Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 3, 435–436

ON THE EXISTENCE OF UNIVERSAL SERIES WITH SPECIAL PROPERTIES

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Abstract. An arbitrary system of Lebesgue measurable and almost everywhere finite functions $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ such that there exists a universal series with respect to Φ is considered. A theorem asserting that for any sequence of real numbers $(c_n)_{n=1}^{\infty}$ there exist two universal series with respect to Φ such that every c_n is a product of two corresponding coefficients of these two universal series is formulated.

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be an arbitrary system of Lebesgue measurable and almost everywhere finite functions defined on $[a, b]$.

Definition 1. A series

$$\sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \tag{1}$$

is called a universal series with respect to Φ in the sense of subsequences of partial sums of this series, if for any Lebesgue measurable function $f(x)$ defined on $[a, b]$ and finite or infinite at any point of $[a, b]$ there exists a strictly increasing sequence of natural numbers $(m_k)_{k=1}^{\infty} \uparrow \infty$ such that the equality:

$$\lim_{k\to\infty} \sum_{n=1}^{m_k} \alpha_n \varphi_n(x) = f(x)$$

holds for almost all $x \in [a, b]$.

In what follows, for the sake of brevity, a universal series (1) with respect to Φ in the sense of subsequences of partial sums of (1) is called a universal series with respect to Φ and measurable is applied instead of Lebesgue measurable.

D. E. Menshoff was the first who established the existence of universal trigonometric series and proved that any trigonometric series is a sum of two universal trigonometric series (see [3]). Namely, he proved that for any sequence of real numbers $(c_n)_{n=1}^{\infty}$ there exist two universal trigonometric series with coefficients $\left(\alpha_n^{(1)}\right)_{n=1}^{\infty}$ and $\left(\alpha_n^{(2)}\right)_{n=1}^{\infty}$, respectively, such that for every natural number $n \geq 1$ the following equality

$$c_n = \alpha_n^{(1)} + \alpha_n^{(2)}$$

holds.

A. A. Talaljan proved (see [2, Theorem 9.2.11]) that for any complete and orthonormal system Φ there exists a universal series (1) with respect to Φ such that $\lim_{n\to\infty} \alpha_n = 0$.

Various aspects of the theory of universal series are presented in the article of K. G. Grosse-Erdman [1].

In [4], the above-mentioned result of Menshoff on trigonometric series is generalized for the series with respect to any system Φ of measurable and almost everywhere finite functions such that there exists a universal series with respect to Φ, in particular, for the series with respect to any complete and orthonormal system Φ (see [4, Theorem 1 and Theorem 2]).

The above-indicated results of [3] and [4] hold true not only for the sums of corresponding coefficients of the above-mentioned two universal series, but also for the products of corresponding coefficients of certain two universal series. Namely, the following theorem holds.

2020 *Mathematics Subject Classification.* 40A30, 40A05.
Key words and phrases. System of measurable functions; Universal series; Trigonometric series.

Theorem 1. *Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be an arbitrary system of measurable and almost everywhere finite functions defined on $[a,b]$ and $(c_n)_{n=1}^{\infty}$ be any sequence of real numbers, then a necessary and sufficient condition for the validity of the equality*

$$c_n = \alpha_n^{(1)} \cdot \alpha_n^{(2)}$$

for every natural number $n \geq 1$, where $\sum\limits_{n=1}^{\infty} \alpha_n^{(1)} \varphi_n(x)$ and $\sum\limits_{n=1}^{\infty} \alpha_n^{(2)} \varphi_n(x)$ are certain universal series with respect to Φ, is the existence of a universal series with respect to Φ.

A consequence of Theorem 1 and of the above-mentioned theorem of A. A. Talaljan is the following

Theorem 2. *Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be any complete and orthonormal system of functions defined on $[a,b]$, then for any sequence of real numbers $(c_n)_{n=1}^{\infty}$, there exist two universal series $\sum\limits_{n=1}^{\infty} \alpha_n^{(1)} \varphi_n(x)$ and $\sum\limits_{n=1}^{\infty} \alpha_n^{(2)} \varphi_n(x)$ with respect to Φ such that the equality*

$$c_n = \alpha_n^{(1)} \cdot \alpha_n^{(2)}$$

holds for every natural number $n \geq 1$.

Proposition 1. *For any system Φ of measurable and almost everywhere finite functions defined on $[a,b]$ such that there exists a universal series with respect to Φ, in particular, for any complete and orthonormal system Φ, there exist two series*

$$\sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \varphi_n(x)$$

with respect to Φ such that every one of them is a universal series with respect to Φ.

Acknowledgement

This work was supported by the Shota Rustaveli National Scientific Foundation of Georgia through the Grant No. FR-18-2499.

References

1. K. G. Grosse-Erdman, Universal families and hypercyclyc operators. *Bulletin of the American Mathematical Society* **36** (1999), no. 3, 345–381.
2. S. Kacmarz, H. Steinhaus, *Theorie der Ortogonalreihen.* Warszawa/Lwów, 1935; Russian translation: Moscow, 1958.
3. D. E. Menchoff, Sur les sommes partielles des séries trigonométriques. (Russian) *Rec. Math. N.S.* **20(62)** (1947), 197–238.
4. Sh. Tetunashvili, Functional series representable as a sum of two universal series. (Russian) *translated from Dokl. Akad. Nauk* 477 (2017), no. 3, 276–277 *Dokl. Math.* **96** (2017), no. 3, 578–579.

(Received 16.02.2020)

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