ON THE SECONDARY COHOMOLOGY OPERATIONS

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Abstract. The new secondary cohomology operations are constructed. These operations together with the Adams operations are intended to calculate the mod $p$ cohomology algebra of loop spaces. In particular, the kernel of the loop suspension map is explicitly described.

1. Introduction

Let $X$ be a topological space and $H^*(X; \mathbb{Z}_p)$ be the cohomology algebra in the coefficients $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ where $\mathbb{Z}$ is the integers and $p$ is a prime. Given $n \geq 1$, let $P_n^*(X) \subset H^*(X; \mathbb{Z}_p)$ be the subset of elements of finite height $P_n^*(X) = \{x \in H^*(X; \mathbb{Z}_p) | x^{n+1} = 0, n \geq 1\}.$

Let $\mathcal{P}_1 : H^m(X; \mathbb{Z}_p) \to H^{mp-p+1}(X; \mathbb{Z}_p)$ denote the Steenrod cohomology operation. Given $n, r \geq 1$, we construct the maps

\[ \psi_{r,1} : H^{2m+1}(X; \mathbb{Z}_p) \to H^{2mp^{r+1}+1}(X; \mathbb{Z}_p)/\text{Im} \mathcal{P}_1, \quad p > 2, \quad (1.1) \]

and

\[ \psi_{r,n} : P_n^m(X) \to H^{(m(n+1)-2)p^{r+1}+1}(X; \mathbb{Z}_p)/\text{Im} \mathcal{P}_1 \quad (m \text{ is even when } p > 2) \quad (1.2) \]

in which $\psi_{1,p^{r+1}} = \psi_k$ is the Adams secondary cohomology operation for $p$ odd or $p = 2$ and $k > 1$ (cf. [1-3]). Note that when $n > 1$, these maps are linear for $n + 1 = p^k$, $k \geq 1$ (e.g., $H^*(X; \mathbb{Z}_p)$ is a Hopf algebra). Let $\Omega X$ be the (based) loop space on $X$. Let $\sigma : H^*(X; \mathbb{Z}_p) \to H^{*+1}(\Omega X; \mathbb{Z}_p)$ be the loop suspension map. Theorem 2 (cf. [3]) explicitly describes $\text{Ker} \sigma$ in terms of the operations $\mathcal{P}_1$ and $\psi_{1,n}$ and higher order Bockstein homomorphisms $\beta_k$ associated with the short exact sequence

\[ 0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^{k+1}} \to \mathbb{Z}_{p^k} \to 0. \]

The calculation of the loop space cohomology algebra $H^*(\Omega X; \mathbb{Z}_p)$ in terms of generators and relations will appear elsewhere.

2. The Secondary Cohomology Operations $\psi_{r,n}$

The secondary cohomology operations are constructed by using the integral filtered model of a space $X$ considered in [4].

2.1. The Hirsch filtered models of a space. Given a commutative graded algebra (cga) $H$, there are two kinds of Hirsch resolutions

\[ \rho_a : (R_a H, d) \to H \quad \text{and} \quad \rho : (RH, d) \to H, \]

the absolute Hirsch resolution $R_a H$ and the minimal Hirsch resolution $RH$, respectively. The first $R_a H$ is endowed, besides the Steenrod cochain operation $E_{1,1} = -\omega$, the cup-one product, with the higher order operations $E_{p,q}, p, q \geq 1$, as they usually exist in the cochain complex $C^*(X; \mathbb{Z})$; the second $RH$ is, in fact, endowed only with the cup-one measuring the non-commutativity of the cup product $\cdot = \omega$. In general, the operations $E_{p,q}$ appear to measure the deviations of the cup-one product from being the left and right derivations with respect to the cup product. But in $RH$ the freeness of the multiplicative structure enables us to fix the relationship between the cup and cup-one operations.
products by explicit formulas, while the relation between $RH$ and the cochain complex $C^*(X;\mathbb{Z})$ is fixed via zig-zag Hirsch maps

$$(RH, d_h) \xrightarrow{z} (R_a H, d_h) \xrightarrow{f} C^*(X;\mathbb{Z}). \quad (2.1)$$

In fact, $RH = R_a H/J$ for a certain Hirsch ideal $J \subset R_a H$. Thus, the Hirsch algebra $(RH, d_h)$, being generated only by the $\varpi_1$-product, becomes an efficient tool for calculating of the loop cohomology algebra.

