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BANACH SPACE VALUED FUNCTIONALS OF THE WIENER PROCESS

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Abstract. The problem of representation of the Banach space-valued functionals of the one- dimensional Wiener process by the Ito stochastic integral is considered. Earlier, in [5] we have developed this problem in case the joint distribution of the Wiener process and its functional is Gaussian. In this article we consider the general case: firstly, for the weak second order Banach space-valued functional the generalized random process is found as an integrand. Further, for the one-dimensional functional of the Wiener process the sequence of adapted step functions converging to the integrand function, generalizing the corresponding result for the Gaussian case, is obtained (see [2]); the sequence of adapted step functions of generalized random elements converging to the integrand generalized random process is constructed for a Banach space-valued functional.

In developing the Ito stochastic analysis in a Banach space the main goal of the problem is to construct the stochastic integral in an arbitrary separable Banach space. This problem is considered in the following cases: (a) the integrand adapted to the σ-algebra generated by the Wiener process is Banach space-valued and the stochastic integral is constructed by the one-dimensional Wiener process; (b) the integrand adaptive process is operator-valued (from the Banach space to the Banach space), and the stochastic integral is constructed by the Wiener process in a Banach space; (c) the integrand adaptive process is operator-valued (from the Hilbert space to the Banach space), and the stochastic integral is constructed by the cylindrical Wiener process in a Hilbert space. In all the above-mentioned cases the main difficulties are the same. Therefore, to realize simply all these difficulties, in the previous article [5] and here we consider the first case (the Wiener process is one-dimensional).

Using traditional methods, it becomes possible to find the suitable conditions that guarantee the construction of the Ito stochastic integral in a Banach space only in a very narrow class of Banach spaces. This class is called the class of UMD Banach spaces (for survey, see [8]). In our approach, the generalized stochastic integral for a wide class of adapted Banach space-valued random processes is constructed and the problem of the existence of the stochastic integral is reduced to the problem of decomposability of the generalized random element (cylindrical random element, or random linear function) (see [4]).

In this article we consider the problem of representation of the functional of the Wiener process by the stochastic integral in an arbitrary separable Banach space. This problem is, in some sense, opposite to the problem of the existence of the stochastic integral: in this case we have the Banach space-valued random element and the problem of finding the integrand Banach space-valued adapted process whose stochastic integral is this random element. In [5], we considered this problem in the case, where the joint distribution of the Wiener process and its functional is Gaussian. In [3], this problem is considered for the case of UMD Banach space, where under special conditions the Wiener functional is represented by the stochastic integral and the Clark–Ocone formula of representation of the functional of the Wiener process is generalized.

Let X be a real separable Banach space, X^* be its conjugate, and (Ω, B, P) a probability space.

Remember that the continuous linear operator $T : X^* \to L_2(\Omega, B, P)$ is called the generalized random element (GRE) Denote by $\mathcal{M}_1 := L(X^*, L_2(\Omega, B, P)$ the Banach space of GRE with the norm

$$\|T\|^2 = \sup_{\|x^*\| \leq 1} E(Tx^*)^2.$$

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We can realize the weak second order random element ξ as an element of $\mathcal{M}_1$, $T_\xi x^* = \langle \xi, x^* \rangle$ (the boundedness of this operator follows by the closed graph theorem), but not conversely: in an infinite-dimensional Banach space, for any $T : X^* \to L_2(\Omega, B, P)$, there does not always exist the random element $\xi : \Omega \to X$ such that $Tx^* = \langle \xi, x^* \rangle$ for all $x^* \in X^*$. The problem of the existence of such random element is well known as the problem of decomposability (radonizability) of the GRE. Denote by $\mathcal{M}_2$ the linear normed space of all random elements of weak second order with the norm

$$\|\xi\|^2 = \sup_{\|x^*\| \le 1} E\langle \xi, x^* \rangle^2.$$

Thus we have $\mathcal{M}_2 \subset \mathcal{M}_1$.

The family of linear operators $(T_t)_{t\in[0,1]}$ is called the weak second order generalized random process (GRP) if $T_t x^*$ is $B([0,1]) \times B(\Omega)$ measurable and

$$\|T_t\|^2 \equiv \sup_{\|x^*\| \le 1} \int_0^1 E(T_t x^*)^2 dt < \infty.$$

Denote by $\mathcal{M}_1^{(\lambda,P)}$ the Banach space of such GRP.

