proof of Theorem 2.4 in [12], we obtain the assertion (2). □

Remark 2.5. The Calderon–Zygmund decomposition stay valid if we replace $\mathbb{R}_+$ by a fixed interval H_0 for $f \in L_{p,\lambda}(H_0)$.

2.2. **The John-Nirenberg type theorem.** Having stated some basic facts about BMO_G, we now turn to a deeper property of BMO_G functions, that is, their exponential integrability. As we saw in Example 2.5, the function $f(\mathrm{ch}x) = \log(\mathrm{sh}x)$ is in BMO_G.

This function is exponentially integrable over any segment $[a, b]$ of $\mathbb{R}_+$ in the sense that

$$\int_a^b e^{|f(\mathrm{ch}x)|}\mathrm{sh}^{2\lambda}x dx < \infty.$$

It turns out that this is a general property of BMO_G functions, and this is the content of the next theorem.

Theorem 2.6. *For all $f \in BMO_G(\mathbb{R}_+)$, for all interval $H = H(0, r)$ and $\alpha > 0$, we have*

$$|\{x \in H : |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| > \alpha\}|_\lambda$$
$$\leq e|H|e^{-\frac{A\alpha}{\|f\|_{BMO_G}}} \text{ with } A = \left(2^{(2\lambda+1)n}e\right)^{-1}.$$

The proof of this theorem is based on the Calderon–Zygmund decomposition and is the same as that of Theorem 7.1.6 in [9].

Corollary 2.7. *For all $0 < p < \infty$ and $H = H(0, r)$, one has*

$$\sup_{r>0}\left(\frac{1}{|H|_\lambda}\int_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}ydy\right)^{\frac{1}{p}} \lesssim \|f\|_{BMO_G}. \tag{2.5}$$

Proof. In fact

$$\begin{aligned}
&\frac{1}{|H|_\lambda}\int_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}ydy\\
=&\frac{p}{|H|_\lambda}\int_0^\infty\left(\int_0^{|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|}\alpha^{p-1}dx\right)\mathrm{sh}^{2\lambda}ydy\\
=&\frac{p}{|H|_\lambda}\int_0^\infty \alpha^{p-1}\left(\int_{\{x\in H:|A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|>\alpha\}}\mathrm{sh}^{2\lambda}ydy\right)dx\\
=&\frac{p}{|H|_\lambda}\int_0^\infty \alpha^{p-1}\left|\{x \in H : |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)| > \alpha\}\right|_\lambda dx\\
\leq&\frac{p}{|H|_\lambda}e|H|_\lambda\int_0^\infty \alpha^{p-1}e^{-\frac{A\alpha}{\|f\|_{BMO_G}}}dx\\
=&pe\frac{\Gamma(p)}{A^p}\|f\|_{BMO_G} = \frac{e}{A^p}\Gamma(p+1)\|f\|_{BMO_G},
\end{aligned}$$

where $A = \left(e^{(2\lambda+1)n}e\right)^{-1}$. □

Since inequality (2.7) can be reversed for $p > 1$ via Hölder's inequality (see [14, Theorem 3.3]), we obtain the following important $L_{p,\lambda}$ characterization of BMO_G norms.

Corollary 2.8. *For all $1 < p < \infty$, we have*

$$\sup_{x,r\in\mathbb{R}_+}\left(\frac{1}{|H|_\lambda}\int_H |A^\lambda_{\mathrm{ch}y} f(\mathrm{ch}x) - f_H(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}ydy\right)^{\frac{1}{p}} \approx \|f\|_{BMO_G}.$$

3. Commutators of Gegenbauer Fractional Integrals

In this section we study the $(L_{p,\lambda}, L_{q,\lambda})$ boundedness of commutators of the Gegenbauer fractional integrals J_G^α, where

$$J_G^\alpha f(\mathrm{ch}x) = \int_0^\infty \frac{A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x)}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy, \ \alpha < \gamma \leq 2\lambda + 1.$$