Denote $H^* = H^*(X;\mathbb{Z})$. Given a prime $p$, let $t_{z_p} : RH \to RH \otimes \mathbb{Z}_p$ be the standard map. For $z = [c] \in H^*(X;\mathbb{Z}_p)$ with $c \in RH \otimes \mathbb{Z}_p$, let $x_0 := t_{z_p}^{-1}(c)$. If $c \in P_n^*(X)$, then in $RH$ there is the equality $dx_1 = x_0^p x_1 = 0 \mod p$, some $x_1 \in RH$, and $p$ does not divide $\lambda$. Note that the essential idea can be seen for $n = 1$ (the case $n > 1$ is somewhat technically difficult only). Each $z \in P_n^*$ produces an infinite sequence of elements $(x_m)_{m \geq 0}$ in $RH$ given by the following formulas:

$$dx_{2k+1} = \sum_{i_1, \ldots, i_{n+1} = k} (-1)^{|z|} \lambda x_{2i_1} \cdots x_{2i_{n+1}} + \sum_{i+j = 2k-1} x_{2i+1} x_{2j+1} + p\tilde{x}_{2k+1},$$

$$dx_{2k} = \sum_{i+j = 2k-1} (-1)^{|z|} x_{i} x_j + p\tilde{x}_{2k}, \quad i, j, k \geq 0.$$

(The signs are fixed for $|z|$ and $n+1$ to be not simultaneously odd above.) In particular, when $z$ is odd dimensional and $n, \lambda = 1$, one gets for $k, i, j \geq 0$:

$$dx_k = \sum_{i+j=k-1} x_i x_j + p\tilde{x}_k.$$

In turn, the sequence $(x_m)_{m \geq 0}$ by means of the $\varpi_1$-product induces four kinds of infinite sequences $b_{k,\ell}^{i_1, i_2} \in \left\{b_{k,\ell}^{i_1, i_2}, b_{k,\ell}^{i_1, i_2}, b_{k,\ell}^{i_1, i_2}, b_{k,\ell}^{i_1, i_2}\right\}$ in $RH$ for $n \geq 1$ (more precisely, one sequence $(b_{k,\ell})_{k, \ell \geq 1}$ when $n = 1$) with $b_{k,\ell} := b_{k,\ell}^{1,1}$ (when $n > 1$, while $k, \ell \geq 1$ when $n = 1$), $b_{2i,j}^1 = b_{2i,j}^{1,1}$, $i, j \geq 1$, defined by the recursive formulas: $b_{1,1}^1 = (-1)^{|z|} b_{1,1}^1$ for $(k, \ell) = (1, 1)$, and

$$db_{1,1}^1 = \begin{cases} 2x_1 + \lambda x_0 \varpi_1 x_0^n, & |z| \text{ is odd,} \\ x_0 \varpi_1 x_0^n, & |z| \text{ is even,} \end{cases}$$

(in the latter case, we, in fact, have $b_{1,1}^1 = \sum_{i+j=n-1} x_i b_{1,1}^1 x_j^j$, and for $k, \ell \geq 1$:

$$db_{k,\ell}^{*,*} = \begin{cases} (-1)^{|z|} \alpha_{k,\ell}^{*,*} x_{k+\ell-1}^{(s)} + x_{k-1}^{(r)} \varpi_1 x_{\ell-1}^{(r)} \\ + \sum_{0 \leq r < k, 0 \leq m \leq \ell} \left((-1)^{r+|z|} \alpha_{r,m}^{*,*} x_{k-r,\ell-m}^{s,*} x_{r+m-1}^{(s)} - (-1)^{r} \alpha_{r,m}^{(s)} x_{r-1}^{(r)} \varpi_1 x_{\ell-1}^{(r)}\right) + \tilde{b}_{k,\ell}^{*,*} \end{cases}. \quad (2.2)$$