The Banach space-valued stochastic process $f(t, \omega)$, $t \in [0, 1]$ is called a weak second order random process, if for all $x^* \in X^*$,

$$\int_0^1 E\langle f(t, \omega), x^* \rangle^2 dt < \infty.$$

The weak second order random process realizes the GRP $T_f : X^* \to L_2([0, 1] \times \Omega)$: $T_f x^* = \langle f(t, \omega), x^* \rangle$.

Denote by $\mathcal{M}_2^{(\lambda,P)}$ the normed linear spaces of $f(t, \omega)$, $t \in [0, 1]$, with the norm

$$\sup_{\|x^*\| \le 1} \left(\int_0^1 E\langle f(t, \omega), x^* \rangle^2 dt \right)^{\frac{1}{2}}.$$

We have $\mathcal{M}_2^{(\lambda,P)} \subset \mathcal{M}_1^{(\lambda,P)}$.

Let $(W_t)_{t\in[0,1]}$ be a real-valued Wiener process. Denote by F_t^W the σ-algebra generated by the random variables $(W_s)_{s\le t}$ ($F_t^W = \sigma(W_s, s \le t)$), which are completed by P-null sets. Suppose that ξ is F_1^W-measurable weak second order random element i.e., ξ is the functional of the Wiener process. Our main aim is to represent the random element ξ by the Ito stochastic integral

$$\xi = E\xi + \int_0^1 f(t, \omega) dW_t,$$

where $f(t, \omega)$ is the Banach space-valued F_t^W-adapted random process, but this is, in general, impossible. We have the following positive result: For all weak second order Wiener functional we always have integrand as a GRP, that is, an element of the Banach space $\mathcal{M}_1^{(\lambda,P)}$. In developing this problem, we considered firstly in [5] the case, where ξ is a Gaussian random element which together with the Wiener process generates the mutually Gaussian system. Even in this case we have constructed an example (see [5, Example 1]), where a) the integrand function $f(t)$ (in this case the integrand is nonrandom) is X-valued; b) the integrand function is not X-valued, but it is X^{**}-valued and c) the integrand function is not X^{**}-valued, but it is a GRE $T : X^* \to L_2[0, 1]$.

The following result gives representation of the Banach space-valued functional of the Wiener process by the stochastic integral from the F_t^W-adapted GRP.

Proposition 1. *Let ξ be a Banach space-valued weak second order functional of the one-dimensional Wiener process. There exists the F_t^W-adapted GRP $T : X^* \to L_2([0,1] \times \Omega)$ such that for all $x^* \in X^*$*

$$\langle \xi, x^* \rangle = E\langle \xi, x^* \rangle + \int_0^1 Tx^*(t, \omega) dW_t. \tag{0.1}$$

Proof. Let ξ be a Banach space-valued weak second order functional of the one-dimensional Wiener process. For any $x^* \in X^*$, $\langle \xi, x^* \rangle$ is one-dimensional functional of the Wiener process. By the one-dimensional theorem, there exists the unique F_t^W-adapted one-dimensional random process $f(x^*, t, \omega)$ such that

$$\langle \xi, x^* \rangle = E\langle \xi, x^* \rangle + \int_0^1 f(x^*, t, \omega) dW_t.$$

Consider the map $T : X^* \to L_2([0,1], \Omega)$, $Tx^* = f(x^*, t, \omega)$. It is easy to see that T is a linear operator. Further,

$$\begin{aligned} \infty > \sup_{\|x^*\| \le 1} E\langle \xi - E\xi, x^* \rangle^2 &= \sup_{\|x^*\| \le 1} E\Big(\int_0^1 f(x^*, t, \omega) dW_t \Big)^2 \\ &= \sup_{\|x^*\| \le 1} \int_0^1 Ef(x^*, t, \omega)^2 dt = \sup_{\|x^*\| \le 1} \int_0^1 E(T_t x^*)^2 dt. \end{aligned}$$

That is, $T : X^* \to L_2([0,1], \Omega)$ is bounded, and therefore, this is the GRP. □