We will also illustrate that the boundedness of commutators of J_G^α may characterize the $BMO_G(\mathbb{R}_+)$ spaces. First, we will give some related results. Suppose that $b \in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)$, then the commutator generated by the function b and the J_G^α is defined as follows:

$$\begin{aligned} J_G^{b,\alpha} f(\mathrm{ch}x) &= b(\mathrm{ch}x) J_G^\alpha f(\mathrm{ch}x) - J_G^\alpha (bf)(\mathrm{ch}x) \\ &= \int_0^\infty \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b(\mathrm{ch}x)]}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy. \end{aligned}$$

This implies that

$$\begin{aligned} J_G^{b,\alpha} f(\mathrm{ch}x) = \lim_{r\to 0} \Bigg\{ &[b_H(\mathrm{ch}x) - b(\mathrm{ch}x)] \int_r^\infty \frac{A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x)}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \\ &+ \int_r^\infty \frac{A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy \Bigg\}, \end{aligned}$$

where $H = H(0, r)$.

Since $b \in BMO_G(\mathbb{R}_+)$, by Theorem 4.1 and Corollary 2.1 in [12], the first term tends to zero a.e. and

$$J_G^{b,\alpha} f(\mathrm{ch}x) = \int_0^\infty \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)]}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy.$$

The $k-th$ order commutator of the J_G^α we define as follows:

$$J_G^{b,\alpha,k} f(\mathrm{ch}x) = \int_0^\infty \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)]^k}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy.$$

Theorem 3.1. *Suppose that $0 < \alpha < \gamma \leq 2\lambda + 1$, $1 < p < \frac{\gamma}{\alpha}$ and let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$. Then $J_G^{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$, if and only if $b \in BMO_G(\mathbb{R}_+)$.*

Proof. Sufficiency. Let $0 < \alpha < \gamma \leq 2\lambda + 1$, $1 < p < \frac{\gamma}{\alpha}$ and $b \in BMO_G(\mathbb{R}_+)$, we get

$$\begin{aligned} J_G^{b,\alpha,k} f(\mathrm{ch}x) &= \left(\int_0^r + \int_r^\infty \right) \frac{[A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)]^k}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda f(\mathrm{ch}x) \mathrm{sh}^{2\lambda} y dy \\ &= J_1(r) + J_2(r). \end{aligned} \tag{3.1}$$

Consider $J_1(r)$. By Hölder's inequality, we have

$$\begin{aligned} |J_1(r)| &\leq \left(\int_0^r \frac{|A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq}}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \right)^{\frac{1}{q}} \left(\int_0^r \frac{A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)|^p}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \right)^{\frac{1}{p}} \\ &= J_{1.1}(r) \cdot J_{1.2}(r). \end{aligned} \tag{3.2}$$

We estimate $J_{1.1}(r)$. One has

$$J_{1.1}(r) \le \Bigg(\sum_{k=0}^{\infty} \int\limits_{2^{-(k+1)}r}^{2^{-k}r} \frac{|A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq}}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\le \Bigg(\sum_{k=0}^{\infty} \frac{(\mathrm{sh}\frac{r}{2^{k+1}})^{\alpha}}{(\mathrm{sh}\frac{r}{2^{k+1}})^{\gamma}} \int\limits_{0}^{2^{-k}r} |A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq} \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\le (\mathrm{sh}r)^{\frac{\alpha}{q}} \|b\|^k_{BMO_G} \Bigg(\sum_{k=0}^{\infty} 2^{-(k+1)\alpha} \Bigg) \lesssim (\mathrm{sh}r)^{\frac{\alpha}{q}} \|b\|_{BMO_G}. \tag{3.3}$$