with the convention $x_{-1} \varpi_1 x_{m} = x_{m} \varpi_1 x_{-1} = -x_m$, and $\alpha_{s,t} := \alpha_{s,t}^{1,1} = \alpha_{s,t}^{n,n} = \alpha_{s,t}^{1,n} = \alpha_{s,t}^{n,1}$; in particular, for $|z|$ odd:

$$\alpha_{s,t} = \begin{cases} \left(\frac{s+t}{s}\right), & n = 1, \\ \left(\frac{s+t}{s}\right) / 2, & n > 1 \text{ and } s, \ell \text{ are even, mod } p, \\ \left(\frac{s+t-1}{s}\right) / 2, & n > 1 \text{ and } s \text{ is even and } t \text{ is odd, mod } p, \\ 0, & n > 1 \text{ and } s, t \text{ are odd, mod } p, \end{cases}$$
and for \(|x|\) even:

\[
\alpha_{s,t} = \begin{cases} 
\binom{(s+t)/2}{s/2}, & n \geq 1 \text{ and } s, \ell \text{ are even,} \mod p, \\
\binom{(s+t-1)/2}{s/2}, & n \geq 1 \text{ and } s \text{ is even and } t \text{ is odd,} \mod p, \\
0, & n \geq 1 \text{ and } s, t \text{ are odd,} \mod p.
\end{cases}
\]

Therefore, when \(|z|\) is odd and \(n, \lambda = 1\), formula (2.2) takes the form

\[
db_{k,\ell} = \binom{k+\ell}{k} x_{k+\ell-1} + x_{k-1} \varpi_{x-1} - \sum_{1 \leq r < k} \sum_{1 \leq m < \ell} \binom{r+m}{r} b_{k-r,\ell-m} x_{r+m-1} + (x_{r-1} \varpi_{x-1} m_{m-1}) b_{k-r,\ell-m} - \sum_{1 \leq r < k} (b_{k-r,\ell} + b_{k-r,\ell} x_{r-1} - x_{r-1} (b_{k-r,\ell} + b_{k-r,\ell})) + p\tilde{b}_{k,\ell}.
\]

The values of the perturbation \(h\) on \(x_q\) and \(b_{k,\ell}^{\ast,\ast}\) are, in fact, purely determined by the transgressive terms \(y_{q+1} := h_\ast x_q\})_{\ast} = R \ast \pi R\ast \ast 1_H\) and \(c_{k,\ell}^{\ast,\ast} := h(\tilde{b}_{k,\ell}^{\ast}) |_{\ast} = R \ast \pi R\ast \ast 1_H\), respectively. Namely,

\[
h_{x_q} = \sum_{\sum_{i=1}^{\ell} r_i \geq q-1, r_i \geq 0, 0 \leq m < q} -b_{x_{\sum_{i=1}^{\ell} r_i} + \sum_{i=1}^{\ell} y_{x_{r_i}} + p h_{x_q} + p h_{x_q}
\]

and denoting \(\gamma_{k,\ell} = \alpha_{k,\ell}^{\ast,\ast,\ast} \cdots \alpha_{k,\ell}^{\ast,\ast} \) and \(m_{[q]} = m_1 + \cdots + m_\ell\),

\[
h_{x_q} = \sum_{1 \leq r < k, 1 \leq \ell < \ell-1} \gamma_{k,\ell} x_{k+\ell-1} \varpi_{x-1} - c_{k,\ell}^{\ast,\ast} + \sum_{1 \leq r < k} \sum_{1 \leq m < \ell} b_{r,\ell}^{\ast,\ast} h_{x_{k-r,\ell-m}} + c_{k,\ell}^{\ast,\ast} + p h_{x_{k,\ell}}.
\]