Remark 1. The representation (0.1) of the Wiener functional is unique for any x^* $dt \otimes dP$-almost everywhere. Indeed, if we have two representations of ξ,

$$\langle \xi, x^* \rangle = E\langle \xi, x^* \rangle + \int_0^1 T_1 x^*(t, \omega) dW_t = E\langle \xi, x^* \rangle + \int_0^1 T_2 x^*(t, \omega) dW_t,$$

then

$$\begin{aligned} 0 &= \sup_{\|x^*\| \le 1} E\Big(\int_0^1 (T_1 x^*(t, \omega) - T_2 x^*(t, \omega)) dW_t \Big)^2 \\ &= \sup_{\|x^*\| \le 1} E \int_0^1 (T_1 x^*(t, \omega) - T_2 x^*(t, \omega))^2 dt. \end{aligned}$$

For any GRP $T : X^* \to L_2([0,1] \times \Omega)$ from $\mathcal{M}_1^{(\lambda, P)}$, the correlation operator of T is called the linear, bounded operator from X^* to X^{**}, $R_T = T^* T$.

Proposition 2. *If for any functional of the Wiener process ξ,*

$$\langle \xi, x^* \rangle = \int_0^1 Tx^*(t, \omega) dW_t,$$

then $R_T = T^* T$ maps X^* onto X.

Proof. For any x^* and y^*,

$$\langle R_T x^*, y^*\rangle = \langle T^*Tx^*, y^*\rangle = \langle Tx^*, Ty^*\rangle = \int_0^1 ETx^*(t,\omega)Ty^*(t,\omega)dt$$
$$= E\Big(\int_0^1 Tx^*(t,\omega)dW_t \times \int_0^1 Ty^*(t,\omega)dW_t\Big)$$
$$= E(\langle(\xi - E\xi), x^*\rangle \times \langle(\xi - E\xi), y^*\rangle) = \langle R_\xi x^*, y^*\rangle,$$

where R_ξ is a covariance operator of ξ, which maps X^* onto X (see [7, Theorem 3.2.1]). Therefore, R_T maps X^* onto X. □

As is known (see [2, Theorem 5.6]), for the one-dimensional case, if the joint distribution of the Wiener process and its one-dimensional functional is Gaussian, then the sequence of step functions

$$f_n(t) = \sum_{i=0}^{2^n-1} 2^n E(\xi - E\xi)(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t)$$

converges in $L_2[0,1]$ to the integrand function $f(t)$, $\int_0^1 f^2(t)dt < \infty$ and

$$\xi_n = E\xi + \int_0^1 f_n(t)dW_t$$

converges in $L_2(\Omega, B, P)$ to

$$\xi = E\xi + \int_0^1 f(t)dW_t.$$

First, we give the generalization of this theorem for an arbitrary (non Gaussian) case when the functional of the Wiener process is one-dimensional.

Theorem 1. *Let the square integrable random variable ξ be a functional of the Wiener process. The sequence of step functions*

$$f_n(t,\omega) = \sum_{i=0}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n}, \frac{i+1}{2^n}]}(t) \tag{0.2}$$

converges in $L_2([0,1],\Omega)$ to the F^W_t-adapted random process $f(t,\omega)$ and

$$\xi = \lim_{n\to\infty} \int_0^1 f_n(t,\omega)dW_t = \int_0^1 f(t,\omega)dW_t.$$

Proof. First of all, we prove the following

Lemma 1. *Let $\xi = \int_0^1 f(t,\omega)dW(t)$ be a real-valued functional of the Wiener process. Then for any $0 \le a \le b$,*

$$E((\xi_b - \xi_a)(W_b - W_a)/F^W_a) = E\Big(\int_a^b f(t,\omega)dt/F^W_a\Big),$$

where $\xi_t = E(\xi/F^W_t) = \int_0^t f(s,\omega)dW(s)$.

Proof of Lemma 1. Consider the left part of the equality and denote $m \equiv (b-a)^{-1}$. Remember that by Lemma 1.1.3 from [7], for any $f(t,\omega) \in L_2([0,1]\times\Omega)$, the sum

$$\sum_{i=1}^{2^n-1} 2^n \int\limits_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f(s,\omega) ds I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$

converges to $f(t,\omega)$ in $L_2([0,1]\times\Omega)$.