Now we estimate $J_{1.2}(r)$. One has

$$J_{1.2}(r) \le \Bigg(\sum_{k=0}^{\infty} \frac{1}{(\mathrm{sh}\frac{r}{2^{k+1}})^{\gamma-\alpha}} \int\limits_{2^{-(k+1)}r}^{2^{-k}r} A^{\lambda}_{\mathrm{ch}y} |f(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{p}}$$

$$\le (\mathrm{sh}r)^{\frac{\alpha}{p}} \left(M_G|f(\mathrm{ch}x)|^p\right)^{\frac{1}{p}}. \tag{3.4}$$

Taking into account (3.3) and (3.4) in (3.2), we get

$$|J_1(r)| \lesssim (\mathrm{sh}r)^{\alpha} \|b\|^k_{BMO_G} \left(M_G|f(\mathrm{ch}x)|^p\right)^{\frac{1}{p}}.$$

Consider $J_2(r)$. By Hölder's inequality, we have

$$|J_2(r)| \le \int\limits_{r}^{\infty} |A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k \frac{A^{\lambda}_{\mathrm{ch}y}|f(\mathrm{ch}x)|}{(\mathrm{sh}y)^{\gamma-\alpha}} \mathrm{sh}^{2\lambda} y dy$$

$$\le \Bigg(\int\limits_{r}^{\infty} \frac{|A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq}}{(\mathrm{sh}y)^{(\gamma-\alpha)q}} \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\times \Bigg(\int\limits_{r}^{\infty} A^{\lambda}_{\mathrm{ch}y} |f(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{p}} \le J_2'(r) \|f\|_{L_{p,\lambda}}. \tag{3.5}$$

For $J_2'(r)$, we have

$$J_2'(r) \le \Bigg(\sum_{k=0}^{\infty} \int\limits_{2^k r}^{2^{k+1}r} \frac{|A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq}}{(\mathrm{sh}y)^{(\gamma-\alpha)q}} \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\le \Bigg(\sum_{k=0}^{\infty} \frac{(\mathrm{sh}2^k)^{\gamma-(\gamma-\alpha)q}}{(\mathrm{sh}2^k)^{\gamma}} \int\limits_{0}^{2^{k+1}r} |A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^{kq} \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}.$$

By property (5), we have $|b_H(\mathrm{ch}x) - b_{2^k H}(\mathrm{ch}x)| \le 2k\|b\|_{BMO_G}$. Then

$$J_2'(r) \lesssim (\mathrm{sh}r)^{\frac{\gamma}{q}+\alpha-\gamma} \Bigg(\sum_{k=0}^{\infty} \frac{(2^k)^{\gamma-(\gamma-\alpha)q}}{(\mathrm{sh}2^k)^{\gamma}} \int\limits_{0}^{2^{k+1}r} |A^{\lambda}_{\mathrm{ch}y} b(\mathrm{ch}x) - b_{2^k H}(\mathrm{ch}x)|^{kq} \mathrm{sh}^{2\lambda} y dy$$

$$+ \sum_{k=0}^{\infty} \frac{(2^k)^{\gamma-(\gamma-\alpha)q}}{(\mathrm{sh}2^k)^{\gamma}} \int\limits_{0}^{2^{k+1}r} |b_H(\mathrm{ch}x) - b_{2^k H}(\mathrm{ch}x)|^{kq} \mathrm{sh}^{2\lambda} y dy \Bigg)^{\frac{1}{q}}$$

$$\lesssim (\mathrm{sh}r)^{\alpha-\frac{\gamma}{p}} \|b\|_{BMO_G} \Bigg(\sum_{k=0}^{\infty} \frac{k}{(2^k)^{(\gamma-\alpha)q-\gamma}} \Bigg)^{\frac{1}{q}} \lesssim (\mathrm{sh}r)^{\alpha-\frac{\gamma}{q}} \|b\|_{BMO_G}, \tag{3.6}$$

since

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma}\Leftrightarrow\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{\gamma}\Leftrightarrow\frac{1}{q}=\frac{\gamma-\alpha p}{\gamma p}$$
$$\Leftrightarrow q=\frac{\gamma p}{\gamma-\alpha p}\Leftrightarrow(\gamma-\alpha)q-\gamma=\frac{(\gamma-\alpha)\gamma p}{\gamma-\alpha p}-\gamma>0\Leftrightarrow\frac{(\gamma-\alpha)\gamma p}{\gamma-\alpha p}>\gamma$$
$$\Leftrightarrow(\gamma-\alpha)p>\gamma-\gamma p\Leftrightarrow\gamma p>\gamma\Leftrightarrow p>1.$$