Furthermore, by means of \(b_{k,\ell}\), we define the elements \(b_{k,\ell} \in RH\) as follows. Fix the integer \(k \geq 1\). Denote \(b_{k,k} = b_{k,k}\) and \(\tau_{k,k} = 1\). If \(b_{k,mk}\) has already been constructed for \(1 \leq m < q\) and \(\tau_{k,qk} := \alpha_{k,(q-1)k} \cdots \alpha_{k,2k} \alpha_{k,k}\), let

\[
b_{k,qk} = \tau_{k,qk} b_{k,qk} - x_{k-1} \varpi_{x-1} b_{k,(q-1)k} = \tau_{k,qk} b_{k,qk} - \tau_{k,(q-1)k} x_{k-1} \varpi_{x-1} b_{k,qk} - \tau_{k,2k} x_{k-1} \varpi_{x-1} b_{k,qk} - x_{k-1} \varpi_{x-1} b_{k,qk} - \tau_{k,qk} b_{k,qk}.
\]

Then

\[
dh_{b_{k,qk}} = \tau_{k,qk} x_{k+qk-1} + x_{k-1} \varpi_{x-1} + u_{k,qk} + p\tilde{b}_{k,qk} + h_{b_{k,qk}} = \tau_{k,qk} x_{k+qk-1} + x_{k-1} \varpi_{x-1} + u_{k,qk} + \tau_{k,qk} c_{k,qk}.
\]

where \(u_{k,qk} := u_{k,qk} + p\tilde{b}_{k,qk} + (h_{b_{k,qk}} - \tau_{k,qk} c_{k,qk})\) and \(u_{k,qk}\) is expressed by \(x_i\) and \(b_{s,t}\) with \((s,t) \leq (k,qk)\).

a) Let \(p\) be odd. Set \(k = p^r\) and \(q = p - 1\) in (2.4), and define (1.1) for \(z \in H^{2m+1}(X; \mathbb{Z}_p)\) and \(r \geq 1\) by

\[
\psi_{r,1}(z) = \left[f_{z,p}(x_{p^r-1} + w_{p^r,1} p^{r-1})\right];
\]

b) Let \(p\) and \(m\) be not odd simultaneously. Set \(k = 2p^{r-1}\) and \(q = p - 1\) in (2.4), and define (1.2) for \(z \in F_m(X)\) and \(r, n \geq 1\) by

\[
\psi_{r,n}(z) = \left[f_{z,p}(x_{2p^{r-1}-1} + w_{2p^{r-1},1} + p^{r-1})\right].
\]
Theorem 1. For any map \( f : X \to Y \), the following diagrams

\[
\begin{align*}
H^{2m+1}(X; \mathbb{Z}_p) &\xrightarrow{\psi_{r,n}} H^{2mp_{r+1}+1}(X; \mathbb{Z}_p)/\text{Im } \mathcal{P}_1 \\
\left. f^* \right\uparrow &\quad \left. f^* \right\uparrow \\
H^{2m+1}(Y; \mathbb{Z}_p) &\xrightarrow{\psi_{r,n}} H^{2mp_{r+1}+1}(Y; \mathbb{Z}_p)/\text{Im } \mathcal{P}_1
\end{align*}
\]

and

\[
\begin{align*}
P^n_m(X) &\xrightarrow{\psi_{r,n}} H^{(m(n+1)-2)p_{r+1}}(X; \mathbb{Z}_p)/\text{Im } \mathcal{P}_1 \\
\left. f^* \right\uparrow &\quad \left. f^* \right\uparrow \\
P^n_m(Y) &\xrightarrow{\psi_{r,n}} H^{(m(n+1)-2)p_{r+1}}(Y; \mathbb{Z}_p)/\text{Im } \mathcal{P}_1
\end{align*}
\]

commute.

Sketch of the proof. Define the cohomology operations on \( H^*(C^*(X; \mathbb{Z}_p)) \) by means of the canonical operations \( \{E_{p,q}\}_{p,q \geq 1} \) on the cochain complex \( C^*(X; \mathbb{Z}_p) \) ([4]) that agree with \( \psi_{r,n} \) on \( H^*(RH, d_h) \) via zig-zag maps (2.1).