Next, we have

$$E((\xi_b-\xi_a)(W_b-W_a)/F_a^W) = \lim_{n\to\infty} E\Bigg(\Bigg(\Bigg(\sum_{i=1}^{2^n-1}(2^n m \int\limits_{(a+\frac{(i-1)}{2^n m})\vee 0}^{a+\frac{i}{2^n m}} f(s,\omega)ds)$$

$$\times(W_{a+\frac{(i+1)}{2^n m}} - W_{a+\frac{i}{2^n m}})\Bigg)\Bigg(\sum_{i=0}^{2^n-1}(W_{a+\frac{(i+1)}{2^n m}} - W_{a+\frac{i}{2^n m}})\Bigg)\Bigg)/F_a^W\Bigg)$$

$$= \lim_{n\to\infty} E\Bigg(\sum_{i=1}^{2^n-1}\Bigg(\Bigg(2^n m \int\limits_{(a+\frac{(i-1)}{m2^n})\vee 0}^{a+\frac{i}{m2^n}} f(s,\omega)ds\Bigg)/F_a^W\Bigg)$$

$$\times E(W_{a+\frac{(i+1)}{m2^n}} - W_{a+\frac{i}{m2^n}})^2\Bigg) = \lim_{n\to\infty}\sum_{i=1}^{2^n-1} E\Bigg(2^n m \int\limits_{(a+\frac{(i-1)}{m2^n}\vee 0}^{a+\frac{i}{m2^n}} f(s,\omega)ds)/F_a^W\Bigg)\frac{1}{m2^n}$$

$$= \lim_{n\to\infty} E\Bigg(\sum_{i=1}^{2^n-1}\int\limits_{a+\frac{i}{m2^n}}^{a+\frac{(i+1)}{m2^n}}\Bigg(2^n m \int\limits_{(a+\frac{(i-1)}{m2^n}\vee 0}^{a+\frac{i}{m2^n}} f(s,\omega)ds)I_{(a+\frac{i}{m2^n},a+\frac{(i+1)}{m2^n}]}(t)\Bigg)dt/F_a^W\Bigg)$$

$$= \lim_{n\to\infty} E\Bigg(\sum_{i=1}^{2^n-1}\Bigg(2^n m \int\limits_{(a+\frac{(i-1)}{m2^n}\vee 0}^{a+\frac{i}{m2^n}} f(s,\omega)ds)\frac{1}{m2^n}\Bigg)/F_a^W\Bigg)$$

$$= \lim_{n\to\infty} E\Bigg(\sum_{i=1}^{2^n-1}\int\limits_{(a+\frac{(i-1)}{m2^n})\vee 0}^{a+\frac{i}{m2^n}} f(s,\omega)ds/F_a^W\Bigg) = E\Bigg(\int\limits_a^{a+\frac{1}{m}} f(t,\omega)dt/F_a^W\Bigg),$$

as

$$E\Bigg(\int\limits_{(a+\frac{(i-1)}{m2^n})\vee 0}^{a+\frac{i}{m2^n}} f(s,\omega)ds(W_{a+\frac{(i+1)}{2^n m}} - W_{a+\frac{i}{2^n m}})(W_{a+\frac{(j+1)}{2^n m}} - W_{a+\frac{j}{2^n m}})/F_a^W\Bigg) = 0,$$

when $i \neq j$.

Thus, the proof of the lemma 1 is completed.

Consider now the following sum

$$\sum_{i=1}^{2^n-1} 2^n \int\limits_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega) ds I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

According to Lemma 1.1.3 of [7], this sum converges likewise to $f(t,\omega)$ in $L_2([0,1]\times\Omega)$. Therefore,

$$\sum_{i=1}^{2^n-1} 2^n \int\limits_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f(s,\omega) ds I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t) - \sum_{i=1}^{2^n-1} 2^n \int\limits_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega) ds) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$

tends to 0 in $L_2([0,1]\times\Omega)$. That is,

$$\int_0^1 E\Big(\sum_{i=1}^{2^n-1} 2^n\Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f(s,\omega)ds - \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega)ds\Big) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)\Big)^2 dt \to 0.$$

Hence,

$$\sum_{i=1}^{2^n-1} 2^{2n} E\Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f(s,\omega)ds - \int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega)ds\Big)^2 \Big(\frac{1}{2^n}\Big) \to 0.$$

Therefore,

$$\sum_{i=1}^{2^n-1} 2^n E\Big(E\Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f(s,\omega)ds/F^W_{\frac{i}{2^n}}\Big) - E\Big(\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega)ds/F^W_{\frac{i}{2^n}}\Big)\Big)^2 \to 0.$$

That is,

$$\int_0^1 E\Big(\sum_{i=1}^{2^n-1} 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} f(s,\omega)ds I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$
$$-\sum_{i=1}^{2^n-1} 2^n E\Big(\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega)ds/F^W_{\frac{i}{2^n}}\Big) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)\Big)^2 \to 0.$$

But the first sum converges to the $f(t,\omega)$. Therefore the sequence of F^W_t-adapted step functions

$$\sum_{i=1}^{2^n-1} 2^n E\Big(\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega)ds/F^W_{\frac{i}{2^n}}\Big) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$

converges to $f(t,\omega)$ in $L_2([0,1]\times\Omega)$.