From (3.6) and (3.5), we have

$$|J_2(r)|\lesssim(\mathrm{sh}r)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}\|b\|_{BMO_G}. \tag{3.7}$$

Taking into account (3.5) and (3.7) in (3.1), we obtain

$$|J_G^{b,\alpha,k}f(\mathrm{ch}x)|\lesssim\left[(\mathrm{sh}r)^{\alpha}\left(M_G|f(\mathrm{ch}x)|^p\right)^{\frac{1}{p}}+(\mathrm{sh}r)^{\alpha-\frac{\gamma}{p}}\|f\|_{L_{p,\lambda}}\right]\|b\|_{BMO_G}^k.$$

The right-hand side attains its minimum for

$$\mathrm{sh}r=\left(\frac{\gamma-\alpha p}{\alpha}\frac{\|f\|_{L_{p,\lambda}}}{(M_G|f|^p(\mathrm{ch}x))^{\frac{1}{p}}}\right)^{\frac{p}{\gamma}},$$

and we have

$$\begin{aligned}|J_G^{b,\alpha,k}f(\mathrm{ch}x)|\lesssim&\left\{\left[\frac{\|f\|_{L_{p,\lambda}}}{(M_G|f|^p(\mathrm{ch}x))^{\frac{1}{p}}}\right]^{\frac{\alpha p}{\gamma}}\left(M_G|f|^p(\mathrm{ch}x)\right)^{\frac{1}{p}}\right.\\&\left.+\left[\frac{\|f\|_{L_{p,\lambda}}}{(M_G|f|^p(\mathrm{ch}x))^{\frac{1}{p}}}\right]^{-\frac{p}{q}}\|f\|_{L_{p,\lambda}}\right\}\|b\|_{BMO_G}\\&=\left(M_G|f|^p(\mathrm{ch}x)\right)^{\frac{1}{p}}\|f\|_{L_{p,\lambda}}^{1-\frac{p}{q}}\|b\|_{BMO_G}^k,\end{aligned}$$

since

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma}\Leftrightarrow 1-\frac{p}{q}=\frac{\alpha p}{\gamma}.$$

From this and Theorem 2.2 in [12], we have

$$\int_0^\infty|J_G^{b,\alpha,k}f(\mathrm{ch}x)|^q\mathrm{sh}^{2\lambda}x\,dx\lesssim\|M_G|f|^p\|_{L_{p,\lambda}}\|f\|_{L_{p,\lambda}}^{q-p}\|b\|_{BMO_G}^{kq}$$
$$\lesssim\|f\|_{L_{p,\lambda}}^q\|b\|_{BMO_G}^{kq}.$$

Thus, we obtain

$$\|J_G^{b,\alpha,k}f(\mathrm{ch}x)\|_{L_{q,\lambda}}\lesssim\|f\|_{L_{p,\lambda}}\|b\|_{BMO_G}^k.$$

Necessity. Let $1<p<\frac{\gamma}{\alpha}$, $f\in L_{p,\lambda}(\mathbb{R}_+)$, and let $J_G^{b,\alpha,k}$ act boundedly from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$, i.e.,

$$\|J_G^{b,\alpha,k}f(\mathrm{ch}x)\|_{L_{q,\lambda}}\lesssim\|f\|_{L_{p,\lambda}}. \tag{3.8}$$