Let \( \mathcal{D}^* := H^+(X; \mathbb{Z}_p) \cdot H^+(X; \mathbb{Z}_p) \subset H^*(X; \mathbb{Z}_p) \) be the decomposables and \( \mathcal{P}_1^{(m)} \) denote \( m \)-fold composition \( \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_1 \).

Theorem 2. Let \( H^*(X; \mathbb{Z}_p) \) be a Hopf algebra. Given \( r \geq 1 \), let \( p(r) \) denote the largest integer such that \( p^{p(r)} \) divides the factorial \( p! \). Let \( \mathcal{I}^* \subset H^*(X; \mathbb{Z}_p) \) be the subset of indecomposables defined for \( a \in \mathcal{I}^*, z \in H^*(X; \mathbb{Z}_p) \) and the integer \( \kappa_z \geq 1 \) such that \( \beta_{p(t)} \mathcal{P}_1^{(t)}(z) = \beta_{p(t)} \mathcal{P}_1^{(t-1)}\psi_{1,n}(z) = 0 \mod \mathcal{D}^* \) for \( t < \kappa_z \) and

a) For \( p > 2 \):

\[
a = \begin{cases} \\
\beta_{p(\kappa_z)} \mathcal{P}_1^{(\kappa_z)}(z), & n = 1 \text{ and } z \text{ is odd dimensional,} \\
\beta_{p(\kappa_z)} \mathcal{P}_1^{(\kappa_z-1)}\psi_{1,n}(z), & n > 1 \text{ and } z \text{ is even dimensional.}
\end{cases}
\]

b) For \( p = 2 \):

\[
a = \beta_{2(\kappa_z)} \mathcal{P}_1^{(\kappa_z-1)}\psi_{1,n}(z), \quad n \geq 1.
\]

Then \( \text{Ker } \sigma = \mathcal{I}^* \cup \mathcal{D}^* \).

Proof. The map \( \tau : RH \otimes \mathbb{Z}_p \to \tilde{V} \otimes \mathbb{Z}_p, a \otimes 1 \to \overline{a} \otimes 1 \) realizes the loop suspension map \( \sigma \) as (cf. [4])

\[
\sigma : H^m(X; \mathbb{Z}_p) \approx H^m(RH \otimes \mathbb{Z}_p, d_h) \xrightarrow{\mathcal{P}_1} H^{m-1}(\tilde{V} \otimes \mathbb{Z}_p, \tilde{d}_h) \approx H^{m-1}(\Omega X; \mathbb{Z}_p).
\]

The inclusion \( \mathcal{D}^* \subset \text{Ker } \sigma \) immediately follows from the above definition of \( \sigma \). Let \( a \in \text{Ker } \sigma \) be indecomposable. Then for \( y \in RH \) with \( \{t_{\mathbb{Z}_p}(y)\} = a \), there is the sequence \( (x_m)_{m \geq 0} \) in \( RH \) and \( r \geq 1 \) such that

\[
d_h(x_{m-1}) = y + u_{m-1} \pmod{p}, \\
d_h(x_i) = u_i \pmod{p}, \quad u_i \in \mathcal{D}^*, \quad i < m 
\]

for

\[
m = \begin{cases} \\
2p^r, & p \text{ and } |x_0| \text{ are odd,} \\
2p^{r-1}, & \text{otherwise.}
\end{cases}
\]

Let \( z = \frac{p^{p(r)}}{p^r} \{t_{\mathbb{Z}_p}(x_0)\} \). Denote \( \kappa_z := r \). Then taking into account (2.3) and the coefficients \( \theta_{k,qk} \) of \( x_{k+qk-1} \) in (2.4) for \( q = p - 1 \) and \( k = p^r \) and \( k = 2p^{r-1} \), \( 1 \leq t \leq \kappa_z \), we establish the equalities of Items a) – b) as desired. Hence, \( a \subset \mathcal{I}^* \). The implication \( \mathcal{I}^* \cup \mathcal{D}^* \subset \text{Ker } \sigma \) is obvious. \( \Box \)
REFERENCES


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