Now we can construct the sequence of step functions $f_n(t,\omega)$, $n\in N$, the stochastic integral of which converges to ξ: let us consider

$$\xi_n = \sum_{i=1}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) \times (W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) = \int_0^1 f_n(t,\omega)dW_t,$$

where

$$f_n(t,\omega) = \sum_{i=1}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$
$$\sum_{i=1}^{2^n-1} 2^n E\Big(\int_{\frac{i}{2^n}}^{\frac{i+1}{2^n}} f(s,\omega)ds/F^W_{\frac{i}{2^n}}\Big) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t). \tag{0.3}$$

Then we have $f_n(t,\omega)\to f(t,\omega)$ in $L_2([0,1]\times\Omega)$ and

$$\int_0^1 f_n(t,\omega)dW_t \to \int_0^1 f(t,\omega)dW_t \quad \text{in } L_2(\Omega). \qquad \square$$

Remark 2. In case when the joint distribution of functional of the Wiener process and of the Wiener process is Gaussian, then

$$f_n(t) = \sum_{i=0}^{2^n-1} 2^n E\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$

$$= \sum_{i=0}^{2^n-1} 2^n E((\xi_{\frac{i}{2^n}} + (\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}}) + (\xi - \xi_{\frac{i+1}{2^n}})))(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$

$$= \sum_{i=0}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$

$$= \sum_{i=0}^{2^n-1} 2^n E(((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}}))/F^W_{\frac{i}{2^n}} I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

Therefore formula (0.3) is the generalization of formula (0.2) for an arbitrary (nonGaussian) case.

Let now ξ be a Banach space-valued functional of the Wiener process. As in the one-dimensional case, consider the sequence of step functions

$$f_n(t,\omega) = \sum_{i=0}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

The random step function $f_n(t,\omega)$ does not always exist as a X-valued random process, because the conditional expectation $E((\xi_t - \xi_s)(W_t - W_s)/F^W_s)$ for the weak second order random element $\xi_t - \xi_s$ does not exist, in general. Nevertheless, we can consider the GRE

$$Tx^* = E(\langle(\xi_t - \xi_s), x^*\rangle(W_t - W_s)/F^W_s).$$

From Proposition 1 and Theorem 1 we immediately obtain the following

Proposition 3. *For any weak second order Banach space-valued functional of the Wiener process $\xi : \Omega \to X$, there exists the sequence of step generalized random functions $(T_n)_{n\in N}$, such that for all $x^* \in X^*$,*

$$\int_0^1 E(T_n x^* - Tx^*)^2 dt \to 0,$$

when $n \to \infty$, where T is the GRP such that

$$\langle \xi, x^*\rangle = E\langle \xi, x^*\rangle + \int_0^1 Tx^*(t,\omega)dW_t.$$

Proof. For any $x^* \in X^*$, let $T_n x^* = \langle f_n(t,\omega), x^*\rangle$. By Theorem 1, $T_n x^* \to Tx^*$, when $n \to \infty$. By Proposition 1, we have

$$\langle \xi, x^*\rangle = E\langle \xi, x^*\rangle + \int_0^1 Tx^*(t,\omega)dW_t. \qquad \square$$

The following theorem is a generalization of the one-dimensional Theorem 1 for the Banach space-valued functionals of the one-dimensional Wiener process.