In what follows, the function f will be assumed positive and monotonically increasing. The dilation function $f_t(\mathrm{ch}x)$ will be defined as follows:

$$\begin{aligned}&f(\mathrm{ch}(\mathrm{th}t)x)\le f_t(\mathrm{ch}x)\le f(\mathrm{ch}(\mathrm{cth}t)x),\quad 0<t<1,\\&f(\mathrm{ch}(\mathrm{th}t)x)\le f_t(\mathrm{ch}x)\le f(\mathrm{ch}(\mathrm{sh}t)x),\quad 1\le t<\infty.\end{aligned} \tag{3.9}$$

Using (3.9) for $0<t<1$, we obtain

$$\|f_t\|_{L_{p,\lambda}}=\left(\int_0^\infty|f_t(\mathrm{ch}x)|^p\mathrm{sh}^{2\lambda}x\,dx\right)^{\frac{1}{p}}\le\left(\int_0^\infty|f(\mathrm{ch}(\mathrm{cth}t)x)|^p\mathrm{sh}^{2\lambda}x\,dx\right)^{\frac{1}{p}}$$

$[(\mathrm{cth}t)x=u,x=(\mathrm{th}t)u]$

$$=(\mathrm{th}t)^{\frac{1}{p}}\Big(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}(\mathrm{th}t)udu\Big)^{\frac{1}{p}}$$

$$\le(\mathrm{th}t)^{\frac{2\lambda+1}{p}}\Big(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}udu\Big)^{\frac{1}{p}}$$

$$=(\mathrm{th}t)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}=\Big(\frac{\mathrm{sh}t}{\mathrm{ch}t}\Big)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}$$

$$\lesssim\frac{1}{(\mathrm{ch}t)^{\frac{2\lambda+1}{p}-(\alpha+\frac{2\lambda+1-\gamma}{p})}}\|f\|_{L_{p,\lambda}}\lesssim(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}. \tag{3.10}$$

On the other hand,

$$\|f_t\|_{L_{p,\lambda}}=\Big(\int_0^\infty |f_t(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}xdx\Big)^{\frac{1}{p}}\ge\Big(\int_0^\infty |f(\mathrm{ch}(\mathrm{th}t)x)|^p \mathrm{sh}^{2\lambda}xdx\Big)^{\frac{1}{p}}$$

$[(\mathrm{th}t)x=u, x=(\mathrm{cth}t)u]$

$$=(\mathrm{cth}t)^{\frac{1}{p}}\Big(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}(\mathrm{cth}t)udu\Big)^{\frac{1}{p}}$$

$$\le(\mathrm{cth}t)^{\frac{2\lambda+1}{p}}\Big(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}udu\Big)^{\frac{1}{p}}$$

$$=(\mathrm{cth}t)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}=\Big(\frac{\mathrm{ch}t}{\mathrm{sh}t}\Big)^{\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}$$

$$\lesssim\frac{1}{(\mathrm{sh}t)^{\frac{2\lambda+1}{p}-(\alpha+\frac{2\lambda+1-\gamma}{p})}}\|f\|_{L_{p,\lambda}}\lesssim(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\|f_t\|_{L_{p,\lambda}}\approx(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}},\quad 0<t<1. \tag{3.12}$$

Now let $1\le t<\infty$. Then from (3.9), we have

$$\|f_t\|_{L_{p,\lambda}}=\Big(\int_0^\infty |f_t(\mathrm{ch}x)|^p \mathrm{sh}^{2\lambda}xdx\Big)^{\frac{1}{p}}\ge\Big(\int_0^\infty |f(\mathrm{ch}(\mathrm{th}t)x)|^p \mathrm{sh}^{2\lambda}xdx\Big)^{\frac{1}{p}}$$

$[(\mathrm{th}t)x=u, x=(\mathrm{cth}t)u]$

$$=(\mathrm{cth}t)^{\frac{1}{p}}\Big(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}(\mathrm{cth}t)udu\Big)^{\frac{1}{p}}$$

$$\le(\mathrm{cth}t)^{\frac{2\lambda+1}{p}}\Big(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}udu\Big)^{\frac{1}{p}}$$

$$\lesssim(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}. \tag{3.13}$$

On the other hand,

$$\|f_t\|_{L_{p,\lambda}}\le\Big(\int_0^\infty |f_t(\mathrm{ch}(\mathrm{sh}t)x)|^p \mathrm{sh}^{2\lambda}xdx\Big)^{\frac{1}{p}}$$

$[(\mathrm{sh}t)x=u, x=\frac{u}{\mathrm{sh}t}]$

$$=(\mathrm{sh}t)^{-\frac{1}{p}}\left(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}\frac{u}{\mathrm{sh}t}du\right)^{\frac{1}{p}}$$

$$\le(\mathrm{sh}t)^{-\frac{2\lambda+1}{p}}\left(\int_0^\infty |f(\mathrm{ch}u)|^p \mathrm{sh}^{2\lambda}u du\right)^{\frac{1}{p}}$$

$$\le(\mathrm{sh}t)^{\alpha+\frac{2\lambda+1-\gamma}{p}-\frac{2\lambda+1}{p}}\|f\|_{L_{p,\lambda}}=(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}}. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\|f_t\|_{L_{p,\lambda}}\approx(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}}\|f\|_{L_{p,\lambda}},\ 1\le t<\infty. \tag{3.15}$$

From (3.12) and (3.15),

$$\|f_t\|_{L_{p,\lambda}}\approx(\mathrm{sh}t)^{\alpha-\frac{\gamma}{q}},\ 0<t<\infty. \tag{3.16}$$

Further, from (3.9) for $0<t<1$, we have

$$\|J_G^{b,\alpha,k}f_t\|_{L_{q,\lambda}}=\left(\int_0^\infty |J_G^{b,\alpha,k}f_t(\mathrm{ch}x)|^q \mathrm{sh}^{2\lambda}x dx\right)^{\frac{1}{q}}$$

$$\le\left(\int_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}(\mathrm{cth}t)x)|^q \mathrm{sh}^{2\lambda}x dx\right)^{\frac{1}{q}}$$

$[(\mathrm{cth}t)x=u, x=(\mathrm{th}t)u]$

$$=(\mathrm{th}t)^{\frac{1}{q}}\left(\int_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda}(\mathrm{th}t)u du\right)^{\frac{1}{q}}$$

$$\le(\mathrm{th}t)^{\frac{2\lambda+1}{q}}\left(\int_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda}u du\right)^{\frac{1}{q}}$$

$$=(\mathrm{th}t)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}=\left(\frac{\mathrm{sh}t}{\mathrm{ch}t}\right)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}$$

$$\lesssim\frac{1}{(\mathrm{ch}t)^{\frac{2\lambda+1}{q}}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}\lesssim\frac{1}{(\mathrm{ch}t)^{\frac{\gamma}{q}}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}$$

$$\lesssim(\mathrm{sh}t)^{-\frac{\gamma}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}. \tag{3.17}$$

On the other hand,

$$\|J_G^{b,\alpha,k}f_t\|_{L_{q,\lambda}}\ge\left(\int_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}(\mathrm{th}t)x)|^q \mathrm{sh}^{2\lambda}x dx\right)^{\frac{1}{q}}$$

$[(\mathrm{th}t)x=u, x=(\mathrm{cth}t)u]$

$$=(\mathrm{cth}t)^{\frac{1}{q}}\left(\int_0^\infty |J_G^{b,\alpha,k}f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda}(\mathrm{cth}t)u du\right)^{\frac{1}{q}}$$

$$\ge(\mathrm{cth}t)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}=\left(\frac{\mathrm{ch}t}{\mathrm{sh}t}\right)^{\frac{2\lambda+1}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}$$

$$\ge\frac{1}{(\mathrm{sh}t)^{\frac{2\lambda+1}{q}}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}\lesssim(\mathrm{sh}t)^{-\frac{\gamma}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}}. \tag{3.18}$$