Theorem 2. *Let ξ be a Banach space-valued F^W_1 measurable weak second order random element such that in the representation* (0.1) *the GRP $T \in \mathcal{M}_1^{\lambda,P}$ $T : [0,1] \to \mathcal{M}_1$ is separable-valued and*

$$\int_0^1 \|T\|^2_{\mathcal{M}_1} < \infty.$$

There exists the sequence of F_t^W-adapted step functions $T_n(t,\omega)$, $n \in N$ converging in $\mathcal{M}_1^{\lambda,P}$ to the F_t^W-adapted GRP $T : X^* \to L_2([0,1],\Omega)$ such that the sequence of the stochastic integrals

$$\int_0^1 T_n x^*(t,\omega) dW_t$$

converges to

$$\int_0^1 T x^*(t,\omega) dW_t = \langle \xi - E\xi, x^* \rangle \quad in \ \ \mathcal{M}_1.$$

Proof. By Proposition 1, for the weak second order random element ξ, there exists the unique GRP $T : X^* \to L_2([0,1],\Omega)$ such that for all $x^* \in X^*$

$$\langle \xi, x^* \rangle = E\langle \xi, x^* \rangle + \int_0^1 T x^*(t,\omega) dW_t.$$

Consider

$$\langle T_n(t,\omega), x^* \rangle = \sum_{i=0}^{2^n-1} 2^n E((\langle \xi_{\frac{i+1}{2^n}}, x^* \rangle - \langle \xi_{\frac{i}{2^n}}, x^* \rangle)(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

By Theorem 1, $\langle T_n(t,\omega), x^* \rangle$, $n \in N$ converges in $L_2([0,1],\Omega)$ to the one-dimensional functional of the Wiener process $Tx^*(t,\omega)$ and we have

$$T_n x^*(t,\omega) = \sum_{i=1}^{2^n-1} 2^n \Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} T x^*(s,\omega) ds \Big) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

Further, it is easy to see that $T_n \in \mathcal{M}_1^{\lambda,P}$ for all $n \in N$ and $\|T_n\| \le \|T\|$. Indeed,

$$\|T_n\|^2 = \sup_{\|x^*\| \le 1} \int_0^1 E\Big(\sum_{i=1}^{2^n-1} 2^n \Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} T x^*(s,\omega) ds I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t) \Big)^2 dt \Big)$$

$$= \sup_{\|x^*\| \le 1} \int_0^1 E\Big(\sum_{i=1}^{2^n-1} 2^{2n} \Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} T x^*(s,\omega) ds \Big)^2 I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t) \Big) dt$$

$$= \sup_{\|x^*\| \le 1} \Big(\sum_{i=1}^{2^n-1} 2^{2n} \frac{1}{2^n} E\Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} T x^*(s,\omega) ds \Big)^2 \Big)$$

$$\le \sup_{\|x^*\| \le 1} \sum_{i=1}^{2^n-1} E \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} (T x^*(s,\omega))^2 ds$$

$$= \sup_{\|x^*\| \le 1} E \int_0^{1-\frac{1}{2^n}} (T x^*(s,\omega))^2 ds \le \|T x^*(t,\omega)\|^2.$$

If $T \in \mathcal{M}_1^{\lambda,P}$ is a continuous function $T : [0,1] \to \mathcal{M}_1$, then $T_n \to T$ in $\mathcal{M}_1^{\lambda,P}$. Really,

$$\|T - T_n\|^2 = \sup_{\|x^*\| \le 1} E \int_0^1 \Big(\sum_{i=1}^{2^n-1} 2^n \Big(\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} T x^*(s,\omega) ds I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t) \Big) - T x^*(t,\omega) \Big)^2 dt$$

$$= \sup_{\|x^*\|\le 1} E\int_0^1 \Big(\sum_{i=1}^{2^n-1}\Big(2^n\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} Tx^*(s,\omega)ds - Tx^*(t,\omega)\Big)I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)\Big)^2 dt$$

$$= \sup_{\|x^*\|\le 1} E\int_0^1 \Big(\sum_{i=1}^{2^n-1}\Big(2^n\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} Tx^*(s,\omega)ds - Tx^*(t,\omega)\Big)\Big)^2 I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)dt$$

$$= \sup_{\|x^*\|\le 1} E\sum_{i=1}^{2^n-1}\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}}\Big(Tx^*(t,\omega) - \Big(2^n\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} Tx^*(s,\omega)ds\Big)\Big)^2 dt$$

$$\le \sum_{i=1}^{2^n-1}\sup_{\|x^*\|\le 1} E\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}}\Big(Tx^*(t,\omega) - \Big(2^n\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} Tx^*(s,\omega)ds\Big)\Big)^2 dt$$

$$\le \sum_{i=1}^{2^n-1}\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}}\sum_{i=1}^{2^n-1}\sup_{\|x^*\|\le 1} E\Big(2^n\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}}(Tx^*(t,\omega) - Tx^*(s,\omega)ds)\Big)^2 dt$$