From (3.17) and (3.18), we have

$$\|J_G^{b,\alpha,k}f_t\|_{L_{q,\lambda}}\approx(\mathrm{sh}t)^{-\frac{\gamma}{q}}\|J_G^{b,\alpha,k}f\|_{L_{q,\lambda}},\quad 0<t<1. \tag{3.19}$$

Now let $1 \le t < \infty$. Then from (3.9), we get

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \ge \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}(\mathrm{sh}t)x)|^q \mathrm{sh}^{2\lambda} x dx \right)^{\frac{1}{q}}$$

$$[(\mathrm{sh}t)x = u, x = \frac{u}{\mathrm{sh}t}]$$

$$= (\mathrm{sh}t)^{-\frac{1}{q}} \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda} \frac{u}{\mathrm{sh}t} du \right)^{\frac{1}{q}}$$

$$\le (\mathrm{sh}t)^{-\frac{2\lambda+1}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} \le (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}}. \tag{3.20}$$

On the other hand,

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \ge \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}(\mathrm{th}t)x)|^q \mathrm{sh}^{2\lambda} x dx \right)^{\frac{1}{q}}$$

$$[(\mathrm{th}t)x = u, x = (\mathrm{cth}t)u]$$

$$= (\mathrm{cth}t)^{\frac{1}{q}} \left(\int_0^\infty |J_G^{b,\alpha,k} f(\mathrm{ch}u)|^q \mathrm{sh}^{2\lambda} (\mathrm{cth}t) u du \right)^{\frac{1}{q}}$$

$$\ge (\mathrm{cth}t)^{\frac{2\lambda+1}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} \ge (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}}. \tag{3.21}$$

From (3.20) and (3.21), we have

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \approx (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}}, \ 1 \le t < \infty. \tag{3.22}$$

Combining (3.19) and (3.22), we obtain

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \approx (\mathrm{sh}t)^{-\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}}, \ 0 < t < \infty. \tag{3.23}$$

Taking into account inequality (3.8), as well as (3.23) and (3.16), we obtain

$$\|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}} \approx (\mathrm{sh}t)^{\frac{\gamma}{q}} \|J_G^{b,\alpha,k} f_t\|_{L_{q,\lambda}}$$

$$\lesssim (\mathrm{sh}t)^{\frac{\gamma}{q}} \|f_t\|_{L_{q,\lambda}} \lesssim (\mathrm{sh}t)^{\alpha - \frac{\gamma}{q} + \frac{\gamma}{q}} \|f\|_{L_{q,\lambda}} = (\mathrm{sh}t)^{\alpha - \gamma(\frac{1}{q} - \frac{1}{q})} \|f\|_{L_{q,\lambda}}.$$

If $\frac{1}{q} - \frac{1}{q} < \frac{\alpha}{\gamma}$, then, as $t \to 0$, we have

$$\|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} = 0 \text{ for all } f \in L_{q,\lambda}(\mathbb{R}_+).$$

If $\frac{1}{q} - \frac{1}{q} > \frac{\alpha}{\gamma}$ then, as $t \to \infty$,

$$\|J_G^{b,\alpha,k} f\|_{L_{q,\lambda}} = 0 \text{ for all } f \in L_{q,\lambda}(\mathbb{R}_+),$$

which cannot be true.

Therefore,

$$\frac{1}{q} - \frac{1}{q} = \frac{\alpha}{\gamma}. \qquad \square$$

4. Commutators of the Gegenbauer Fractional Maximal Operator

Let $b \in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)$, then the $k-th$ order commutator $M_G^{b,\alpha,k}$ generated by the function b and M_G^α is defined as follows:

$$M_G^{b,\alpha,k} f(\mathrm{ch}x) =$$

$$= \sup_{r \in \mathbb{R}_+} \frac{1}{|H|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_H |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy, k = 1, 2, \dots,$$

where $H = H(0, r)$.

Theorem 4.1. *Suppose that* $0 < \alpha < \gamma \le 2\lambda + 1$, $1 < p < \frac{\gamma}{\alpha}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$. *Then the commutator* $M_G^{b,\alpha,k}$ *is bounded from* $L_{p,\lambda}(\mathbb{R}_+)$ *to* $L_{q,\lambda}(\mathbb{R}_+)$, *if and only if* $b \in BMO_G(\mathbb{R}_+)$.

Proof. Let $b \in BMO_G(\mathbb{R}_+)$. For the fixed $x \in \mathbb{R}_+$ and $r > 0$, we have

$$J_G^{b,\alpha,k}|f(\mathrm{ch}x)| = \int\limits_{\mathbb{R}_+} \frac{|A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$

$$\ge \int\limits_0^r \frac{|A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k}{(\mathrm{sh}y)^{\gamma-\alpha}} A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$

$$\ge \frac{1}{(\mathrm{sh}y)^{\gamma-\alpha}} \int\limits_0^r |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$

$$\approx \frac{1}{|H|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_H |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)|^k A_{\mathrm{ch}y}^\lambda |f(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy. \tag{4.1}$$

Taking supremum for $r > 0$ on both sides of (4.1), we obtain

$$M_G^{b,\alpha,k} f(\mathrm{ch}x) \lesssim J_G^{b,\alpha,k}(|f|)(\mathrm{ch}x), \quad \forall \in \mathbb{R}_+.$$

Thus, when $b \in BMO_G(\mathbb{R}_+)$, from this and Theorem 3.1, we have

$$\|M_G^{b,\alpha,k} f(\mathrm{ch}x)\|_{L_{q,\lambda}(\mathbb{R}_+)} \lesssim \|f\|_{L_{p,\lambda}(\mathbb{R}_+)}.$$

On the other hand, suppose that $M_G^{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$. Choose any interval H in $\mathbb{R}_+$,

$$\frac{1}{|H|_\lambda} \int\limits_H |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy$$

$$\approx \frac{1}{|H|_\lambda^2} \int\limits_H |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)| \mathrm{sh}^{2\lambda} y dy \cdot \int\limits_H A_{\mathrm{ch}y}^\lambda \chi_H(\mathrm{ch}x) \mathrm{sh}^{2\lambda} x dx$$

$$\approx \frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}} \int\limits_H \left(\frac{1}{|H|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_H |A_{\mathrm{ch}y}^\lambda b(\mathrm{ch}x) - b_H(\mathrm{ch}x)| \cdot A_{\mathrm{ch}y}^\lambda \chi_H(\mathrm{ch}x) \mathrm{sh}^{2\lambda} x dx \right) \mathrm{sh}^{2\lambda} y dy$$

$$\approx \frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}} \int\limits_H M_G^{b,\alpha}(\chi_H(\mathrm{ch}x)) \mathrm{sh}^{2\lambda} y dy$$

$$\lesssim \frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}} \left(\int\limits_H \mathrm{sh}^{2\lambda} y dy \right)^{\frac{1}{q'}} \left(\int\limits_H M_G^{b,\alpha}(\chi_H(\mathrm{ch}x)) \mathrm{sh}^{2\lambda} y dy \right)^{\frac{1}{q}}$$

$$\lesssim \frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_\lambda|^{\frac{1}{q'}} \|M_G^{b,\alpha} \chi_H\|_{L_{q,\lambda}(H)} \lesssim \frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_\lambda|^{\frac{1}{q'}} \|\chi_H\|_{L_{q,\lambda}(H)}$$

$$\lesssim \frac{1}{|H|_\lambda^{1+\frac{\alpha}{\gamma}}} |H_\lambda|^{\frac{1}{q'}} |H_\lambda|^{\frac{1}{p}} \lesssim 1.$$

Thus $b \in BMO_G(\mathbb{R}_+)$. □

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