$$\le \sum_{i=1}^{2^n-1}\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}}\sum_{i=1}^{2^n-1}\Big(2^{2n}\frac{1}{2^n}\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}}\sup_{\|x^*\|\le 1} E(Tx^*(t,\omega) - Tx^*(s,\omega))^2 ds\Big)dt < \varepsilon,$$

as the function $T : [0,1] \to \mathcal{M}_1$ is continuous, for any $\varepsilon > 0$ and sufficiently large n,

$$\sup_{\|x^*\|\le 1} E(Tx^*(t,\omega) - Tx^*(s,\omega))^2 ds)) < \varepsilon,$$

when $|t-s| < \frac{1}{2^n}$.

Consider now an arbitrary separable-valued $T : [0,1] \to \mathcal{M}_1$. Any fixed $x^* \in X^*$ and $g \in L_2(\Omega)$, generates the linear continuous functional $f : \mathcal{M}_1 \to R$,

$$f(x^*, g)(T) = \int_\Omega Tx^*(\omega)g(\omega)dP.$$

The set of such functionals separates the points of the Banach space $\mathcal{M}_1$. As $T : [0,1] \to \mathcal{M}_1$ is separable-valued and $f(x^*, g)T(t)$ is measurable, by the Pettis theorem (see [6, Proposition 1.1.10]), $T : [0,1] \to \mathcal{M}_1$ is measurable. As $\int_0^1 \|T(t)\|dt < \infty$, the Bochner integral $\int_s^t T(t)dt$ exists for all $0 \le s < t \le 1$. Let $T(t)$ be a bounded function. Consider $T_m(t) := m\int_{(t-\frac{1}{m})\vee 0}^t T(s)ds$, $m \in N$. $T_m(t) \to T(t)$ a.s. (see [1, Corollary 2 of Theorem 3.8.5]). By the Lebesgue theorem, $\int_0^1 \|T_m(t) - T(t)\|^2 dt \to 0$. As $T_m(t)$ is continuous for all $m \in N$, there exists the sequence of F_t^W- adapted step functions T_{mn}, $n \in N$ such that $\int_0^1 \|T_{mn}(t) - T_m(t)\|^2 dt \to 0$. Therefore we can choose the sequence of step functions $(T_n)_{n\in N}$ such that $\int_0^1 \|T_n(t) - T(t)\|^2 dt \to 0$. It is now easy to get the sequence of step functions converging to the arbitrary separable-valued $T : [0,1] \to \mathcal{M}_1$, with $\int_0^1 \|T(t)\|^2 dt < \infty$. □

Remark 3. By Proposition 1, for the X-valued weak second order functional of the Wiener process the integrand $T(t,\omega)$ belongs to the Banach space $\mathcal{M}_1^{\lambda,P}$. The existence of step functions converging to the integrand we prove in the case for $T \in L_2([0,1], \mathcal{M}_1)$ which is separable-valued. We prove the convergence in $L_2([0,1], \mathcal{M}_1)$, but there arises the question whether this theorem is true for $\mathcal{M}_1^{\lambda,P}$ without the above restrictions? The answer is unknown.

Remark 4. If the sequence $f_n(t,\omega)$, $n \in N$ is such that the members of it as X-valued random processes exist (for example, the functional ξ has strong p-th moment for any $p > 1$), then from

Theorem 2 it follows that the integrand process $T(t,\omega)$ belongs to $\mathcal{M}_2^{(\lambda,P)} \subset \mathcal{M}_1^{(\lambda,P)}$. If the sequence $f_n(t,\omega)$, $n \in N$, of X-valued random processes converges in X, then the integrand process is X-valued and in this case we have the representation of the Banach space-valued functional by the stochastic integral from the Banach space F_t^W-adapted X-valued random process.

Remark 5. It is easy to see that

$$\sum_{i=0}^{2^n-1} 2^n E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t)$$
$$= \sum_{i=0}^{2^n-1} 2^n E(\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}) I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(t).$$

As

$$E(\xi(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}})$$
$$= E(((\xi - \xi_{\frac{i+1}{2^n}}) + (\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}}) + \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}})$$
$$= E((\xi_{\frac{i+1}{2^n}} - \xi_{\frac{i}{2^n}})(W_{\frac{i+1}{2^n}} - W_{\frac{i}{2^n}})/F^W_{\frac{i}{2^n}}).$$

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